ECE 301 - Lecture #19

Fourier Transform of Aperiodic Signals (Continuous-Time)

Convergence

$$\boxed{\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \\ X(j\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \end{aligned}}$$

for any general Aperiodic signal $x(t)$,

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

when is $\hat{x}(t) = x(t)$?

If $\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$

then $\int_{-\infty}^{\infty} |e(t)|^2 \, dt = 0$

where $e(t) = \hat{x}(t) - x(t)$

Dirichlet Conditions

* They guarantee that $\hat{x}(t) = x(t)$ except at discontintinuities

1. $\int_{-\infty}^{\infty} |x(t)| < \infty$
2. Finite number of maxima and minima in any finite interval
3. Finite number of discont. in any finite interval. Each of the discont. is finite

Example 4.1

$$x(t) = e^{-at} u(t), \quad a > 0$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

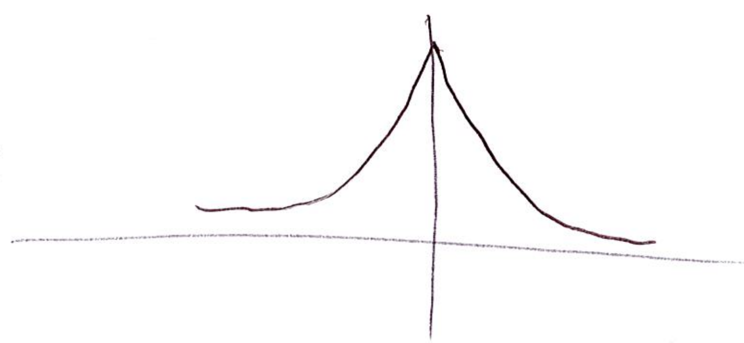
$$= \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$= -\frac{1}{a+j\omega}\, e^{-(a+j\omega)t} \Big|_{0}^{\infty}$$

$$= \frac{1}{a+j\omega}, \quad a > 0$$

Example 4.2

$$x(t) = e^{-a|t|}, \quad a > 0$$



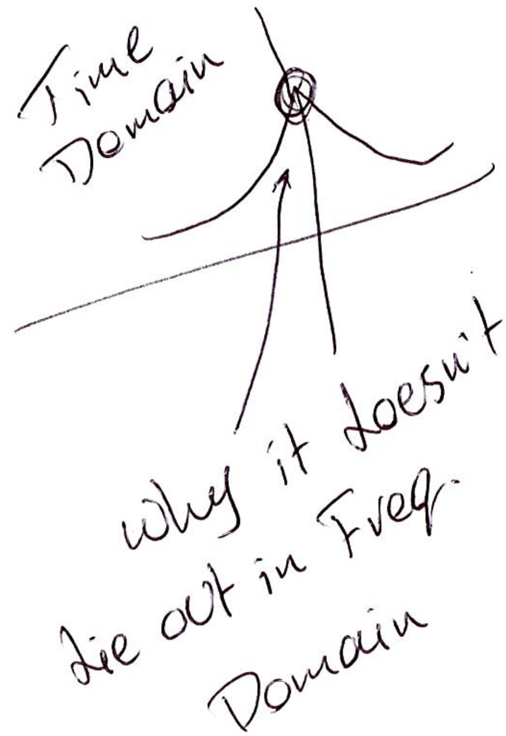
$$X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|}\, e^{-j\omega t}\, dt$$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|}\, e^{-j\omega t}\, dt$$

