ECE 301 - Lecture #13

Fourier Series Representation of Periodic Signals

Complex exponentials are Eigenfunctions of LTI Systems

$x(t) = e^{st}$

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

$$y(t) = \int h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= \int h(\tau)\, e^{st} e^{-s\tau}\, d\tau$$

$$= e^{st} \underbrace{\int h(\tau) e^{-s\tau}\, d\tau}_{H(s)}$$

Set of harmonically related complex exponentials

$$\phi_k(t) = e^{jk\omega_0 t}$$

$$\phi_1(t) = e^{j\omega_0 t}$$

Fundamental freq. ω_0

Fundamental Period $T_0 = \frac{2\pi}{\omega_0}$

$$\phi_2(t) = e^{j2\omega_0 t}$$

Fund. freq. $2\omega_0$

Fund Period

$$\frac{2\pi}{2\omega_0} = \frac{T_0}{2}$$

$$\phi_3(t) \Rightarrow \frac{T_0}{3}$$

$$\phi_k(t) \Rightarrow \frac{T_0}{|k|}, \quad |k| \geq 1$$

Fund Period of $\phi_k(t)$

$$x(t) = \sum a_k e^{jk\omega_0 t}$$

Fund. Period $T_0 = \frac{2\pi}{\omega_0}$

Fund. Frequency ω_0

If $a_1 \neq 0$ or $a_{-1} \neq 0$

a_0 : DC component

$$= \sum a_k e^{jk\left(\frac{2\pi}{T}\right)t}$$

Convergence of the Fourier Series

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

As $N \to \infty$, Does $x_N(t)$ converge?

If so, does it converge to the original signal $x(t)$?

$$e_N(t) = x(t) - x_N(t)$$

Error signal

Energy in the error

$$E_N = \int_T |e_N(t)|^2 \, dt$$

* $a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} \, dt$

Minimizes the Energy in the error

* For periodic signals with finite energy over one period

$$\int_T |x(t)|^2 \, dt < \infty$$

$$\Rightarrow a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} \, dt < \infty$$

And, $E_N \rightarrow 0$ as $N \rightarrow \infty$

This does not <u>necessarily</u> mean that $x_N(t) \rightarrow x(t)$ as $N \rightarrow \infty$

Dirichlet Conditions

Original Periodic Signal

They imply that $x(t)$ ~~equ~~ equals its Fourier Series Representation except at values where $x(t)$ is discontinuous.

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

At these points, the series equals the ~~av~~ average value of $x(t)$

Determination of the Fourier Series Coefficients for Continuous-time Signals

$$x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}\, e^{-jn\omega_0 t}$$

$$\int_0^T x(t)\, e^{-jn\omega_0 t} = \int_0^T \sum_{k=-\infty}^{\infty} a_k\, e^{j(k-n)\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \underbrace{\int_0^T e^{j(k-n)\omega_0 t}\, dt}$$

$\Downarrow$

?

$\boxed{a_n T}$

$$\text{If } k-n \neq 0 \Rightarrow \int_0^T e^{j(k-n)\omega_0 t}\, dt = 0$$

$$\text{If } k-n = 0 \Rightarrow \int_0^T e^{j(k-n)\omega_0 t}\, dt = T$$

$$\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = a_n T$$

$$\int_T x(t)\, e^{-jn\omega_0 t}\, dt = a_n T$$

$$\boxed{a_k = \frac{1}{T} \int_T x(t)\, e^{-jk\omega_0 t}\, dt}$$