ECE - 301

HW-3 SOLNS.

1. $\frac{dy(t)}{dt} + 4y(t) = x(t).$

$x(t) = 2e^{5t}u(t).$

$y(t) = y_p(t) + y_n(t).$

$y_p(t) \rightarrow$ Soln. to

$\frac{dy(t)}{dt} + 4\cancel{t}y(t) = x(t).$

$y_p(t) = A e^{5(t)} u(t).$

$\Rightarrow \quad 5 \cdot A e^{5(t)} u(t) + 4A e^{5(t)} u(t) = 2e^{5t}u(t).$

$\Rightarrow \quad 9A e^{5(t)} u(t) = 2e^{5(t)} u(t)$

$\Rightarrow \quad A = 2/9.$

$y_n(t) = \cancel{B} B e^{kt} u(t)$ [$y_n(t)$ is soln. to $\frac{dy(t)}{dt} + 4y(t) = \cancel{x(t)}\ 0$].

$\cancel{y_n}$

$B \cancel{B} k e^{kt} u(t) + \cancel{4A}\ 4B\cancel{k}e^{kt} u(t) = 0$

$\Rightarrow \quad Bk + 4B = 0 \quad \Rightarrow k = -4$

$\therefore y(t) = \left(\frac{2}{9}e^{5t} + Be^{-4t}\right) u(t).$

2. Convergence of Fourier series.

a)

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{j\omega_0 kt}$$

$$e_N(t) = x(t) - x_N(t) \rightarrow \text{Approximation error}$$

$$E_N = \int_T |e_N(t)|^2 dt \rightarrow \text{Energy of error.}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt \quad \text{Minimises error energy}$$

If a signal $x(t)$ has a Fourier series representation

then $E_N \rightarrow 0$ as $N \rightarrow \infty$.

For a periodic signal with finite energy

$$\int_T |x(t)|^2 dt < \infty \Rightarrow a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt < \infty$$

and $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ then $E_N \rightarrow 0$ as $N \rightarrow \infty$

$$e(t) = x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{Then} \int_T |e(t)|^2 dt = 0$$

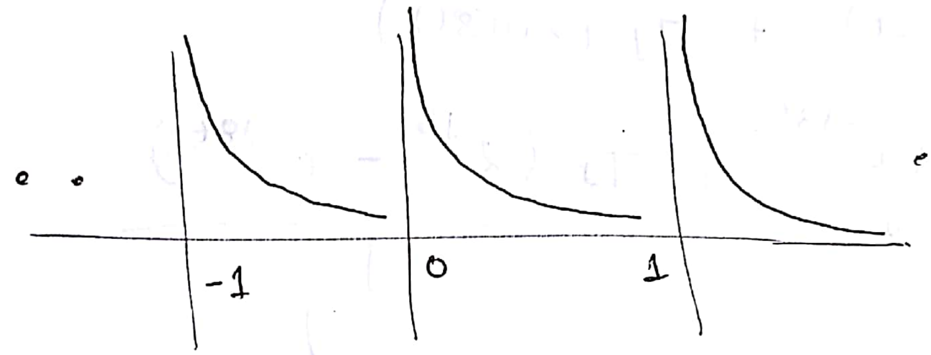
b) Dirichlet conditions

1) $\int_T |x(t)|\, dt < \infty$.

Counter example:

$x(t) = 1/t \quad 0 < t \leq 1$

Repeats with T=1

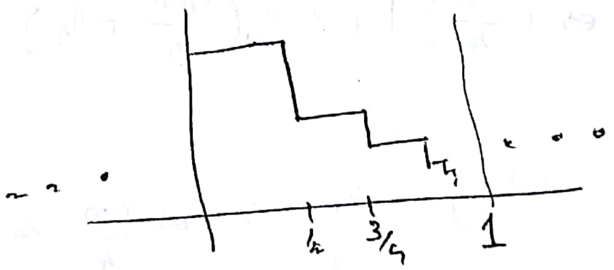


2) In any finite interval in time, there is finite number of maxima or minima.

Counter example: $x(t) = \sin\left(\frac{2\pi}{t}\right)$.