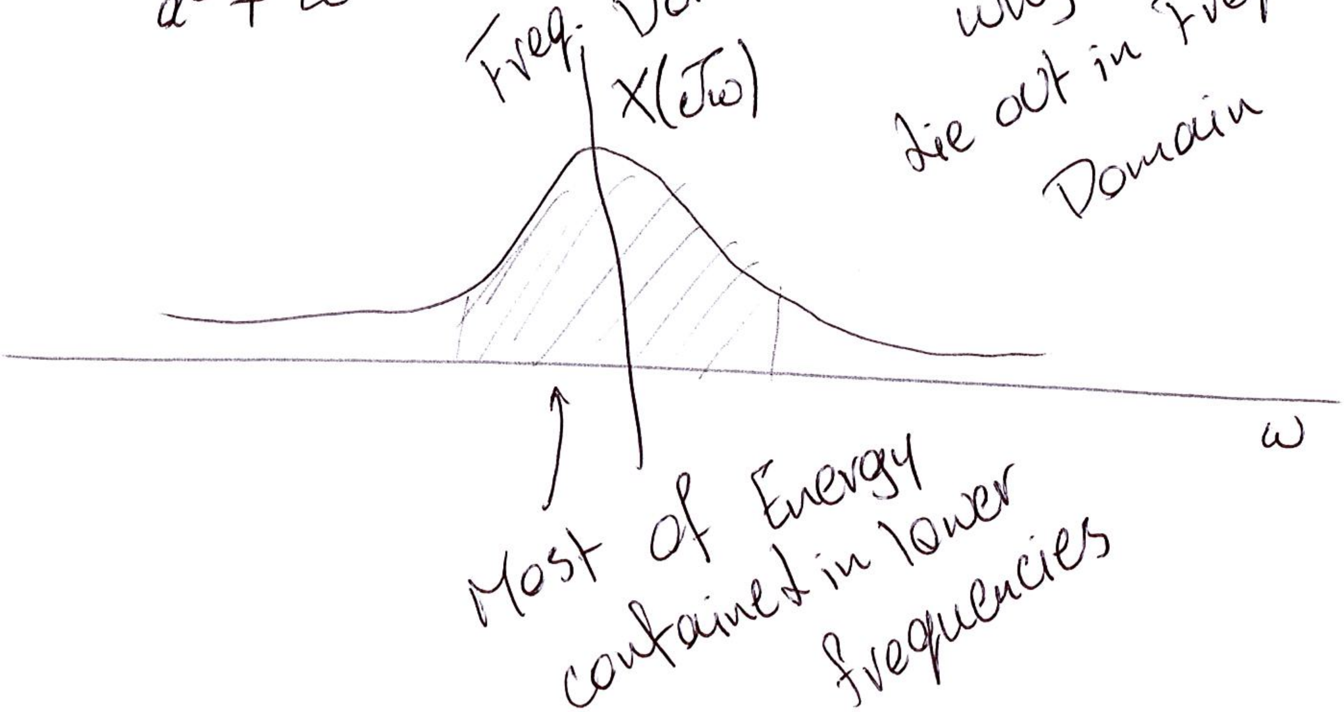
$$= \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt + \int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\, dt$$

$$= -\frac{1}{a+j\omega}\, e^{-(a+j\omega)t}\Big|_{0}^{\infty} + \frac{1}{a-j\omega}\, e^{(a-j\omega)t}\Big|_{-\infty}^{0}$$

$$= \frac{1}{a+j\omega} + \frac{1}{a-j\omega}$$



$$= \frac{2a}{a^2+\omega^2}$$



Example 4.3

$$x(t) = \delta(t)$$

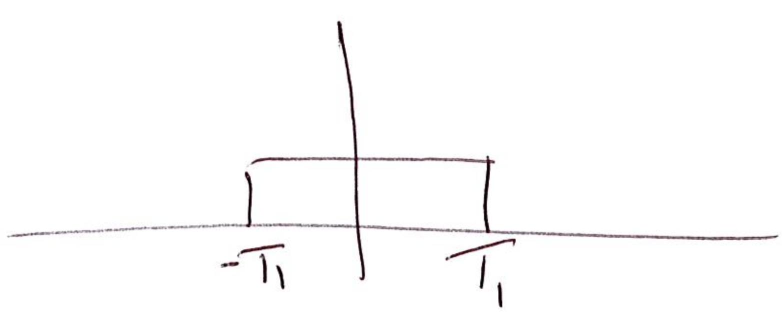
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= 1$$

Example 4.4

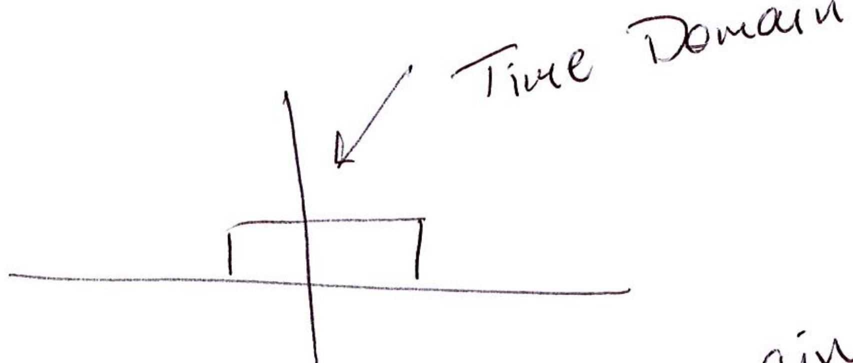
$$x(t) = \begin{cases} 1 , & |t| < T_1 \\ 0 , & |t| > T_1 \end{cases}$$

