HW-6 solns.

1.

a) $$\frac{dy(t)}{dt} + a y(t) = x(t)$$

$$\leftrightarrow j\omega Y(j\omega) + a Y(j\omega) = X(j\omega)$$

$$\Rightarrow \frac{Y(j\omega)}{X(j\omega)} = H(j\omega) = \frac{1}{a+j\omega}$$

b.) $$\frac{1}{a+j\omega} \xleftrightarrow{\text{IFT}} e^{-at} u(t).$$

c) This system is an approximation for a low pass filter

d) The bandwidth decreases if 'a' increases and vice versa

2. CTFS:

a) $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt.$$

DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

b) ~~a_k~~ ~~=~~

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-jk\omega_0 n}$$

$$a_k = \frac{1}{N}\sum_{n=-\langle N\rangle} x[-n] e^{jk\omega_0 n}$$

$\Rightarrow$ The FS coefficients of a_k are $\frac{x[-n]}{N}$

c. $x(t)$ has to be periodic with period 2π for duality to exist.

d. $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\cancel{X(e^{jk}) = \sum_{n=-\infty}^{\infty} x[N] e^{-jkn}}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[-n] e^{j\omega n}.$$

$$\Rightarrow X(e^{jt}) = \sum_{n=-\infty}^{\infty} x[-n] e^{jnt}$$

$\Rightarrow$ FS coefficients of $X(e^{jt})$ are $x[-n]$.

3. $\frac{d^2}{dt^2} y(t) + 6 \frac{d}{dt} y(t) + 8 y(t) = 2x(t)$.

$$(j\omega)^2 Y(j\omega) + 6 j\omega Y(j\omega) + 8 Y(j\omega) = 2X(j\omega)$$

$$\Rightarrow \quad H(j\omega) = \frac{2}{(j\omega)^2 + 6j\omega + 8} = \frac{2}{(j\omega+4)(j\omega+2)}$$

$$= \frac{-1}{j\omega+4} + \frac{1}{j\omega+2}$$

$$\xleftrightarrow{IFT} -e^{-4t}u(t) + e^{-2t}u(t).$$

b). $x(t) = t e^{-4t} u(t)$.

$$X(j\omega) = \frac{1}{(j\omega+4)^2}.$$

$$\Rightarrow \quad Y(j\omega) = X(j\omega) \cdot H(j\omega) = \frac{-1}{(j\omega+4)^3} + \frac{1}{(j\omega+2)(j\omega+4)^2}$$

$$= \frac{1}{4(j\omega+2)} - \frac{1}{4(j\omega+4)} - \frac{1}{2(j\omega+4)^2} - \frac{1}{(j\omega+4)^3}$$

$$= \frac{1}{4} e^{-2t} u(t) - \frac{1}{4} e^{-4t} u(t) - \frac{1}{2} t e^{-4t} u(t)$$

$$- \frac{t^2}{2} e^{-4t} u(t).$$

4. $Im\{X(e^{j\omega})\} = \sin 3\omega - \sin 6\omega$

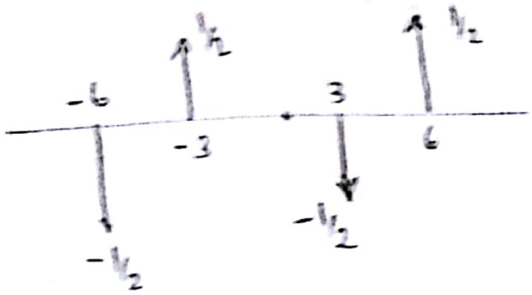
$$= \frac{j\,(e^{3j\omega} - e^{-3j\omega})}{2j} - \frac{j\,(e^{6j\omega} - e^{-6j\omega})}{2j}$$

$$= \frac{e^{j3\omega}}{2} - \frac{e^{-j3\omega}}{2} - \frac{e^{+6j\omega}}{2} + \frac{e^{-6j\omega}}{2}$$

$x[n]$ is real $\Rightarrow$ $x_{odd}[n] \xleftrightarrow{DTFT} Im\{X(e^{j\omega})\}$

