ECE 301 - Lecture #16

Fourier Series Representation of Discrete-time Periodic Signals

$$\phi_k[n] = e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$\phi_{k+N}[n] = e^{j(k+N)\left(\frac{2\pi}{N}\right)n}$$

$$= e^{jk\left(\frac{2\pi}{N}\right)n} \underbrace{e^{j\cancel{N}\frac{2\pi}{\cancel{N}}n}}_{1}$$

$$= e^{jk\left(\frac{2\pi}{N}\right)n} = \phi_k[n]$$

$$\underbrace{\phi_0[n] \quad \phi_1[n] \quad \cdots\cdots \quad \phi_{N-1}[n]}$$

All the elements in the set of harmonically related complex exponentials with fund. freq. $\frac{2\pi}{N}$

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$k=\langle N\rangle$ ↑ N consecutive indices

<u>Fact</u>

$$\sum_{n=\langle N\rangle} e^{jk\left(\frac{2\pi}{N}\right)n} = \begin{cases} N & k=0,\ \pm N,\ \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$\sum_{n=\langle N\rangle} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)rn}$$

$$= \sum_{n=\langle N\rangle} \sum_{k=\langle N\rangle} a_k e^{j(k-r)\left(\frac{2\pi}{N}\right)n}$$

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$$= \sum_{k=\langle N\rangle} \sum_{n=\langle N\rangle} a_k e^{j(k-r)\left(\frac{2\pi}{N}\right)n}$$

$$= \sum_{k=\langle N\rangle} a_k \underbrace{\sum_{n=\langle N\rangle} e^{j(k-r)\left(\frac{2\pi}{N}\right)n}}_{\begin{cases} N & \text{if } k=r \\ 0 & \text{otherwise} \end{cases}}$$

$$= a_r N$$

$$= \sum_{n=\langle N\rangle} x[n] e^{-j\left(\frac{2\pi}{N}\right)rn}$$

DT - FS

$$\Rightarrow \boxed{\begin{aligned} a_k &= \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-j\left(\frac{2\pi}{N}\right)kn} \\ x[n] &= \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \end{aligned}}$$

Example

$x[n] = \sin \omega_0 n$

$\frac{2\pi}{\omega_0}$ is rational $\Longleftrightarrow$ $x[n]$ periodic

Only if it's periodic, we find FS

Let's say that $\omega_0 = \frac{2\pi}{N}$

$$x[n] = \frac{1}{2j} e^{j\left(\frac{2\pi}{N}\right)n} - \frac{1}{2j} e^{-j\left(\frac{2\pi}{N}\right)n}$$

$a_1 = \frac{1}{2j}$ $\qquad$ $a_{-1} = -\frac{1}{2j}$

Let's say that $N = 5$

So we have 5 coefficients

$a_0 = 0 \quad a_1 = \frac{1}{2j} \quad a_2 = 0 \quad a_3 = 0 \quad a_4 = -\frac{1}{2j}$

$x[n] = \sin \omega_0 n$

$$\frac{2\pi}{\omega_0} = \frac{3}{5} = \frac{M}{N}$$

$$x[n] = \frac{1}{2j} e^{j3\left(\frac{2\pi}{N}\right)n} - \frac{1}{2j} e^{-j3\left(\frac{2\pi}{N}\right)n}$$

$$= \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$\Rightarrow \quad a_3 = \frac{1}{2j} \qquad a_{-3} = -\frac{1}{2j}$$

we still have 5 FS coefficients

$$a_0 = 0 \quad a_1 = 0 \quad a_2 = -\frac{1}{2j} \quad a_3 = \frac{1}{2j} \quad a_4 = 0$$

Properties of DT-FS

Same period

$x[n] \xleftrightarrow{FS} a_k$

$y[n] \xleftrightarrow{FS} b_k$

$$A\,x[n] + B\,y[n] \xleftrightarrow{FS} A\,a_k + B\,b_k$$

Time shifting

$$x[n] \xleftrightarrow{FS} a_k$$

$$x[n-n_0] \xleftrightarrow{FS} ?$$

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$x[n-n_0] = \sum_{k=\langle N\rangle} a_k \underbrace{e^{jk\left(\frac{2\pi}{N}\right)(n-n_0)}}_{e^{jk\left(\frac{2\pi}{N}\right)n}\; e^{-jk\left(\frac{2\pi}{N}\right)n_0}}$$

$$x[n-n_0] = \sum_{k=\langle N \rangle} \underbrace{a_k e^{-jk\left(\frac{2\pi}{N}\right)n_0}}_{\text{new FS coeff.}} e^{jk\left(\frac{2\pi}{N}\right)n} \quad (T)$$

$$x[n-n_0] \xleftrightarrow{FS} a_k e^{-jk\left(\frac{2\pi}{N}\right)n_0}$$

Frequency Shifting

$$x[n] \xleftrightarrow{FS} a_k$$

$$e^{jM\left(\frac{2\pi}{N}\right)n} x[n] = e^{jM\left(\frac{2\pi}{N}\right)n} \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$= \sum_{k=\langle N \rangle} a_k e^{j(k+M)\left(\frac{2\pi}{N}\right)n}$$

Let $r = k+M$

$$e^{jM\left(\frac{2\pi}{N}\right)n}\, x[n]$$

$$= \sum_{k=\langle N\rangle} a_k\, e^{j(k+M)\left(\frac{2\pi}{N}\right)n}$$

Let $r = k+M$

$$= \sum_{r=\langle N\rangle} a_{r-M}\, e^{jr\left(\frac{2\pi}{N}\right)n}$$

$$= \sum_{k=\langle N\rangle} \underbrace{a_{k-M}}_{\substack{\text{new FS} \\ \text{coeff.}}}\, e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$e^{jM\left(\frac{2\pi}{N}\right)n}\, x[n] \xleftrightarrow{FS} a_{k-M}$$

Conjugation

$$x[n] \xleftrightarrow{FS} a_k$$

$$x^*[n] \xleftrightarrow{FS} \; ? \quad a^*_{-k}$$

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$x^*[n] = \sum_{k=\langle N \rangle} a^*_k e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$= \sum_{k=\langle N \rangle} \underbrace{a^*_{-k}}_{\text{new FS coeff.}} e^{jk\left(\frac{2\pi}{N}\right)n}$$

Time Reversal

$$x[n] \xleftrightarrow{FS} a_k$$

$$x[n] = \sum_{k=\langle N \rangle} a_k \, e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$x[-n] = \sum_{k=\langle N \rangle} a_k \, \underbrace{e^{jk\left(\frac{2\pi}{N}\right)-n}}_{e^{-jk\left(\frac{2\pi}{N}\right)n}}$$

$$= \sum_{k=\langle N \rangle} \underbrace{a_{-k}}_{\text{new FS coeff.}} \, e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$x[-n] \xleftrightarrow{FS} a_{-k}$$