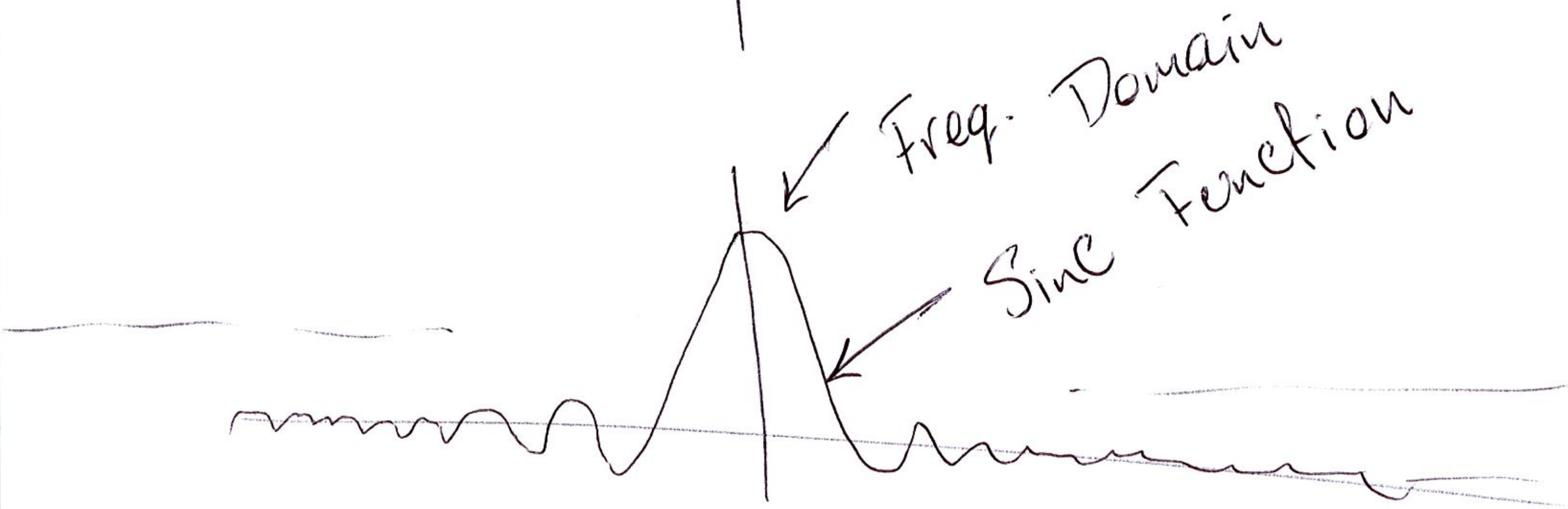


$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T_1}^{T_1} e^{-j\omega t}\, dt$$

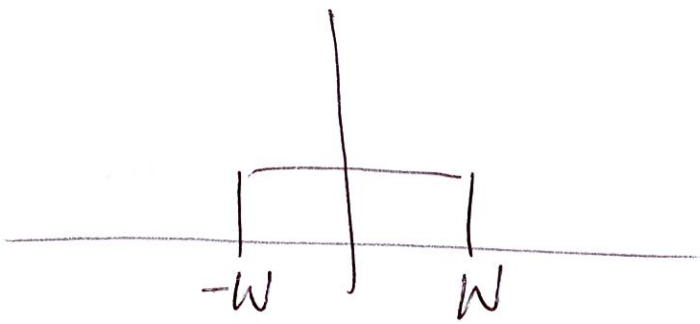
$$= 2\,\frac{\sin \omega T_1}{\omega}$$





Example 4.5

$$X(e^{j\omega}) = \begin{cases} 1 & , |\omega| < W \\ 0 & , |\omega| > W \end{cases}$$



$$x(t) = \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega t}$$

$$= \frac{\sin Wt}{\pi t}$$

$$\mathrm{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$$

$$\frac{2\sin\omega T_1}{\omega} = 2T_1 \,\mathrm{sinc}\left(\frac{\omega T_1}{\pi}\right)$$

$$\frac{\sin Wt}{\pi t} = \frac{W}{\pi}\,\mathrm{sinc}\left(\frac{Wt}{\pi}\right)$$