3. In any finite interval of time, there is a finite number of discontinuities and each of these discontinuities are finite.

Counter example:



If a periodic signal satisfies all 3 condns., then the signal and its FS representation are same except at points of discontinuities.

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3. a) $e^{-j\frac{3\pi t}{4}}$

$\omega_0 = \frac{3\pi}{4}$ $\quad a_{-1} = 1$, $a_k = 0, \; k \neq -1$

b). $\cos(3t) + 7j(\sin(8t))$

$$= \frac{e^{j3t} + e^{-j3t}}{2} + \frac{7\cancel{j}(e^{j8t} - e^{-j8t})}{2\cancel{j}}$$

$T = \frac{2\pi}{3}$ $\quad\quad T = \frac{2\pi}{8}$.

$\text{LCM}\left(\frac{2\pi}{3}, \frac{2\pi}{8}\right) = 2\pi = T_0$

$\omega_0 = \frac{2\pi}{T_0} = 1.$

$\Rightarrow$ $a_3 = \frac{1}{2}$ $\quad a_8 = \frac{7}{2}$ $\quad a_k = 0 \quad k \neq 3, -3, 8, -8.$

$a_{-3} = \frac{1}{2}$ $\quad a_{-8} = \frac{-7}{2}$

c) $\left(1 + 3\cos\left(\frac{\pi n}{4}\right)\right)\left(\sin\left(\frac{3\pi n}{4} + \frac{\pi}{3}\right)\right)$

$$= \sin\left(\frac{3\pi n}{4} + \frac{\pi}{3}\right) + \frac{3}{2} \cdot 2\cos\frac{\pi n}{4}\sin\left(\frac{3\pi n}{4} + \frac{\pi}{3}\right)$$

$$= \sin\left(\frac{3\pi n}{4} + \frac{\pi}{3}\right) + \frac{3}{2}\left[\sin\left(\pi n + \frac{\pi}{3}\right) + \sin\left(\frac{\pi n}{2} + \frac{\pi}{3}\right)\right].$$

$$= \frac{e^{j\frac{3\pi}{4}n} \cdot e^{j\frac{\pi}{3}} \cancel{+} - e^{-j\frac{3\pi}{4}} \cdot e^{-j\pi/3}}{2j} + \frac{3}{2}\, \frac{e^{j\pi n} e^{j\frac{\pi}{3}} - e^{-j\frac{\pi}{n}} e^{-j\pi/3}}{2j}$$

$T = 2$

$$\cancel{T = \frac{\frac{3\pi}{4}}{2\pi} = \frac{3}{8}}$$

$T = \frac{8}{3}$

$$+ \frac{3}{2} \cdot \frac{e^{j\frac{\pi n}{2}} e^{j\frac{\pi}{3}} - e^{-j\frac{\pi n}{2}} e^{-j\pi/3}}{2j}$$

$T = 4$

$T_0 = LCM\left(\frac{8}{3}, 4, 2\right) = 8 \Rightarrow \omega_0 = \frac{\pi}{4}$.

$a_5 = \frac{e^{j\frac{\pi}{3}}}{2j}$ $\quad a_{-3} = \frac{-e^{-j\pi/3}}{2j}$

$a_4 = \frac{3}{4j} e^{j\pi/3}$ $\quad a_{-4} = -\frac{3}{4j} e^{-j\frac{\pi}{3}}$

$a_2 = \frac{3}{4j} \cdot e^{j\frac{\pi}{3}}$ $\quad a_{-2} = \frac{-3}{4j} e^{-j\pi/3}$.

d). $1 + \sin(2\omega_0 t) + 3\cos\left(4\omega_0 t + \pi/3\right)$.

$\cancel{\omega_0} \neq \cancel{2\omega_0}$ $\quad T = \frac{2\pi}{2\omega_0} = \frac{\pi}{\omega_0}$ $\quad T = \frac{2\pi}{4\omega_0} = \frac{\pi}{2\omega_0}$.

$T_0 = LCM\left(\frac{\pi}{\omega_0}, \frac{\pi}{2\omega_0}\right) = \frac{\pi}{\omega_0}$.

