ECE 301 - Lecture #20

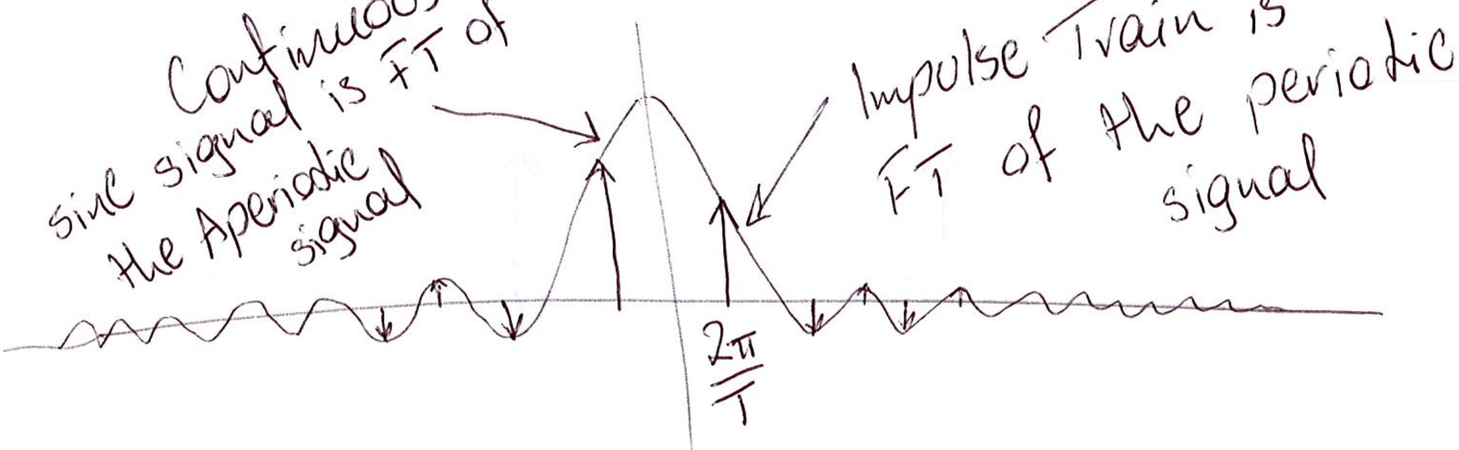
Fourier Transform for Periodic Signals (Continuous-time)

Periodic

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \text{(FS Representation)}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

$T > 2T_1$

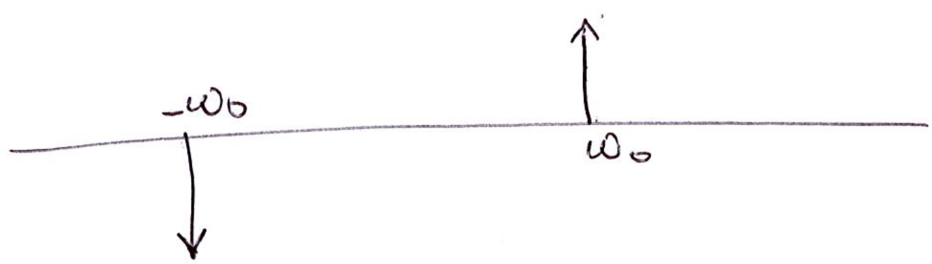


Example 4.7

$$x(t) = \sin \omega_0 t$$

$$a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j}$$

$$X(j\omega)$$

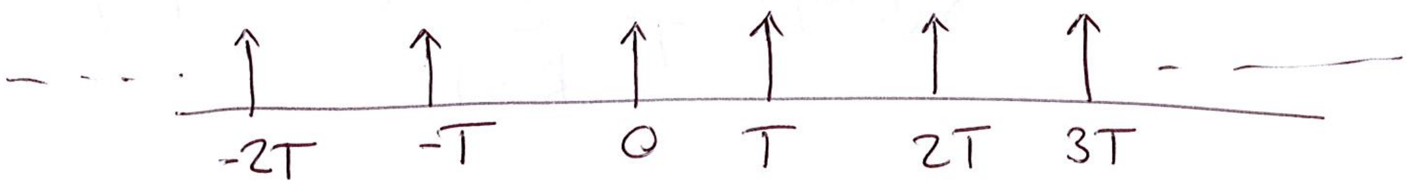


$$x(t) = \cos \omega_0 t$$

$$a_1 = a_{-1} = \frac{1}{2}$$

Example 4-8

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$$



$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}$$

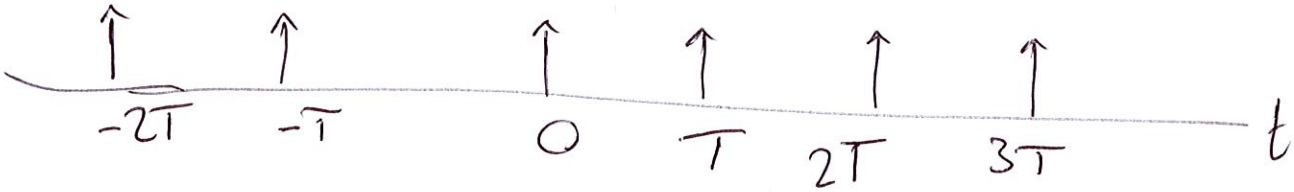
$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta(\omega - k\omega_0)$$

$$= 2\pi \sum_{k=-\infty}^{\infty} a_k\, \delta\left(\omega - \frac{k2\pi}{T}\right)$$

