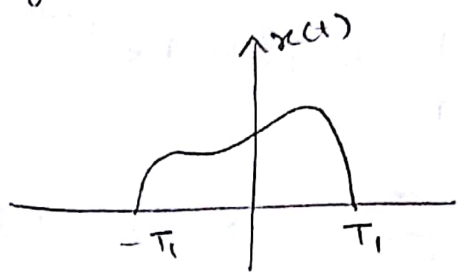
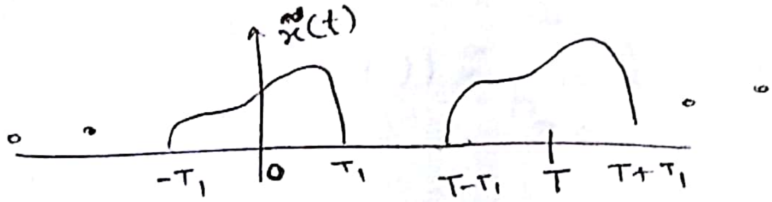
HW-4 solns.

1. Aperiodic signals.

Assume finite duration signal $x(t)$





$$\tilde{x} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt$$

Define $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt.$

$$a_k = \frac{1}{T} X(jk\omega_0)$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} x(jk\omega_0) e^{jk\omega_0 t}.$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} x(jk\omega_0) e^{jk\omega_0 t}.$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} x(kj\omega_0) e^{jk\omega_0 t} \omega_0.$$

$$x(t) = \lim_{T\to\infty} \tilde{x}(t).$$

As $T \to \infty$ $\omega_0 \to 0$

$$x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} x(j\omega) e^{j\omega t} d\omega.$$

$$x(j\omega) = \int_{t=-\infty}^{\infty} x(t) e^{-j\omega t} dt.$$

2. $\underbrace{\cos\left(\frac{3\pi n}{4}+\frac{\pi}{4}\right)}_{a_1[n],\ N=8} \underbrace{\sin\left(\frac{3\pi n}{4}\right)}_{a_2[n],\ N=8} + \underbrace{\sin\left(\frac{5\pi n}{3}+\frac{\pi}{4}\right)}_{b_1[n],\ N=6} \underbrace{\cos\left(\frac{5\pi n}{3}\right)}_{b_2[n],\ N=6}$

$N = \text{LCM}(8, 6) = 24.$

$$\omega_0 = \frac{2\pi}{24} = \frac{\pi}{12}.$$

$$a_1[n] = \cos\left(\frac{3\pi n}{4} + \pi/4\right) = \frac{1}{2} e^{j\left(\frac{3\pi}{4}n\right)} e^{j\pi/4} + \frac{1}{2} e^{-j\frac{3\pi}{4}n} e^{-j\pi/4}.$$

By inspection $a_9 = \frac{1}{2} e^{j\pi/4}$. $a_{-9} = \frac{1}{2} e^{-j\pi/4}$.

$$a_2[n] = \sin\left(\frac{3\pi n}{4}\right) = \frac{1}{2j} e^{j\frac{3\pi}{4}n} - \frac{1}{2j} e^{-j\frac{3\pi}{4}n}.$$

By inspection $b_9 = \frac{1}{2j}$ $b_{-9} = -\frac{1}{2j}$.

Fourier series for $a_1[n] \cdot a_2[n]$

$$= \left(a_9 \delta[k-9] + a_{-9} \delta[k+9]\right) * \left(b_9 \delta[k-9] + b_{-9} \delta[k+9]\right)$$

$$e_k = a_9 b_9 \delta[k-18] + a_9 b_{-9} \delta[k] + a_{-9} b_9 \delta[k] + a_{-9} b_{-9} \delta[k+18].$$

$$e_{18} = e_{-6} = \frac{1}{4j} e^{j\pi/4}. \quad e_0 = \frac{-1}{4j} e^{j\pi/4} + \frac{1}{4j} e^{-j\pi/4}$$

$$= \frac{-1}{2}\left[\frac{1}{2j} e^{j\pi/4} - \frac{1}{2j} e^{-j\pi/4}\right]$$

$$= \frac{-1}{2} \sin\frac{\pi}{4} = \frac{-1}{2\sqrt{2}}.$$

$$e_{-18} = e_6 = \frac{-1}{4j} e^{-j\pi/4}.$$

$$b_1[n] = \sin\left(\frac{5\pi}{3}n + \pi/4\right) = \frac{1}{2j} e^{j\frac{5\pi}{3}n} \cdot e^{j\pi/4} - \frac{1}{2j} e^{-j\frac{5\pi}{3}n} e^{-j\pi/4}$$