$\omega_0' = \frac{2\pi}{T_0} = 2\omega_0$

$$x(t) = 1 + \frac{e^{j2\omega_0 t} - e^{-j2\omega_0 t}}{2j} + \frac{3}{2}\left(e^{j4\omega_0 t} e^{\frac{j\pi}{3}} + e^{-j4\omega_0 t} e^{-\frac{j\pi}{3}}\right).$$

$a_0 = 1$

$a_1 = \frac{1}{2j} \quad a_{-1} = \frac{-1}{2j}$

$a_2 = \frac{3}{2} e^{j\frac{\pi}{3}} \quad a_{-2} = \frac{3}{2} e^{-j\pi/3}.$

4\. a) $x(t) = 5\cos(2\omega_0 t + \pi/3).$

$$= \frac{5}{2}\left(e^{j2\omega_0 t} \cdot e^{j\pi/3} + e^{-j2\omega_0 t} e^{-j\pi/3}\right).$$

$\omega_0' = 2\omega_0$

$a_1 = \frac{5}{2} e^{j\pi/3} \quad a_{-1} = \frac{5}{2} e^{-j\pi/3}.$

b) $x[n] = \sum_{k=-\infty}^{\infty} \delta[n-3k]$. Period $N = 3. \Rightarrow \omega_0 = \frac{2\pi}{3}$

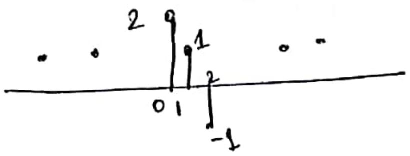
$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\omega_0 n}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} \delta[n-3l]\, e^{-jk\omega_0 n}$$

$$= \frac{1}{3}\sum_{n=0}^{N-1} \delta[n]\, e^{-jk\frac{2\pi}{3}n}$$

$$\boxed{a_k = \frac{1}{3}}.$$

c.



$N = 3 \quad \omega_0 = \frac{2\pi}{3}$.

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \, e^{-jk\omega_0 n}$$

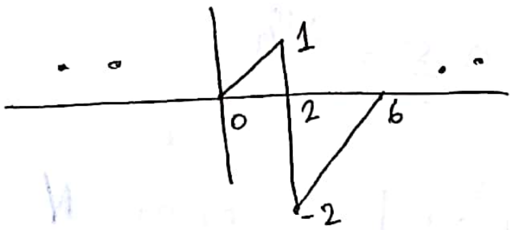
$$= \frac{1}{3} \sum_{n=0}^{2} x[n] \, e^{-jk\frac{2\pi}{3}n}$$

$$a_k = \frac{1}{3}\left(2 + 1e^{-j\frac{2\pi}{3}k} - e^{-j\frac{4\pi}{3}k}\right)$$

$$= \frac{1}{3}\left(2 + e^{-j\frac{2\pi}{3}k} - e^{j\frac{2\pi}{3}k}\right).$$

$$\boxed{a_k = \frac{1}{3}\left(2 + 2j\sin\left(-\frac{2\pi}{3}k\right)\right)}$$

d).



$T = 6 \quad \omega_0 = \frac{\pi}{3}$

$$a_k = \frac{1}{T} \int_T x(t) \, e^{-j\omega_0 k t} \, dt \qquad x(t) = \begin{cases} t/2 & 0 \le t \le 2 \\ t/2 - 3 & 2 \le t \le 6. \end{cases}$$

$$a_k = \frac{1}{6} \int_0^2 \frac{t}{2} e^{-j\omega_0 k t} \, dt + \frac{1}{6} \int_2^6 \left(t/2 - 3\right) e^{-j\omega_0 k t} \, dt.$$

$$\cancel{= \frac{1}{6}\left[\frac{t}{2} \frac{e^{-j\frac{\pi}{3}kt}}{-j\frac{\pi}{3}k}\right]_0^2 - \frac{1}{6}\int_0^2 \frac{1}{2} e^{-j\omega_0 k t} \, dt} + \frac{1}{6}$$

$$a_k = \frac{1}{6}\int_0^6 \frac{t}{2} e^{-j\omega_0 k t} dt + \frac{1}{6}\int_2^6 -3\, e^{-j\omega_0 k t} dt.$$

