ECE 301- Lecture #14

Fourier Series Representation of Continuous-Time Periodic Signals

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad (1)$$

$$\boxed{a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt}$$

Assume $x(t)$ is real

then $x^*(t) = x(t)$

$$\Rightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

Replacing k by $-k$

$$x(t) = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t} \quad (2)$$

Comparing (1) and (2) $\Rightarrow a_k = a_{-k}^*$

Real CT Periodic Signals

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t} \right]$$

Replacing a_{-k} by a_k^*

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t} \right]$$

Conjugates

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{ a_k e^{jk\omega_0 t} \right\}$$

If a_k is in polar form,

$$a_k = A_k e^{j\theta_k}$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{ A_k e^{j(k\omega_0 t + \theta_k)} \right\}$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2 A_k \cos(k\omega_0 t + \theta_k)$$

Alternative Form of FS for CT

real periodic signals

Dirichlet Conditions

Guarantee that $x(t)$ and its FS representation are the same except at points of discontinuity.

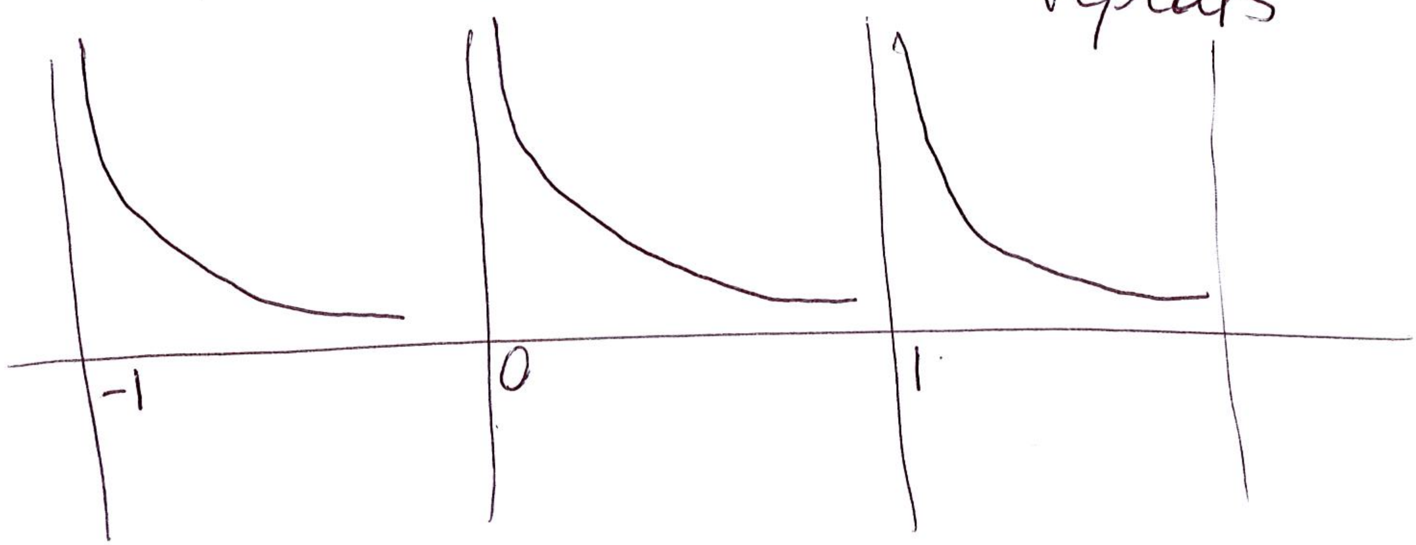
At these points, FS equals the average value of $x(t)$