$$c_{20} = c_{-4} = \frac{1}{2j} e^{j\pi/4}$$

$$c_{-20} = c_4 = \frac{-1}{2j} e^{-j\pi/4}.$$

$$b_2[n] = \cos\left(\frac{5\pi n}{3}\right) = \frac{1}{2} e^{j\frac{5\pi}{3}n} + \frac{1}{2} e^{-j\frac{5\pi}{3}n}.$$

$$d_{20} = d_{-4} = 1/2 \quad ; \quad d_{-20} = d_4 = \frac{1}{2}.$$

$$f_k = \left(c_4\,\delta[k-4] + c_{-4}\,\delta[k+4]\right) * \left(d_4\,\delta[k-4] + d_{-4}\,\delta[k+4]\right)$$

$$= c_4 d_4\,\delta[k-8] + \left(c_{-4}d_4 + c_4 d_{-4}\right)\delta[k] + c_4 d_{-4}\,\delta[k+8]$$

$$f_8 = c_4 d_4 = \frac{-1}{4j} e^{-j\pi/4}.$$

$$f_0 = \frac{1}{4j} e^{j\frac{\pi}{4}} - \frac{1}{4j} e^{-j\frac{\pi}{4}} = \frac{1}{2}\left[\frac{1}{2j} e^{j\pi/4} - \frac{1}{2j} e^{-j\pi/4}\right]$$

$$= \frac{1}{2} \sin\frac{\pi}{4} = \frac{1}{2\sqrt{2}}.$$

$$f_{-8} = c_{-4} d_{-4} = \frac{1}{4j} e^{+j\frac{\pi}{4}}$$

Fourier series of $a_1[n]\,a_2[n] + b_1[n]\,b_2[n]$

$$= e_k + f_k = \begin{cases} \frac{1}{4j} e^{j\frac{\pi}{4}} & k = 16, 18 \\ \frac{-1}{4j} e^{-j\frac{\pi}{4}} & k = 8, 6 \\ 0 & \text{o.w.} \end{cases}$$

Notice $N=24$ is even and $a_k=0$ for all odd k's. $\therefore N=12$. 5

$$g_k = \begin{cases} \frac{1}{4j} e^{j\frac{\pi}{4}}, & k=8,9 \\ \frac{-1}{4j} e^{-j\frac{\pi}{4}}, & k=4,3 \\ 0, & o.w. \end{cases} \qquad N=12.$$

3. a) $\sum_{k=50}^{74} x[n] = N a_0 = 25 \cdot 4 = 100.$

b). Average power $= \frac{1}{N} \sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$

$\therefore$ Average power $= \frac{1}{25} \sum_{k=0}^{24} |a_k|^2$

c). $x[100] = x[0] = \sum_k a_k = 4 + 9(-2) + 10(2) + 5(3)$
$= 4 - 18 + 20 + 15 = 21.$

d) $x[126] = x[1] = \sum_{k=0}^{24} a_k e^{jk\omega_0}$ $\qquad \omega_0 = \frac{2\pi}{25}$

$= 4 + \sum_{k=1}^{9} -2e^{jk\omega_0} + \sum_{k=10}^{19} 2e^{jk\omega_0} + \sum_{k=20}^{24} e^{jk\omega_0}.$

e) $b_k = e^{-jk\omega_0 4} a_k.$

4. a) $x(t) = e^{-3(t-2)} u(t-5)$.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt.$$

$$= \int_{-\infty}^{\infty} e^{-3(t-2)} u(t-5) e^{-j\omega t} dt$$

$$= \int_{5}^{\infty} e^{-3(t-2)} e^{-j\omega t} dt$$

$$= \int_{5}^{\infty} e^{6} \cdot e^{t(-3-j\omega)} dt$$

$$= e^{6} \cdot \left[\frac{e^{t(-3-j\omega)}}{-3-j\omega} \right]_{5}^{\infty}$$

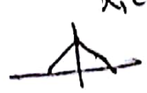
$$= e^{6} \left[0 - \frac{e^{-15-5j\omega}}{-3-j\omega} \right]$$