$\therefore\ x_{odd}[n] = \frac{1}{2}\delta[n+3] - \frac{1}{2}\delta[n-3] - \frac{1}{2}\delta[n+6]$

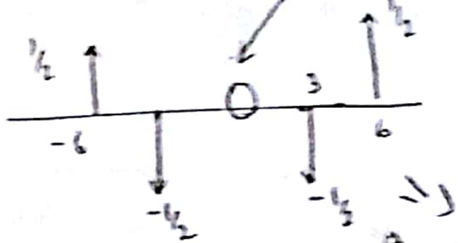
$-\frac{1}{2}\delta[n-6]$



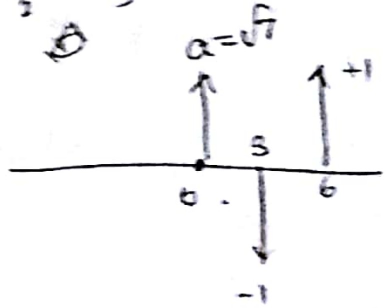
$x[n] = 0 \quad , n<0$

still have figure $x_{even}[0]$.

$\Rightarrow\ x_{even}[n] =$



$\Rightarrow$



$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(e^{j\omega})|^2\, d\omega = 9.$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} |x[n]|^2 = a^2+2 = 9$$

$\Rightarrow\ a = \pm\sqrt{7}$. But $x[0] > 0$

$\Rightarrow a = \sqrt{7}$

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b). $$Y(e^{j\omega}) = \cos\omega$$

(i), (ii) $\Big\}$ $\cos\omega$ is real and even.

$\therefore$ $y[n]$ is also real and even.

(iii) $$\sum_{n=-\infty}^{\infty} y[n] = Y(e^{j\omega})\Big|_{\omega=0} = 1.$$

(iv) $$\sum_{n=-\infty}^{\infty} (-1)^n y[n] = Y(e^{j\omega})\Big|_{\omega=\pi} = -1.$$

5. a) $$x[n] = \frac{1}{4}^{|n|-1}.$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= 4\sum_{n=-\infty}^{-1} \left(\frac{1}{4}\right)^{-n} e^{-j\omega n} + 4\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n} e^{-j\omega n}$$

$$= 4\sum_{n=1}^{\infty} \left(\frac{1}{4} e^{j\omega}\right)^{n} + 4\sum_{n=0}^{\infty} \left(\frac{1}{4} e^{-j\omega}\right)^{n}$$

$$= \frac{4 \cdot \frac{1}{4} e^{j\omega}}{1 - \frac{1}{4} e^{j\omega}} + \frac{4.}{1 - \frac{1}{4} e^{-j\omega}}$$

$$= \frac{4e^{j\omega}}{4 - e^{j\omega}} + \frac{16}{4 - e^{-j\omega}} = \frac{60}{17 - 8\cos\omega}.$$

b. $\cancel{X(e^{j\omega}) = \sum_{n=-N_1}^{N_2} e^{-j\omega n}}$

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} e^{-jk\frac{2\pi}{N} \cdot n}$$

$$= \frac{1}{N} \sum_{n=-N_1}^{\infty} e^{-jk\frac{2\pi}{N} \cdot n} - \frac{1}{N} \sum_{n=N_2}^{\infty} e^{-jk\frac{2\pi}{N} \cdot n}$$

$$= \frac{1}{N} \frac{e^{-jk\frac{2\pi}{N}(-N_1)}}{1 - e^{-j\frac{2\pi}{N}k}} - \frac{1}{N} \frac{e^{-jk\frac{2\pi}{N}(N_2)}}{1 - e^{-j\frac{2\pi}{N}k}}.$$

$$a_k = \frac{e^{jk2\pi\frac{N_1}{N}} - e^{-jk2\pi\frac{N_2}{N}}}{N\left(1 - e^{-j\frac{2\pi}{N}k}\right)}$$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \, \delta(\omega - \omega_0 k) \qquad \omega_0 = \frac{2\pi}{N}.$$

c. $x[n] = \delta[-n] + 5\delta[n-2] + \delta[n+4]$

$$X(e^{j\omega}) = 1 + 5e^{-2j\omega} + e^{4j\omega}.$$