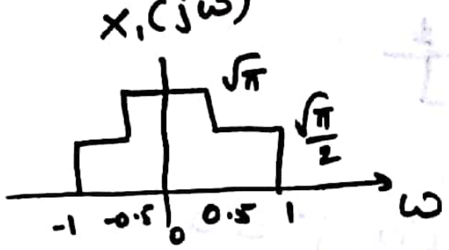
HW 5 Solns.

1

Prof. El Gamal

1. $E = \int_{-\infty}^{\infty} |x(t)|^2$

a)



Parseval's relation:

$$\int_{-\infty}^{\infty} |x(t)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(e^{j\omega})|^2 \, d\omega$$

$$= \frac{1}{2\pi} \left(\int_{-1}^{-0.5} \frac{\pi}{4} \, d\omega + \int_{-0.5}^{0.5} \pi \, d\omega + \int_{0.5}^{1} \frac{\pi}{4} \, d\omega \right).$$

$$= \frac{1}{2\pi} \left[\frac{\pi}{4} (0.5) + \pi + \frac{\pi}{4} \, 0.5 \right]$$

$$= \frac{1}{2\pi} \left(\pi + 2 \cdot \frac{\pi}{8} \right) = \frac{1}{2\pi} \cdot \left(\frac{5\pi}{4} \right). = \frac{5}{8}$$

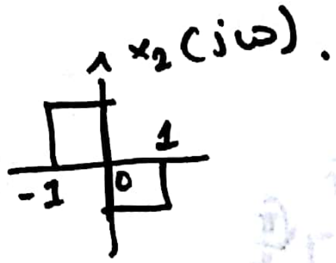
$$D = \frac{d}{dt} x(t) \Big|_{t=0}$$

Let $f(t) = \frac{d}{dt} x(t)$. $\quad D = f(0)$.

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) \, e^{j\omega t} \, d\omega$$

$$\Rightarrow f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega \, x(j\omega)\, d\omega. = 0.$$

2 (ii).



$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |x(j\omega)|^2 d\omega.$$

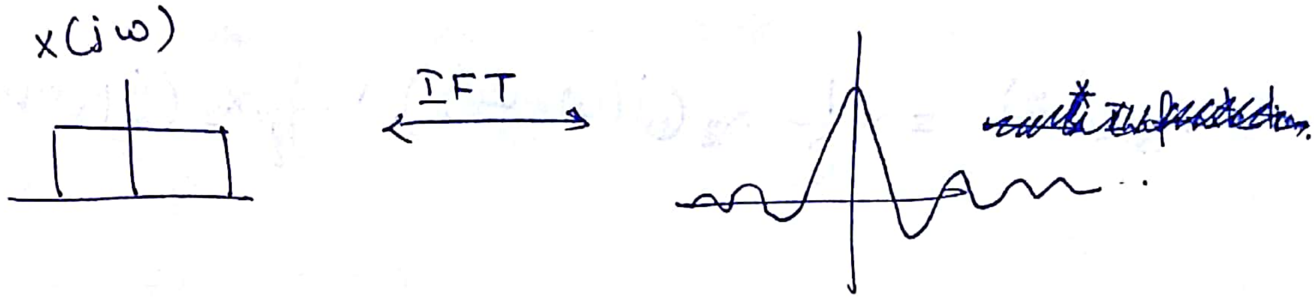
$$= \frac{1}{2\pi} \left[\int_{-1}^{0} \pi \, d\omega + \int_{0}^{1} \pi \, d\omega \right] = 1.$$

$$D = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega \, x(j\omega)\, d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-1}^{0} -\omega\sqrt{\pi}\, d\omega + \int_{0}^{1} \omega\sqrt{\pi}\, d\omega \right]$$

$$= \frac{1}{2\sqrt{\pi}}$$

(2).

