ECE 301 - Lecture #21

If $\quad x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$

then $\quad x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-j\omega)$

Hence, if $x(t)$ is real, then

$$X(j\omega) = X^*(-j\omega)$$

$$\Rightarrow \operatorname{Re}\{X(j\omega)\} = \operatorname{Re}\{X(-j\omega)\}$$

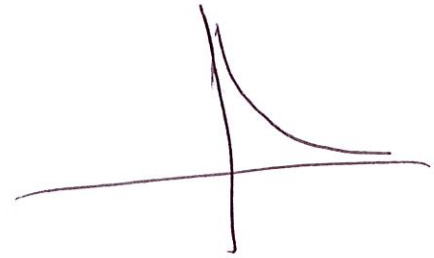
Even part of the signal

$$\operatorname{Im}\{X(j\omega)\} = -\operatorname{Im}\{X(-j\omega)\}$$

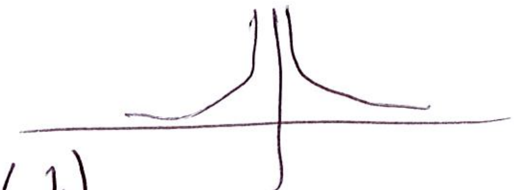
odd part of the signal

Example 4.10

$x_1(t) \rightarrow e^{-at} u(t), \quad a > 0$



$\xleftrightarrow{\mathcal{F}} \frac{1}{a + j\omega}$



$$x(t) = e^{-a|t|} = e^{-at} u(t) + e^{at} u(-t)$$

$$= 2 \left[\frac{e^{-at} u(t) + e^{at} u(-t)}{2} \right]$$

$$= 2 \, \mathcal{E}v\{x_1(t)\}$$

Real signal

$$X(j\omega) = 2 \, \mathrm{Re}\{X_1(j\omega)\} = 2 \, \mathrm{Re}\left\{\frac{1}{a + j\omega}\right\}$$

$$= \frac{2a}{a^2 + \omega^2}$$

Differentiation and Integration

$$\frac{d\,x(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega\, X(j\omega)\, e^{j\omega t}\, d\omega$$

If $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$

then $\frac{d\,x(t)}{dt} \xleftrightarrow{\mathcal{F}} j\omega\, X(j\omega)$

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

Example 4.11

$$g(t) = \delta(t) \xleftrightarrow{\mathcal{F}} G(j\omega) = 1$$

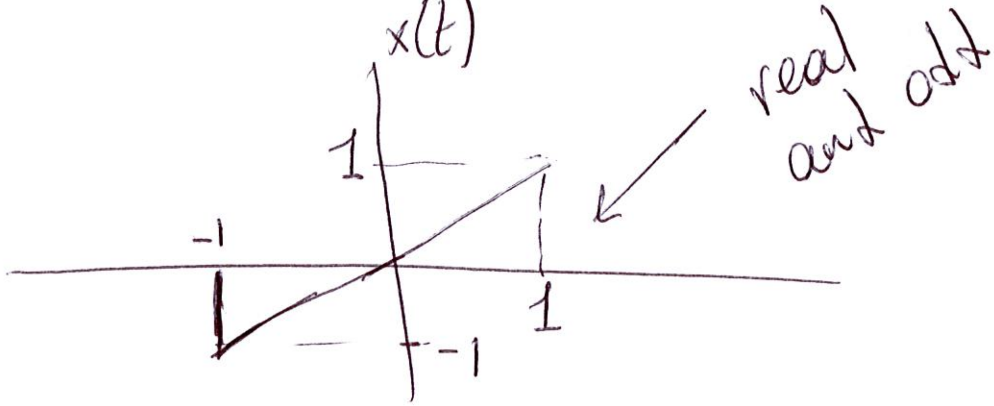
$$x(t) = \int_{-\infty}^{t} g(\tau)\, d\tau \leftarrow u(t)$$

