ECE 301 - Lecture #23

Examples on convolution property

$y(t) = x(t) * h(t)$ ← Impulse response of an LTI system

$Y(j\omega) = X(j\omega)\, H(j\omega)$

↑ Frequency response of the LTI system

Example 4.15

$h(t) = \delta(t - t_0)$ ← shifting by t_0 to the right

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t - t_0)\, e^{-j\omega t}\, dt = e^{-j\omega t_0}$$

$Y(j\omega) = X(j\omega)\, e^{-j\omega t_0}$ $\quad\quad$ $y(t) = x(t - t_0)$

Example 4-16

$$y(t) = \frac{dx(t)}{dt}$$

$$Y(j\omega) = j\omega \, X(j\omega)$$

$$H(j\omega) = j\omega$$ ← Frequency Response of Differentiator

Example 4-17

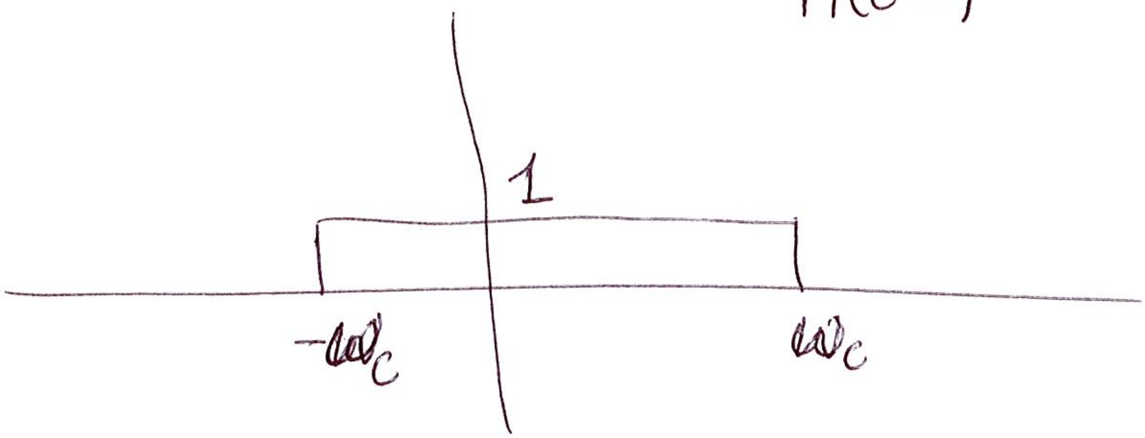
$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

$$H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

$$Y(j\omega) = X(j\omega)\, H(j\omega)$$

$$= \frac{X(j\omega)}{j\omega} + \pi \overset{X(0)}{\cancel{X(j\omega)}}\, \delta(\omega)$$

Example 4-18



$$h(t) = \frac{\sin \omega_c t}{\pi t}$$

$$Y(j\omega) = X(j\omega) H(j\omega)$$

$$= \begin{cases} X(j\omega) , & -\omega_c \leq \omega \leq \omega_c \\ 0 , & |\omega| > \omega_c \end{cases}$$

Low Pass filter (LPF)