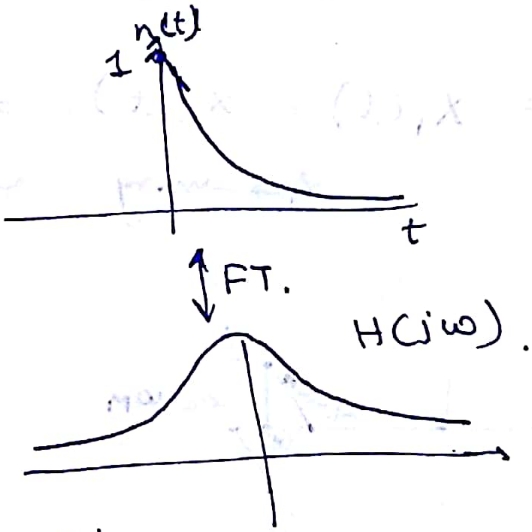


* Non causal
* Oscillatory

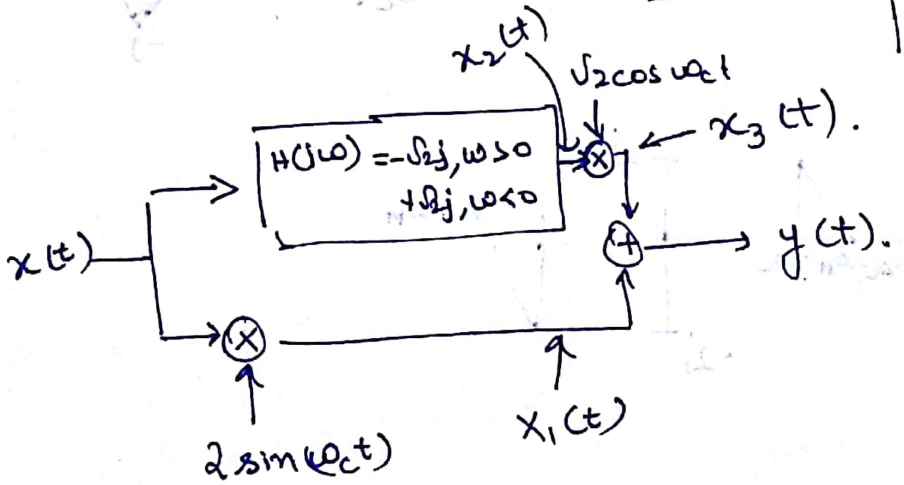
] In time domain.

Eg. of a practical low-pass filter.

$e^{-at} u(t)$. $a>0$

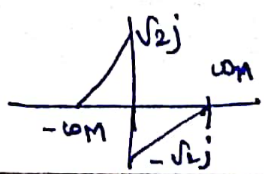


3.



$x_2(t) = x(t) * h(t) \leftrightarrow X_2(j\omega) = X(j\omega) \cdot H(j\omega)$.

$X_2(j\omega)$. =



$x_3(t) = x_2(t) \cdot \sqrt{2}\cos\omega_c t \Rightarrow X(j\omega) = \sqrt{2}\, x_2(t) * FT(\cos\omega_c t)$.

$\Rightarrow X_3(j\omega) = \frac{1}{\sqrt{2}} X_2(j(\omega-\omega_c)) + \frac{1}{\sqrt{2}} X_2(j(\omega+\omega_c))$.

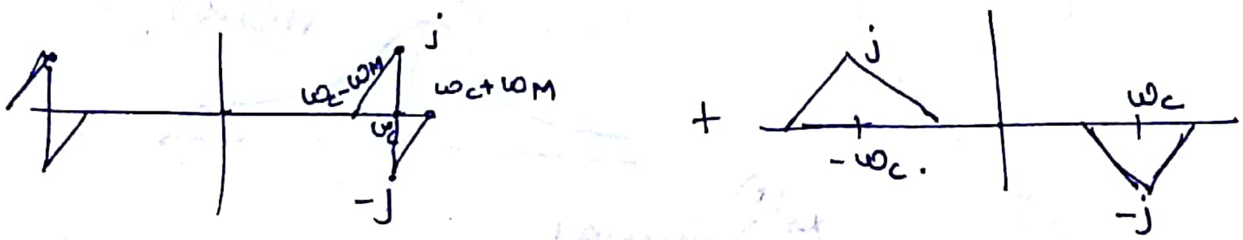
$x_1(t) = x(t) \cdot 2\sin(\omega_c t)$.

