ECE 301 - Lecture #17

Properties of DT- FS (cont'd)

Time Scaling

What happens in CT?

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk(\alpha\omega_0)t}$$

$$x(t) \xleftrightarrow{FS} a_k$$

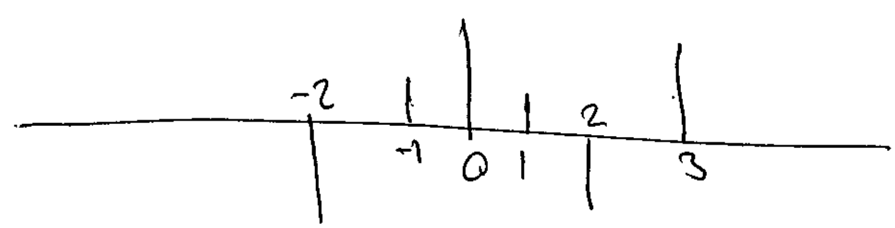
$$x(\alpha t) \xleftrightarrow{FS} ? \; a_k$$

What happens in DT?

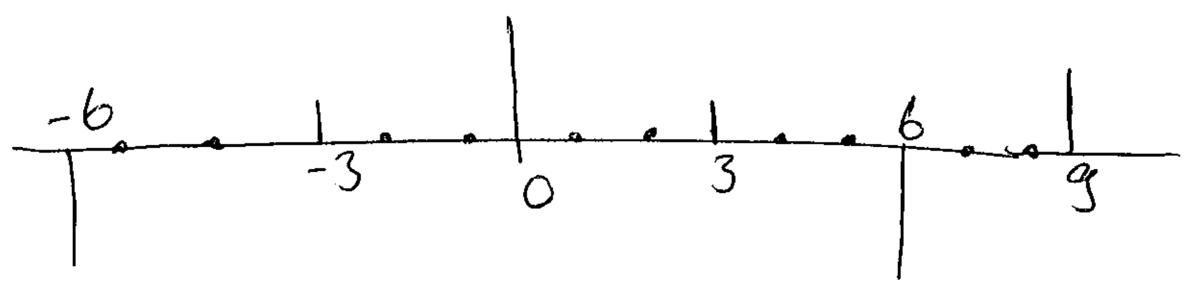
$$x_{(m)}[n] = \begin{cases} x[n/m] & \text{if } n \text{ is a multiple of } m \\ 0 & \text{otherwise} \end{cases}$$

Example ($m=3$)

$x[n]$



$x_{(3)}[n]$



What happens to Fund. Period?

$$N_{(new)} = Nm$$

Now, we have Nm coeffs. $a_0, a_1, \ldots, a_{Nm-1}$

Before, we had N FS coeffs. $a_0, a_1, \ldots, a_{N-1}$

(3)

$$x_{(m)}[n] = \begin{cases} x[n/m] & \text{if } n \text{ is a multiple of } m \\ 0 & \text{otherwise} \end{cases}$$

and $x[n] \overset{FS}{\longleftrightarrow} a_k$

then $x[n/m] \overset{FS}{\longleftrightarrow} \frac{1}{m} a_k$

Parserval's Relation

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2 \quad (*)$$

Let's verify what happens to the L.H.S. and R.H.S. of $(*)$ as we scale $x[n]$ in time by m

L.H.S

Over one period

$$\frac{1}{Nm} \sum |x[n]|^2 = \frac{1}{m} \text{ of L.H.S. of } (*)$$

R.H.S.

$$\sum_{k=\langle Nm\rangle} |a_k|^2 \frac{1}{m^2} = \frac{1}{m^2}\, m \sum_{k=\langle N\rangle} |a_k|^2$$

$$= \frac{1}{m} \text{ R.H.S. of } (*)$$

First Difference

$$x[n] \overset{FS}{\longleftrightarrow} a_k$$

$$x[n]-x[n-1] \overset{FS}{\longleftrightarrow} \ ? \ \left(1-e^{-jk\left(\frac{2\pi}{N}\right)}\right)a_k$$

$$x[n] = \sum_{k=\langle N\rangle} a_k\, e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$x[n-1] = \sum_{k=\langle N\rangle} a_k\, e^{jk\left(\frac{2\pi}{N}\right)(n-1)}$$

$$= \cancel{e} \sum_{k=\langle N\rangle} \underbrace{a_k\, e^{-jk\left(\frac{2\pi}{N}\right)}}_{\text{new FS coeff. for } x[n-1]} e^{jk\left(\frac{2\pi}{N}\right)n}$$

Running Sum

$$x[n] \overset{FS}{\longleftrightarrow} a_k$$

$$\sum_{k=-\infty}^{n} x[k] \overset{FS}{\longleftrightarrow} \left(\frac{1}{1-e^{-jk\left(\frac{2\pi}{N}\right)}} \right) a_k$$

~~Periodic Conv~~

Multiplication

$$x[n] \overset{FS}{\longleftrightarrow} a_k$$

with same period $\longrightarrow y[n] \overset{FS}{\longleftrightarrow} b_k$

$$z[n] = x[n]\,y[n] = \left(\sum_{k_1=\langle N \rangle} a_{k_1} e^{jk_1\left(\frac{2\pi}{N}n\right)} \right)$$

$$\left(\sum_{k_2=\langle N \rangle} b_{k_2} e^{jk_2\left(\frac{2\pi}{N}\right)n} \right)$$

$$z[n] = x[n]\, y[n]$$

$$z[n] \overset{FS}{\longleftrightarrow} \underbrace{\sum_{\ell = \langle N \rangle} a_\ell\, b_{k-\ell}}_{\text{Periodic Convolution}}$$

Periodic Convolution

$$z[n] = \sum_{r = \langle N \rangle} x[r]\, y[n-r] \overset{FS}{\longleftrightarrow} N a_k b_k$$

$$= \sum_{r = \langle N \rangle} \left(\sum_{k = \langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right) r} \right) \left(\sum_{k = \langle N \rangle} b_k e^{jk\left(\frac{2\pi}{N}\right)(n-r)} \right)$$

(n-r)

Fourier Series and LTI Systems

CT if $x(t) = e^{st}$ then $y(t) = H(s)\,e^{st}$

$$H(s) = \int_{-\infty}^{\infty} \underbrace{h(\tau)}_{\text{Impulse response}}\, e^{-s\tau}\, d\tau$$

DT

if $x[n] = z^n$ then $y[n] = H(z)\,z^n$

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$$

CT $s = j\omega$ $x(t) = e^{j\omega t}$

then $y(t) = \underbrace{H(j\omega)}_{\text{Frequency Response}}\, e^{j\omega t}$ and $H(j\omega) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

then $$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$