$\omega_0 = \pi/3$

$$\frac{1}{6}\int_0^6 \frac{t}{2} e^{-j\omega_0 k t} dt = \frac{1}{6}\left[\frac{t}{2}\frac{e^{-j\omega_0 k t}}{-j\omega_0 k}\right]_0^6 - \frac{1}{6}\int_0^6 \cdot\frac{1}{2} e^{-j\omega_0 k t} dt$$

$$= \frac{1}{\cancel{6}}\frac{\cancel{6}}{2}\cdot\frac{e^{-j\frac{\pi}{3}k 6^2}}{-j\omega_0 k.} - \frac{1}{12}\left[\frac{e^{-j\omega_0 k t}}{-j\omega_0 k}\right]_0^6$$

$$= \frac{1}{-2j\omega_0 k} + \frac{1}{12\,4}\left[\frac{e^{-j2\pi k} - 1}{+j\frac{\pi}{3}k}\right].$$

$$= \frac{1}{-2j\omega_0 k} + \frac{1}{j4\pi k}\left[\cancel{e}\ 1 - 1\right] = \frac{1}{-2j\frac{\pi}{3}k} = \frac{-1}{j\frac{2\pi}{3}k}$$

$$\frac{1}{6}\int_2^6 -3\, e^{-j\omega_0 k t} dt = \frac{+1}{2}\left[\frac{e^{-j\omega_0 k t}}{+j\omega_0 k}\right]_2^6$$

$$= \frac{1}{2}\left[\frac{1 - e^{-j\frac{2\pi}{3}k}}{j\frac{2\pi}{3}k}\right].$$

$$= \frac{1 - e^{-j\frac{2\pi}{3}k}}{j\frac{4\pi}{3}k}.$$

$$a_k = \frac{-1}{j\frac{2\pi}{3}k} + \frac{1}{j\frac{4\pi}{3}k} - \frac{e^{-j\frac{2\pi}{3}k}}{j\frac{4\pi}{3}k} = \boxed{\frac{-1 - e^{-j\frac{2\pi}{3}k}}{j\frac{4\pi}{3}k.} = a_k}$$

$$\boxed{a_0 = \frac{-3}{6}} = \boxed{\frac{-1}{2}}$$

5. $x(t) = \begin{cases} 4t-6 , & 0 \le t \le 1 \\ -2t , & 1 \le t \le 2 . \end{cases}$

$$y(t) = \frac{dx}{dt} = \begin{cases} 4 , & 0 \le t \le 1 \\ -2 , & 1 \le t \le 2 . \end{cases}$$

$T = 2.$ $\omega_0 = \pi$

$$a_k = \frac{1}{T} \int_0^2 y(t)\, e^{-j\omega_0 k t}\, dt .$$

$$= \frac{1}{2} \int_0^1 4\, e^{-j\pi k t}\, dt + \frac{1}{2} \int_1^2 (-2)\, e^{-j\pi k t}\, dt .$$

$$= 2 \left[\frac{e^{-j\pi k t}}{-j\pi k} \right]_0^1 - \left[\frac{e^{-j\pi k t}}{-j\pi k} \right]_1^2$$

$$= 2 \left[\frac{e^{-j\pi k} - 1}{-j\pi k} \right] - \left[\frac{1 - e^{-j\pi k}}{-j\pi k} \right] .$$

$$a_k = \frac{3e^{-j\pi k} - 3}{-j\pi k}$$

$$a_0 = \frac{1}{2} \cdot [4-2] = 1$$

For $x(t)$:

$$a_0 = \frac{1}{2}\int_0^2 x(t)\,dt = \frac{1}{2}\left[\int_0^1 (4t-6)\,dt + \int_1^2 (-2t)\,dt\right]$$

$$\cancel{= \frac{1}{2}\left[4 + (-2)\right]} \quad = \frac{1}{2}\left[\left[2t^2 - 6t\right]_0^1 + \left[-t^2\right]_1^2\right].$$

$$= \frac{1}{2}\left[2 - 4 + (-4+1)\right]$$

$$\boxed{a_0 = -\frac{7}{2}}$$