ECE 301 - Lecture #18

Fourier Series and LTI Systems

$$x(t) \xleftrightarrow{FS} a_k$$

$$y(t) \xleftrightarrow{FS} a_k\, H(jk\omega_0)$$

$$H(j\omega) = \int_{\tau=\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$

Impulse Response

Frequency Response

Example 3-16

$$h(t) = e^{-t}u(t)$$

$$H(j\omega) = \int_{\tau=0}^{\infty} e^{-\tau}\, e^{-j\omega\tau}\, d\tau$$

$$= -\frac{1}{1+j\omega}\, e^{-(1+j\omega)\tau}\Big|_0^{\infty}$$

$$= \frac{1}{1+j\omega}$$

(2)

$$H(j\omega) = \frac{1}{1+j\omega} \qquad H(jk\omega_0) = \frac{1}{1+jk\omega_0}$$

$$x(t) = \sum_{k=-3}^{3} a_k e^{jk\omega_0 t} \qquad \omega_0 = 2\pi$$

$$a_0 = 1 \qquad a_1 = a_{-1} = \frac{1}{4}$$

$$a_2 = a_{-2} = \frac{1}{2} \qquad a_3 = a_{-3} = \frac{1}{3}$$

$$y(t) = \sum_{k=-3}^{3} b_k e^{jk2\pi t}$$

$$b_0 = 1 \qquad b_1 = \frac{1}{4}\left(\frac{1}{1+j2\pi}\right)$$

$$b_{-1} = \frac{1}{4}\left(\frac{1}{1-j2\pi}\right)$$

$$b_2 = \frac{1}{2}\left(\frac{1}{1+j4\pi}\right) \qquad b_{-2} = \frac{1}{2}\left(\frac{1}{1-j4\pi}\right)$$

$$b_3 = \frac{1}{3}\left(\frac{1}{1+j6\pi}\right) \qquad b_{-3} = \frac{1}{3}\left(\frac{1}{1-j6\pi}\right)$$

Observations

1- The output of an LTI system has the same period as the input
Fundamental

unless the frequency response has a special property (For example, zeroing out all FS coeffs with odd indices, then the fundamental period is halved)

2- In Example 3.16, $b_k = b_{-k}^*$, for all $k \neq 0$

so $y(t)$ is a real signal

This makes sense because both $x(t)$ and $h(t)$ are real signals

Example 3.17

$$h[n] = \alpha^n u[n], \quad -1 < \alpha < 1$$

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n$$

$$= \frac{1}{1-\alpha e^{-j\omega}}$$

$$x[n] = \cos\left(\frac{2\pi n}{N}\right)$$

$$= \frac{1}{2} e^{j\left(\frac{2\pi}{N}\right)n} + \frac{1}{2} e^{-j\left(\frac{2\pi}{N}\right)n}$$

$$y[n] = \frac{1}{2}\left(\frac{1}{1-\alpha e^{-j\frac{2\pi}{N}}}\right) e^{j\left(\frac{2\pi}{N}\right)n}$$

$$+ \frac{1}{2}\left(\frac{1}{1-\alpha e^{j\frac{2\pi}{N}}}\right) e^{-j\left(\frac{2\pi}{N}\right)n}$$

