

Fourier Series Representation of Periodic Signals

Complex exponentials are Eigenfunctions
of LTI Systems

$$x(t) = e^{st}$$

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= h(t) * x(t) \end{aligned}$$

$$y(t) = \int h(\tau) x(t-\tau) d\tau$$

$$= \int h(\tau) e^{s(t-\tau)} d\tau$$

$$= \int h(\tau) e^{st} e^{-s\tau} d\tau$$

$$= e^{st} \underbrace{\int h(\tau) e^{-s\tau} d\tau}_{H(s)}$$

(2)

Set of harmonically related complex exponentials

$$\phi_k(t) = e^{j k \omega_0 t}$$

$$\phi_1(t) = e^{j \omega_0 t}$$

Fundamental
freq. ω_0

Fundamental
Period $T_0 = \frac{2\pi}{\omega_0}$

$$\phi_2(t) = e^{j 2 \omega_0 t}$$

Fund. freq. $2\omega_0$

Fund. Period

$$\frac{2\pi}{2\omega_0} = \frac{T_0}{2}$$

$$\phi_3(t) \Rightarrow \frac{T_0}{3}$$

$$\phi_k(t) \Rightarrow \frac{T_0}{|k|}, \quad |k| \geq 1$$

↑
Fund. Period of $\phi_k(t)$

(3)

$$x(t) = \sum a_k e^{j k \omega_0 t}$$

Fund. Period $T_0 = \frac{2\pi}{\omega_0}$

Fund. Frequency ω_0

If $a_1 \neq 0$ or $a_{-1} \neq 0$

$a_0 =$ DC component

$$= \sum a_k e^{j k \left(\frac{2\pi}{T}\right) t}$$

Convergence of the Fourier Series

(4)

$$x_N(t) = \sum_{k=-N}^N a_k e^{jkw_0 t}$$

As $N \rightarrow \infty$, Does $x_N(t)$ converge?

If so, does it converge to the original signal $x(t)$?

$$e_N(t) = x(t) - x_N(t)$$

Error signal

Energy in the error

$$E_N = \int_T |e_N(t)|^2 dt$$

$$* a_k = \frac{1}{T} \int_T x(t) e^{-j k \omega_0 t} dt \quad (5)$$

Minimizes the Energy in the error

* For periodic signals with finite energy over one period

$$\int_T |x(t)|^2 dt < \infty$$

$$\Rightarrow a_k = \frac{1}{T} \int_T x(t) e^{-j k \omega_0 t} dt < \infty$$

And, $E_N \rightarrow 0$ as $N \rightarrow \infty$

This does not necessarily mean

that $x_N(t) \rightarrow x(t)$ as $N \rightarrow \infty$

Dirichlet Conditions

Original Periodic Signal (6)

They imply that $x(t)$ ~~equal~~ equals its

Fourier Series Representation except at

$$\sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$

values where $x(t)$ is discontinuous.

At these points, the

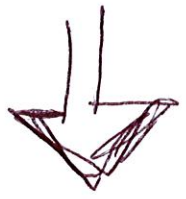
series equals the ~~an~~ average value of $x(t)$

Determination of the Fourier Series

Coefficients for Continuous-time Signals

$$x(t) e^{-j n \omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j k \omega_0 t} e^{-j n \omega_0 t}$$

$$\int_0^T x(t) e^{-j n \omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j (k-n) \omega_0 t} dt$$



?

$$= \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j (k-n) \omega_0 t} dt$$

$$a_n T$$

If $k-n \neq 0 \Rightarrow \int_0^T e^{j (k-n) \omega_0 t} dt = 0$

If $k-n = 0 \Rightarrow \int_0^T e^{j (k-n) \omega_0 t} dt = T$

(8)

$$\int_0^T x(t) e^{-j n \omega_0 t} dt = a_n T$$

$$\int_T x(t) e^{-j n \omega_0 t} dt = a_n T$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-j k \omega_0 t} dt$$