Condition 1

$$\int_T |x(t)|\, dt < \infty$$

Counter Example

$x(t) = \frac{1}{t}$, $0 < t \leq 1$ then repeats



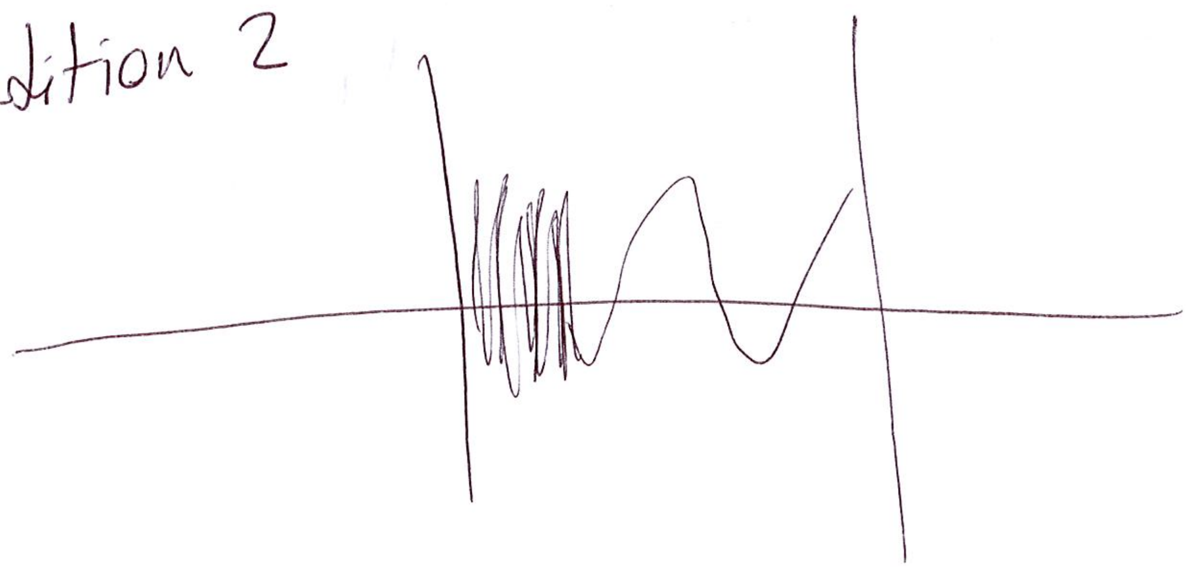
Condition 2

In any finite interval in time, there are a finite number of maxima and minima

Counter Example

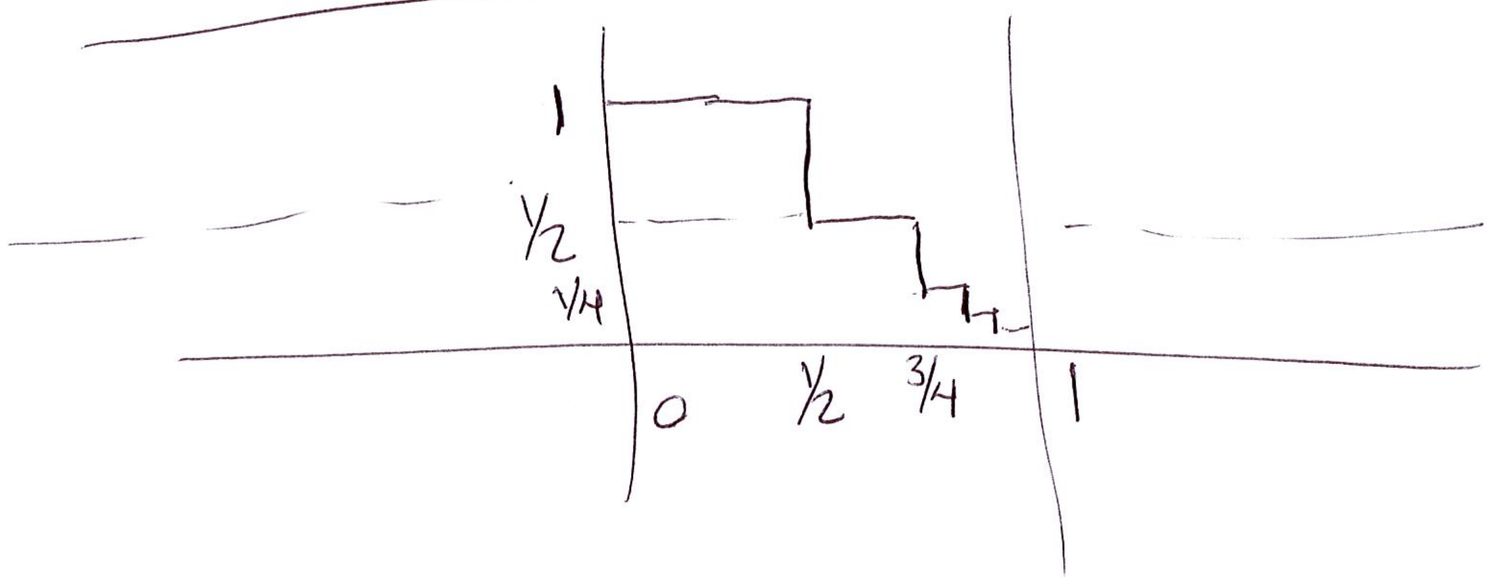
$$x(t) = \sin\left(\frac{2\pi}{t}\right)$$

satisfies condition 1 but not condition 2



In any finite interval of time, there is a finite number of discontinuities, and each of these discontinuities is finite.

Counter Example



Properties of Continuous-time Fourier Series

Linearity

$$x(t) \xleftrightarrow{FS} a_k$$

$$y(t) \xleftrightarrow{FS} b_k$$

same fundamental frequency

$$z(t) = A\,x(t) + B\,y(t) \xleftrightarrow{FS} A\,a_k + B\,b_k$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k\, e^{jk\omega_0 t}$$

$$z(t) = A\,x(t) + B\,y(t) = \sum_{k=-\infty}^{\infty} (A\,a_k + B\,b_k)\, e^{jk\omega_0 t}$$

Time Shifting

$$y(t) = x(t - t_0)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

How does b_k relate to a_k?

$$a_k = \frac{1}{T} \int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$b_k = \frac{1}{T} \int_T y(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T} \int_T x(t - t_0)\, e^{-jk\omega_0 t}\, dt$$

Let $\tau = t - t_0$

$$b_k = \frac{1}{T}\int_T x(\tau)\, \underbrace{e^{-jk\omega_0(\tau + t_0)}}_{e^{-jk\omega_0 \tau}\, \underbrace{e^{-jk\omega_0 t_0}}_{\text{does not depend on } \tau}}\, d\tau$$

$$= \frac{1}{T} e^{-jk\omega_0 t_0} \int_T x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau$$

$$\downarrow$$

$$a_k$$

$$= a_k\, e^{-jk\omega_0 t_0}$$

$$x(t) \xleftrightarrow{FS} a_k$$

$$\Longrightarrow x(t - t_0) \xleftrightarrow{FS} a_k\, e^{-jk\omega_0 t_0}$$

Time Reversal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0 t}$$

Let $m = -k$

$$x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jk\omega_0 m}$$

$$= \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

~~How does~~ b_k ~~relate~~

How does b_k relate to a_k?

$$b_k = a_{-k}$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$a_0 = \frac{1}{T}\int_T x(t)\, dt$$

Average value of $x(t)$

$x(t)$ and $x(-t)$ have the same average value

$$\Rightarrow b_0 = a_0$$