Fourier Transform for Periodic Signals

$$X(j\omega) = 2\pi\,\delta(\omega-\omega_0)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega$$

$$= e^{j\omega_0 t}$$

$$e^{j\omega_0 t} \xleftrightarrow{FT} 2\pi\,\delta(\omega-\omega_0)$$

$$e^{jk\omega_0 t} \xleftrightarrow{FT} 2\pi\,\delta(\omega-k\omega_0)$$

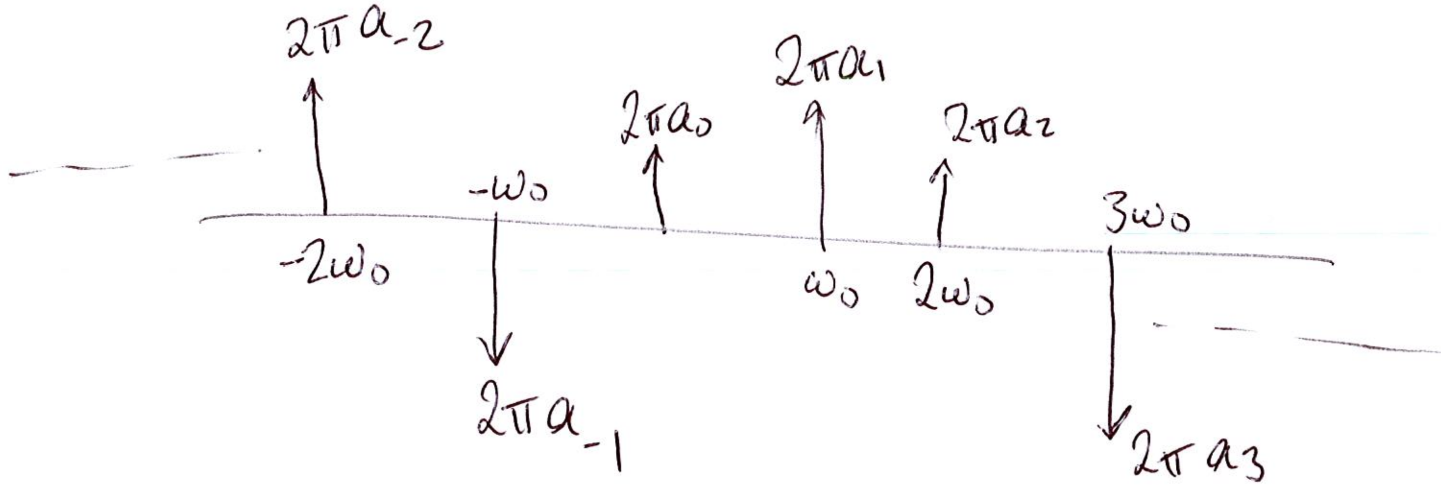
$$a_k\, e^{jk\omega_0 t} \xleftrightarrow{FT} 2\pi\, a_k\, \delta(\omega-k\omega_0)$$

$$\sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} \xleftrightarrow{FT} \sum_{k=-\infty}^{\infty} 2\pi\, a_k\, \delta(\omega-k\omega_0)$$

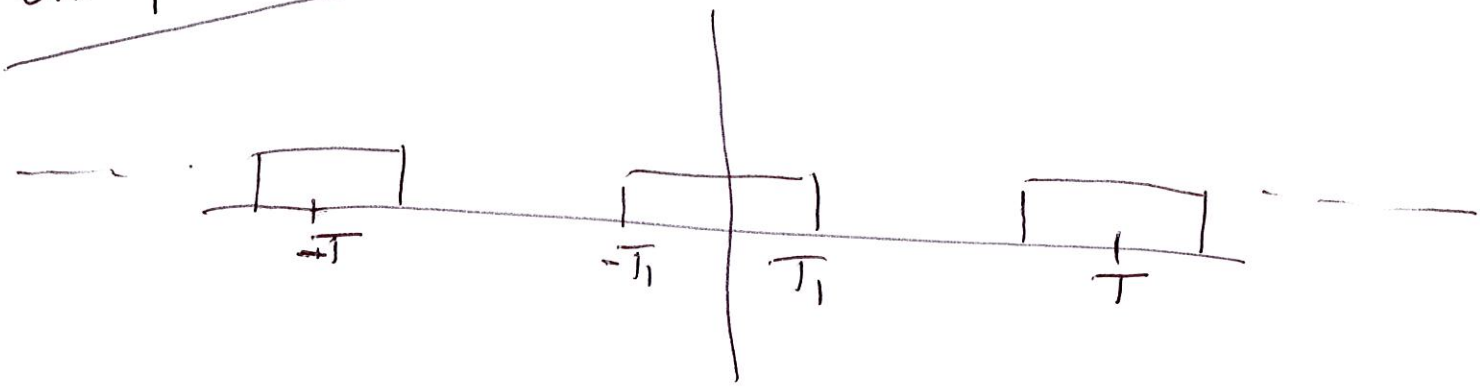
For periodic signals

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$



Example 4-6



$$a_k = \frac{\sin k\omega_0 T_1}{\pi k}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$$