If we write

$$\frac{1}{1-\alpha e^{-j\frac{2\pi}{N}}} = r e^{j\theta}$$

then

$$y[n] = r \cos\left(\frac{2\pi}{N} n + \theta\right)$$

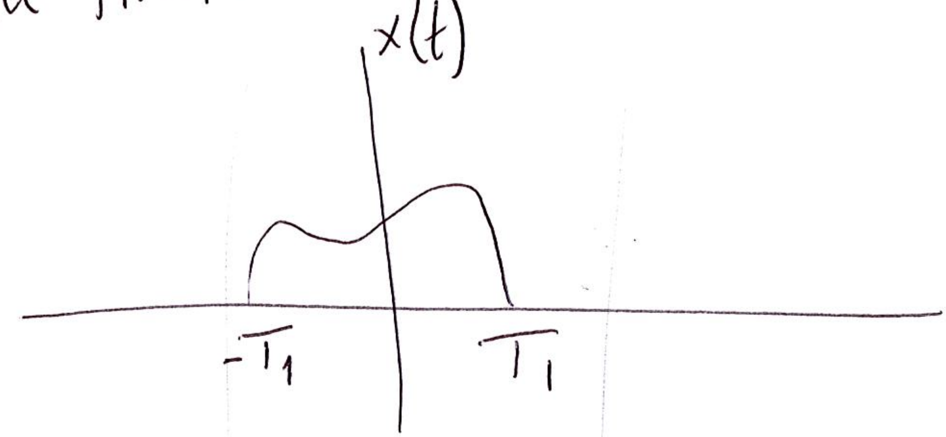
For example if $N=4$

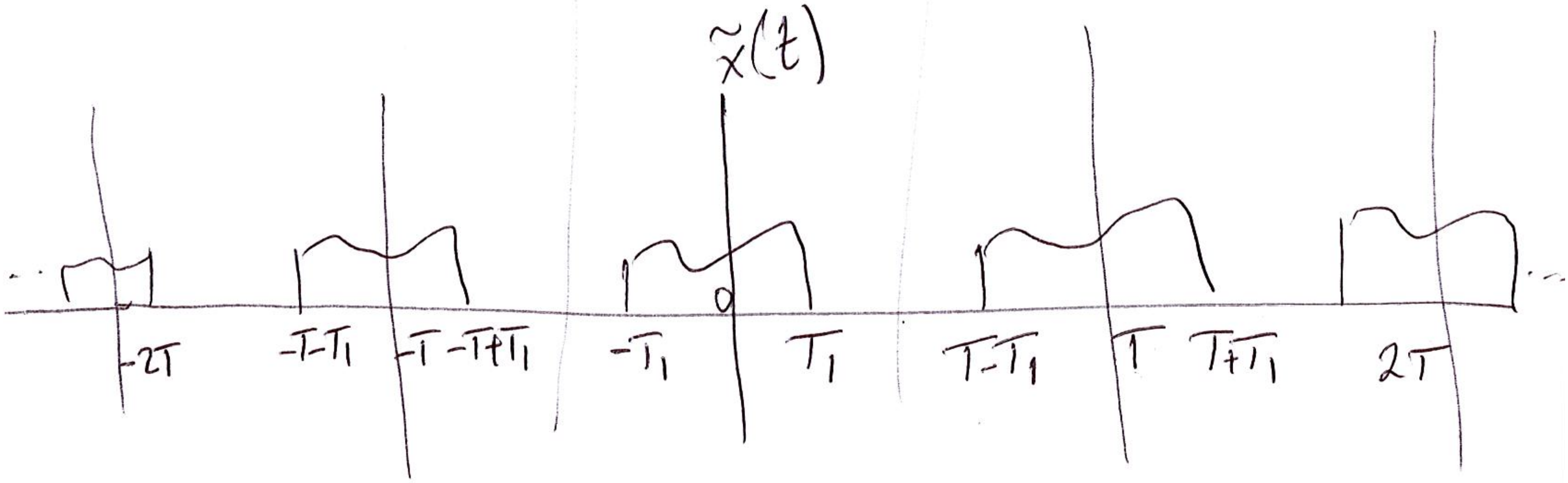
$$\frac{1}{1-\alpha e^{-j\frac{\pi}{2}}} = \frac{1}{1+\alpha j} = \frac{1}{\sqrt{1+\alpha^2}} e^{j\left(-\tan^{-1}(\alpha)\right)}$$

$$y[n] = \frac{1}{\sqrt{1+\alpha^2}} \cos\left(\frac{\pi n}{2} - \tan^{-1}(\alpha)\right)$$

Continuous-time Fourier Transform for Aperiodic Signals

Let's assume that we have a signal $x(t)$ with finite duration





$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt$$

Define $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$a_k = \frac{1}{T} X(jk\omega_0)$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t}$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0)\, e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0)\, e^{jk\omega_0 t}\, \omega_0$$

$$x(t) = \lim_{T\to\infty} \tilde{x}(t)$$

As $T \to \infty$, $\omega_0 \to 0$

Holds for any aperiodic signal satisfying Dirichlet Conditions

$$\boxed{\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \\ X(j\omega) &= \int_{t=-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \end{aligned}}$$

CT Fourier Transform