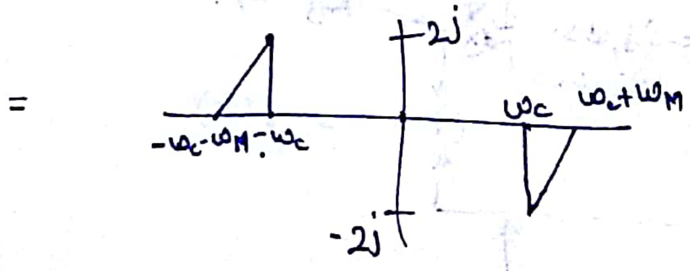
$\Rightarrow X_1(j\omega) = \frac{1}{j}\left(X(j(\omega-\omega_c)) - X(j(\omega+\omega_c))\right)$.

$y(t) = x_1(t) + x_3(t) \Rightarrow Y(j\omega) = X_1(j\omega) + X_3(j\omega)$.

Assuming $\omega_c > \omega_M$.

X_1 :





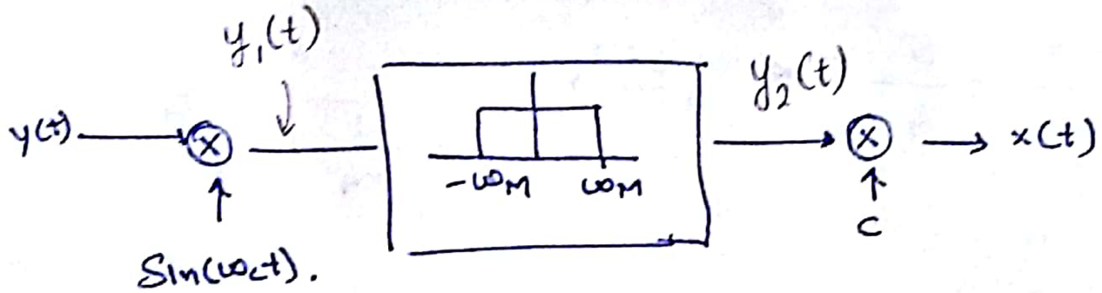
b. $X(j\omega)$ is real and even

$\Rightarrow$ $x(t)$ is also real and even.

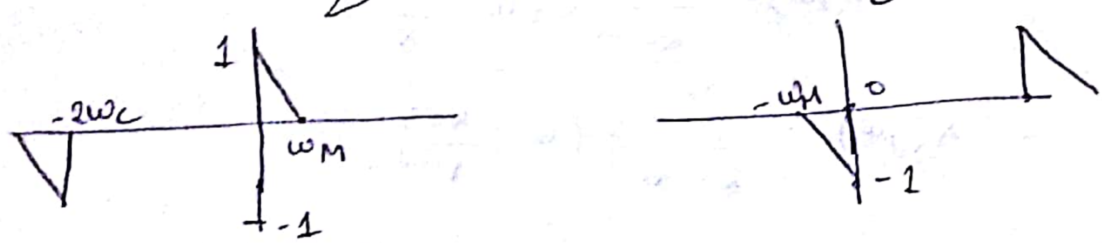
c. Real $(Y(j\omega)) = 0$

$\Rightarrow$ $y(t)$ is real and odd.

d.

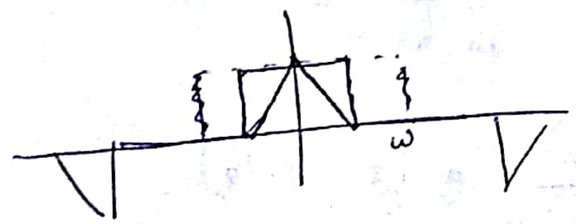


$$Y_1(j\omega) = \frac{1}{2}\left[j Y(j(\omega+\omega_c)) - j Y(j(\omega-\omega_c)) \right].$$

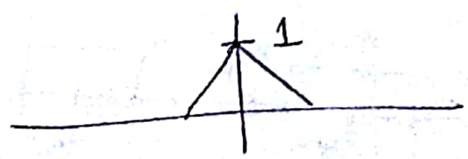


Assume $\omega_M > \omega_c$

$Y_1(j\omega)$



$\Rightarrow$ $Y_2(j\omega)$

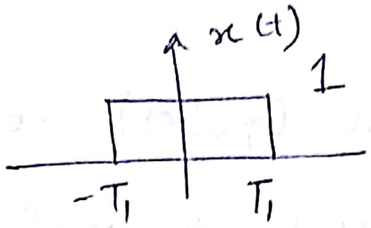


$\Rightarrow$ $c = 1$

4.