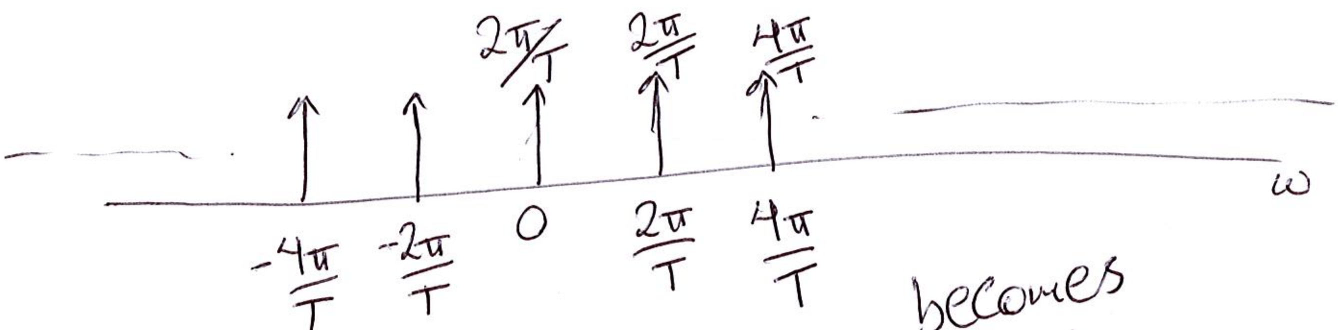
$$= \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

(4)

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



$$X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$



As the period in time domain ~~is~~ becomes larger, the period in frequency domain becomes smaller, and vice versa

If T is very large, signal is very slow, ~~fast of~~ High energy at low frequencies

If T is very small, signal is very fast, Low energy at low frequencies

Properties of the continuous time Fourier Transform

Linearity

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

$$y(t) \xleftrightarrow{\mathcal{F}} Y(j\omega)$$

$$a\,x(t) + b\,y(t) \xleftrightarrow{\mathcal{F}} a\,X(j\omega) + b\,Y(j\omega)$$

Time Shifting

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

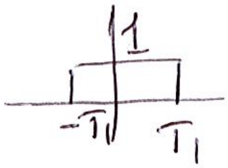
$$x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(j\omega)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

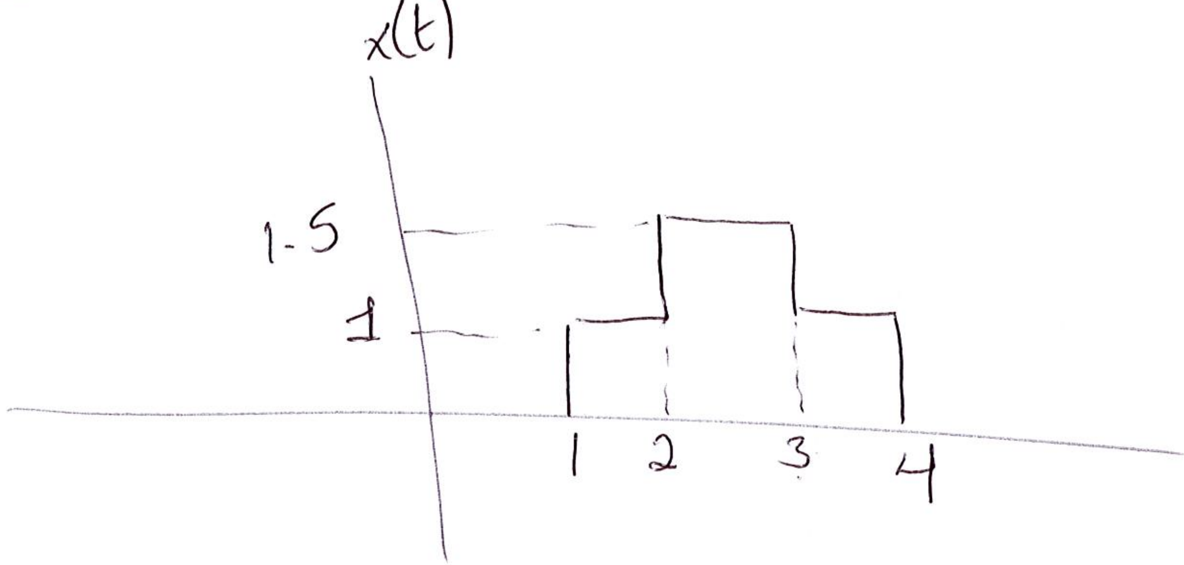
$$x(t-t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega(t-t_0)}\, d\omega$$