$$X(j\omega) = \frac{1}{j\omega} G(j\omega) + \pi\, G(0)\,\delta(\omega)$$

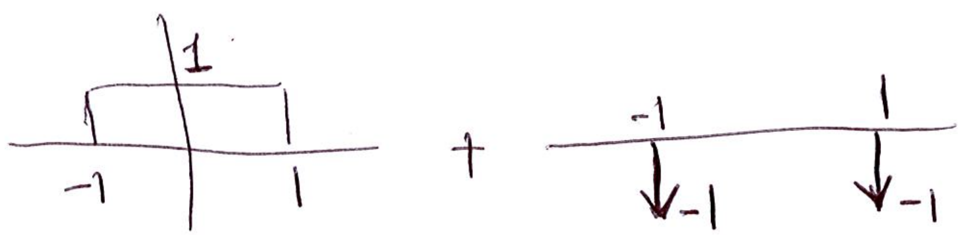
$$= \frac{1}{j\omega} + \pi\,\delta(\omega)$$

Example 4-12

$$g(t) = \frac{d}{dt}\,x(t)$$



$g(t) =$



$$G(j\omega) = \left(\frac{2\sin\omega}{\omega}\right) - e^{j\omega} - e^{-j\omega}$$

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi\,\cancel{G(0)}^{\,0}\,\delta(\omega)$$

$$= \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}$$

purely imaginary and odd

Time and Frequency Scaling

If $x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$

then $x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$

$$\mathcal{F}\{\cancel{x}(at)\} = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

Let $\tau = at$

$$\mathcal{F}\{x(at)\} = \begin{cases} \frac{1}{a} \int_{-\infty}^{\infty} x(\tau)\, e^{-j\left(\frac{\omega}{a}\right)\tau}\, d\tau, & a>0 \\ -\frac{1}{a} \int_{-\infty}^{\infty} x(\tau)\, e^{-j\left(\frac{\omega}{a}\right)\tau}\, d\tau, & a<0 \end{cases}$$

If we compress $x(t)$
then $X(j\omega)$ is stretched
and vice versa

Duality

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x_1(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases} \xleftrightarrow{\mathcal{F}} X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}$$

$$x_2(t) = \frac{\sin Wt}{\pi t} \xleftrightarrow{\mathcal{F}} X_2(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

Example 4.13

$$g(t) = \frac{2}{1+t^2}$$

$$\underbrace{\text{If } X(j\omega) = \frac{2}{1+\omega^2} \quad \text{then} \quad x(t) = e^{-|t|}}$$

we want to exploit this fact in order to find $G(j\omega)$

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{1+\omega^2} e^{j\omega t}\, d\omega$$

$$2\pi\, e^{-|t|} = \int_{-\infty}^{\infty} \frac{2}{1+\omega^2} e^{j\omega t}\, d\omega$$

Exchange t with ω

$$2\pi\, e^{-|\omega|} = \int_{-\infty}^{\infty} \frac{2}{1+t^2} e^{j\omega t}\, dt$$

Replace t with $-t$

$$2\pi\, e^{-|\omega|} = \int_{-\infty}^{\infty} \underbrace{\frac{2}{1+t^2}}_{g(t)} e^{-j\omega t}\, dt$$

$$\mathcal{F}\left\{\frac{2}{1+t^2}\right\} = 2\pi\, e^{-|\omega|}$$

Duality in differentiation

$$\frac{d\,X(j\omega)}{d\omega} = \int_{-\infty}^{\infty} -jt\,x(t)\,e^{-j\omega t}\,dt$$

$$-jt\,x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{d\,X(j\omega)}{d\omega}$$

Duality in Integration

$$-\frac{1}{jt}\,x(t) + \pi\,x(0)\,\delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\omega} x(\eta)\,d\eta$$

Duality in time shifting

$$e^{j\omega_0 t}\,x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X\big(j(\omega-\omega_0)\big)$$