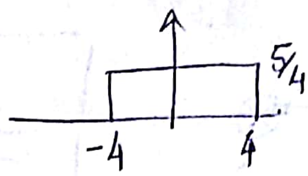
a) $X(j\omega) = \frac{5}{2} \frac{\sin(4\omega)}{\omega}$ Find $x(t)$.

$\frac{2 \sin \omega T_1}{\omega} \xleftrightarrow{IFT}$



Here $T_1 = 4$

$\frac{2 \cdot \sin \omega \cdot 4}{\omega} \cdot \frac{5}{4} \xleftrightarrow{IFT}$



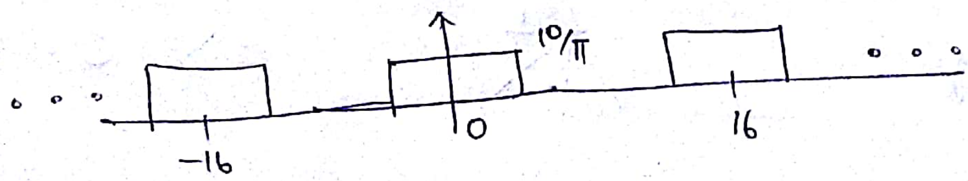
b). $X(j\omega) \longrightarrow \otimes \longrightarrow Y(j\omega)$.

$H(j\omega) = \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{k\pi}{8}\right)$

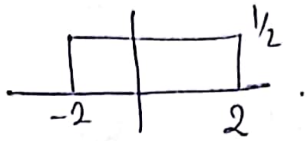
$$Y(j\omega) = X(j\omega) \cdot \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{k\pi}{8}\right).$$

$$\longleftrightarrow y(t) = x(t) * \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \delta(t - k16) \cdot 16$$

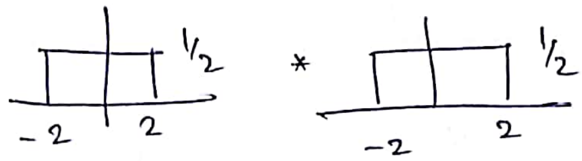
$$\Rightarrow y(t) = \frac{8}{\pi} \sum_{k=-\infty}^{\infty} x(t - k.16).$$



$$\frac{\sin 2\omega}{\omega} \xleftrightarrow{IFT}$$



$$\frac{\sin^2 2\omega}{\omega^2} \xleftrightarrow{IFT}$$



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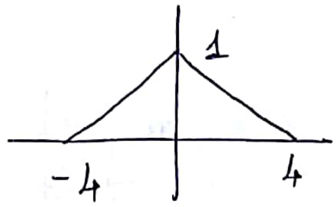
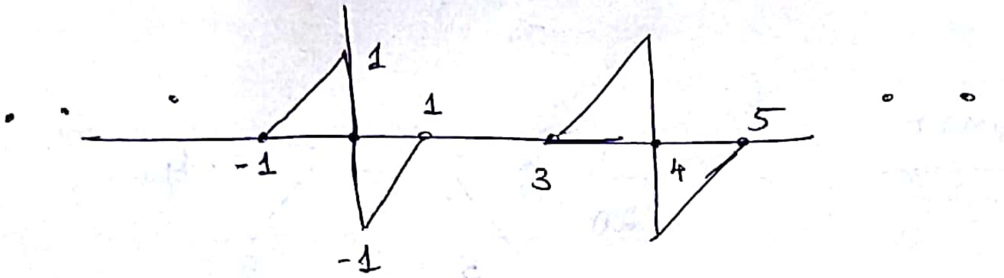
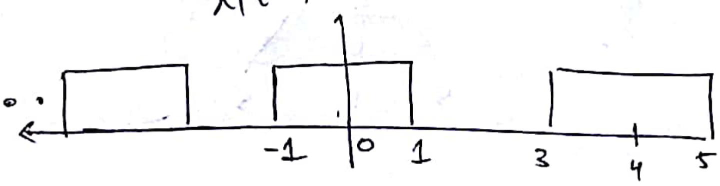


C.