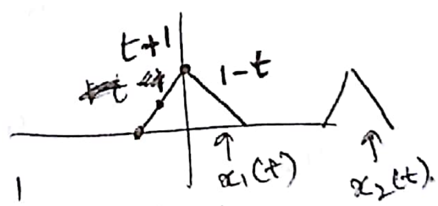
$$= \frac{e^{-9} e^{-5j\omega}}{3+j\omega}.$$

$x(-1) = 0$
$x(0) = 1$

b). $$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

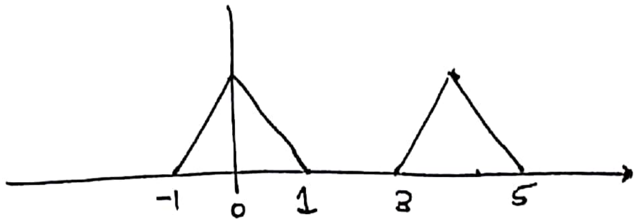
$\int_{-1}$ for $x_1(t)$



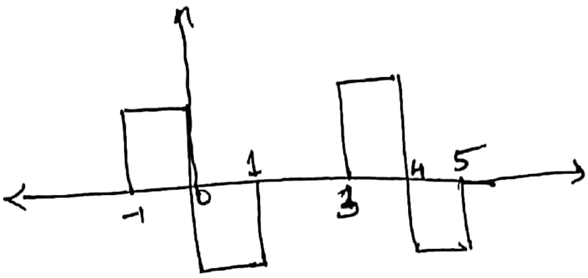


$$X_1(j\omega) = \int_{-1}^{0} (t+1) e^{-j\omega t} dt + \int_{0}^{1} (1-t) e^{-j\omega t} dt.$$

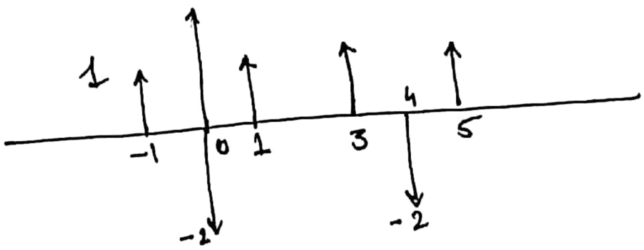
b) $x(t)$



$\frac{dx(t)}{dt}$



$\frac{d^2x(t)}{dt^2}$



$$\frac{d^2x(t)}{dt^2} \xleftrightarrow{FT} e^{j\omega} - 2 + e^{-j\omega} + e^{-3j\omega} - 2e^{-4j\omega} + e^{-5j\omega}$$

$$\frac{d^2x(t)}{dt^2} \xleftrightarrow{FT} X(j\omega)(-\omega^2).$$

$$\Rightarrow X(j\omega) = \frac{e^{j\omega} - 2 + e^{-j\omega} + e^{-3j\omega} - 2e^{-4j\omega} + e^{-5j\omega}}{-\omega^2}.$$

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$$x_1(j\omega) = \frac{e^{j\omega} - e^{-j\omega}}{j\omega}.$$

$$X_2(j\omega) = x_1(j\omega)\, e^{-j\omega \cdot 4}$$

$$X(\omega) = x_1(j\omega) + X_2(j\omega).$$

5. $h(t) = e^{-|t|}$.

$$H(j\omega) = \int_{-\infty}^{0} e^{t} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-t} e^{-j\omega t}\, dt$$

$$= \left[\frac{e^{t(1-j\omega)}}{1-j\omega}\right]_{-\infty}^{0} + \left[\frac{e^{t(-1-j\omega)}}{-1-j\omega}\right]_{0}^{\infty}$$

$$= \frac{1}{1-j\omega} + \frac{-1}{-1-j\omega} = \frac{1}{1-j\omega} + \frac{1}{1+j\omega}$$

$$H(j\omega) = \frac{2}{1+\omega^2}.$$

a) $x_1(t) = \sum_{k=-\infty}^{\infty} \delta(t-4k)$

$\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

$a_k = 1/4.$

b_k of output $= a_k \cdot H(j\omega_0 k)$

$$b_k = \frac{1}{4} \cdot \frac{2}{1+\left(\frac{\pi}{2}k\right)^2}.$$

b). $a_k = \int_{-T/2}^{T/2} x(t)\, e^{-j\omega_0 k t}\, dt.$

$= X_1(j\omega_0 k)$ from Prob. 4.

$b_k = a_k \cdot H(j\omega_0 k).$