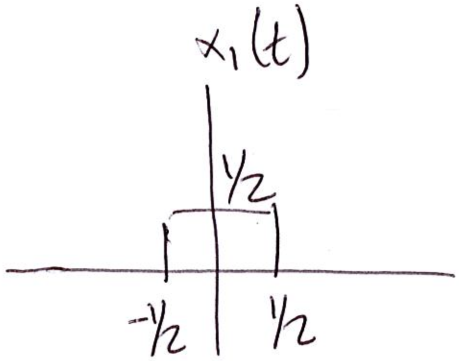
$$= e^{-j\omega t_0} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}$$

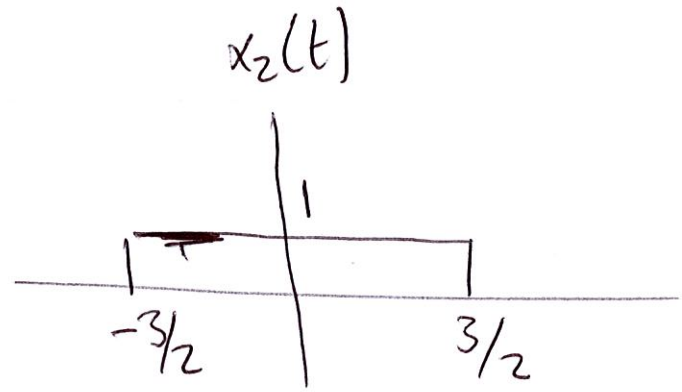
Example 4.9



$$\overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{2\sin(\omega T_1)}{\omega}$$







$$x(t) = x_1(t-2.5) + x_2(t-2.5)$$

$$X(j\omega) = e^{-j\omega\frac{5}{2}} X_1(j\omega) + e^{-j\omega\frac{5}{2}} X_2(j\omega)$$

$$= e^{-j\frac{5\omega}{2}} \left[\frac{\sin(\omega/2)}{\omega} + \frac{2\sin(3\omega/2)}{\omega} \right]$$

Conjugation

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

$$x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-j\omega)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$x^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega)\, e^{-j\omega t}\, d\omega$$

$$x^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{X^*(-j\omega)}\, e^{j\omega t}\, d\omega$$

If $x(t)$ is real, then $X^*(-j\omega) = X(j\omega)$

$$x(t) = e^{-at}\, u(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt = \frac{1}{a + j\omega}$$

$$X(j\omega) = \frac{1}{a+j\omega}$$

$$X^*(j\omega) = \frac{1}{a-j\omega}$$

$$X^*(-j\omega) = \frac{1}{a+j\omega} = X(j\omega)$$

because $x(t) = e^{-at}u(t)$ is a real signal

Let $X(j\omega) = Re\{X(j\omega)\} + j\,Im\{X(j\omega)\}$

If $x(t)$ is real, then

$$Re\{X(j\omega)\} = Re\{X(-j\omega)\}$$

Even signal

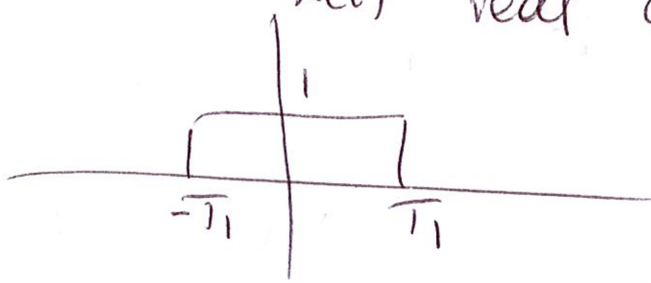
$$Im\{X(j\omega)\} = -Im\{X(-j\omega)\}$$

Odd signal

* If $x(t)$ is real and even
  then $X(j\omega)$ is real and even

* If $x(t)$ is real and odd
  then $X(j\omega)$ is purely imaginary and odd

Example

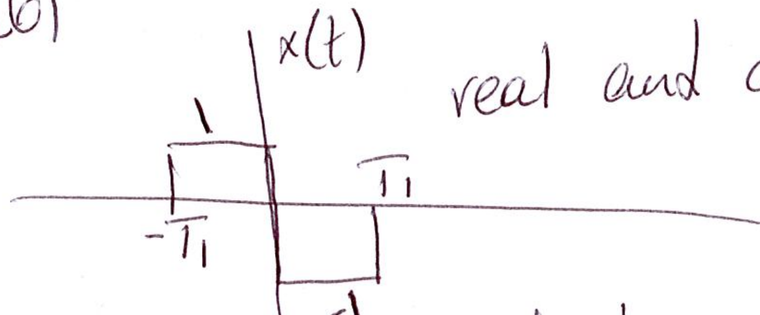


real and even

$$X(j\omega) = \frac{2\sin(\omega T_1)}{\omega}$$

real and even

Bonus (0.3%)
(Due Friday Oct. 26)



real and odd

Find $X(j\omega)$ and show that it is purely imaginary and odd