$x(t)$



$\frac{dx(t)}{dt}$ $x_1(t)$



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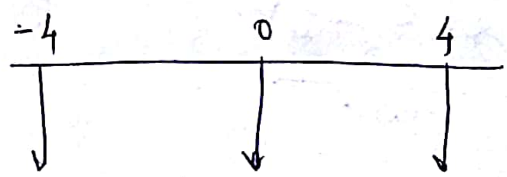
$x_2(t)$



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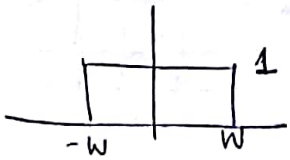
$$FT : -\frac{2\pi}{4} \sum \delta\left(\omega - \frac{2\pi \cdot k}{4}\right).$$

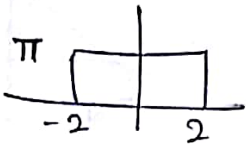
5. a) F.T. of $\dfrac{\sin(2t)\,\sin(3t)}{t^2}$

$$= \frac{\sin 2t}{t} \cdot \frac{\sin 3t}{t}$$

$\dfrac{\sin wt}{\pi t} \longleftrightarrow$

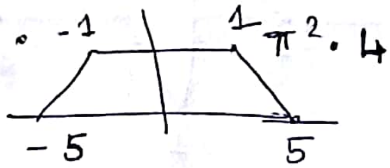


$\dfrac{\sin 2t}{t} \longleftrightarrow$



$\dfrac{\sin 3t}{t} \longleftrightarrow$

$\dfrac{\sin 2t \cdot \sin 3t}{t^2} \longleftrightarrow \dfrac{1}{2\pi} \cdot$



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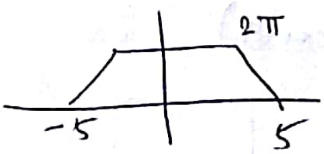
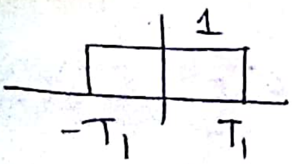
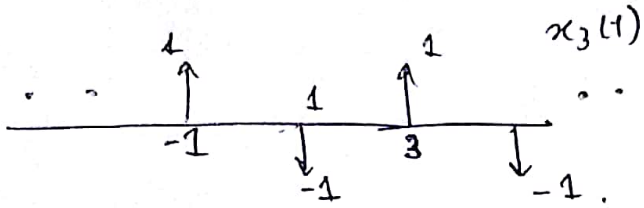


b) $\dfrac{\sin^2(2\omega)}{\omega^2}$



$\longleftrightarrow \dfrac{2\sin \omega T_1}{\omega}$

$\frac{d}{dt} x_1(t)$.



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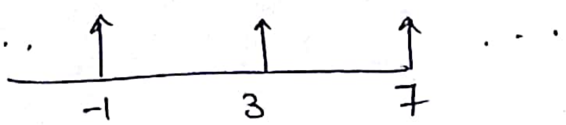

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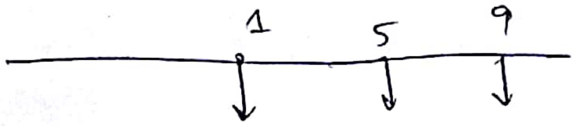

$$X_3(j\omega) = \frac{2\pi}{4} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{4}\right) \cdot e^{j\pi/2}$$

$$- \frac{2\pi}{4} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{4}\right) e^{-j\pi/2}.$$

$$= \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{4}\right) [j + j]$$

$$= j\pi \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{4}\right).$$

$$\frac{dx(t)}{dt} \longleftrightarrow \frac{j\pi}{j\omega} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{\pi k}{2}\right) - \frac{\pi}{2} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{\pi k}{2}\right)$$

$$\Rightarrow x(t) \longleftrightarrow \frac{\pi}{j\omega^2} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{\pi}{2} k\right) - \frac{\pi}{2j\omega} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{\pi k}{2}\right)$$