LPF
in time domain

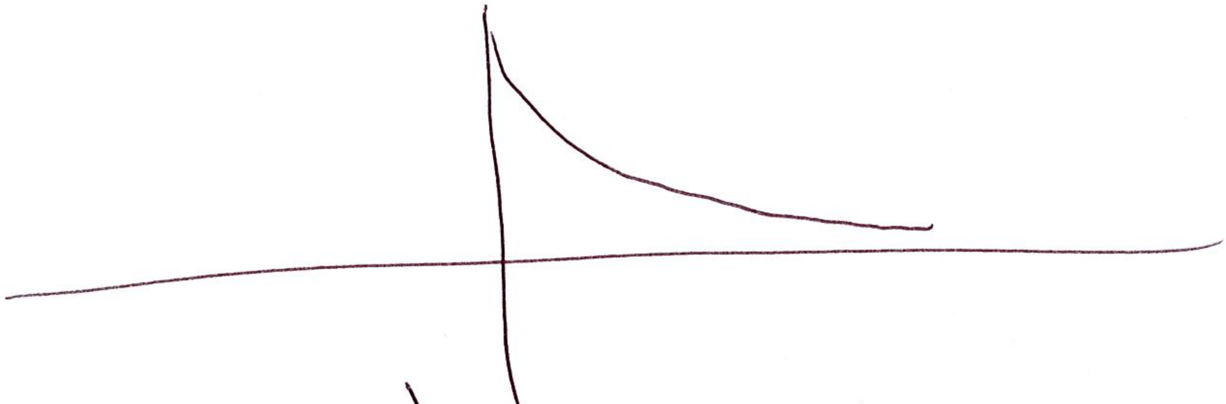
Practical Issues

- Non-Causal, because $h(t) \neq 0$ for $t < 0$

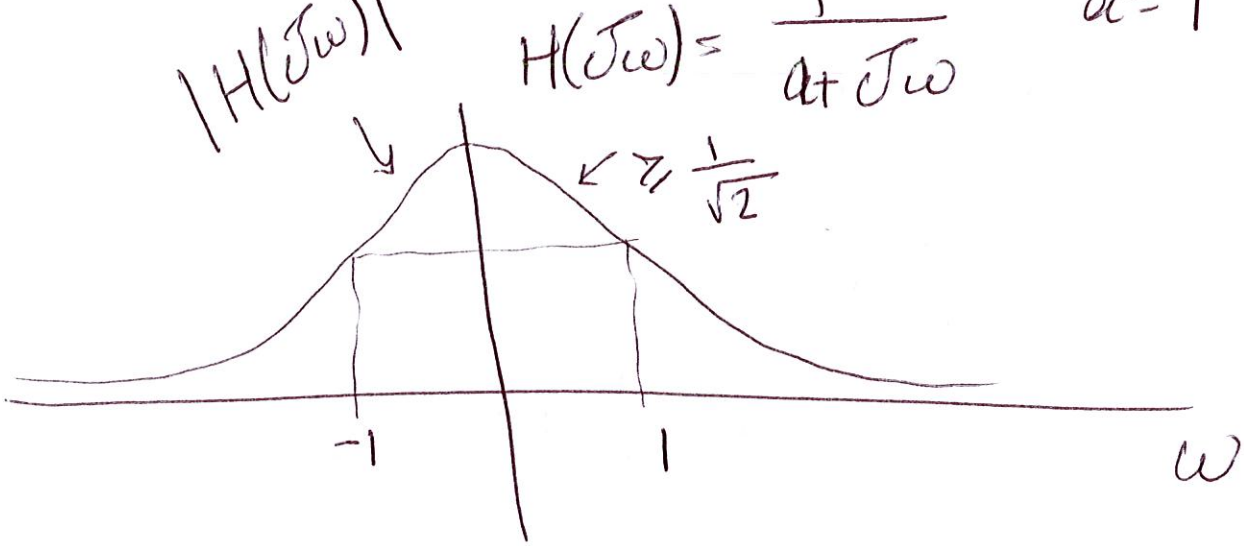
- Oscillatory behavior

Approximation to LPF

$$h(t) = e^{-at}\, u(t)\ , \quad a > 0$$



$$H(j\omega) = \frac{1}{a + j\omega} \qquad a = 1$$



Example 4-19

$$h(t) = e^{-at}\, u(t)\ , \quad a > 0$$

$$x(t) = e^{-bt}\, u(t)\ , \quad b > 0$$

$$H(j\omega) = \frac{1}{a + j\omega}$$

$$X(j\omega) = \frac{1}{b + j\omega}$$

$$Y(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)}$$

Partial fraction expansion

$$Y(j\omega) = \frac{A}{a + j\omega} + \frac{B}{b + j\omega}$$

$$1 = A(b + j\omega) + B(a + j\omega)$$

$$A\, j\omega + B\, j\omega = 0$$

$$\Rightarrow A = -B$$

$$1 = Ab + Ba$$

$$1 = A(b-a)$$

$$\Rightarrow A = \frac{1}{b-a} \qquad B = -\frac{1}{b-a}$$

$$Y(j\omega) = \frac{1}{b-a}\left[\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right]$$

$$y(t) = \frac{1}{b-a}\left[e^{-at} - e^{-bt}\right]u(t)$$

if $b \neq a$

If $b = a$

$$Y(j\omega) = \frac{1}{(a+j\omega)^2} = \frac{1}{-j}\left[\boxed{\frac{d}{d\omega}(a+j\omega)^{-1}}\right]$$

$$-1\,(a+j\omega)^{-2}\cdot j$$

$$= j\,\frac{d}{d\omega}\left[\frac{1}{a+j\omega}\right] \qquad y(t) = t\,e^{-at}\,u(t)$$

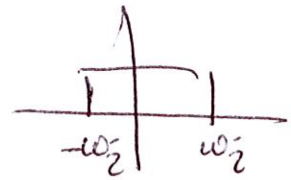
$$\boxed{\begin{aligned} x(t) &\overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \\ -jt\,x(t) &\overset{\mathcal{F}}{\longleftrightarrow} \frac{d}{d\omega}X(j\omega) \end{aligned}}$$

$$x(t) = e^{-at}\,u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega}$$

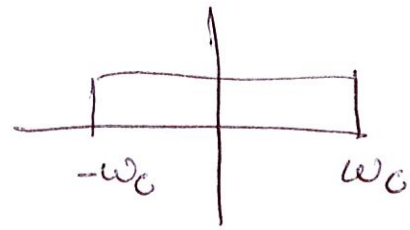
$$\underbrace{j\cdot -j}_{1}\,t\,e^{-at}\,u(t) \overset{\mathcal{F}}{\longleftrightarrow} j\,\frac{d}{d\omega}\left[\frac{1}{a+j\omega}\right]$$

Example 4.20

$$x(t) = \frac{\sin \omega_i t}{\pi t}$$



$$h(t) = \frac{\sin \omega_c t}{\pi t}$$



$$x(t) * h(t) = \begin{cases} \dfrac{\sin \omega_i t}{\pi t} & \text{if } |\omega_i| \leq |\omega_c| \\ \dfrac{\sin \omega_c t}{\pi t} & \text{otherwise} \end{cases}$$

The Multiplication Property

$$r(t) = s(t)\,p(t) \xleftrightarrow{\mathcal{F}} R(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(j\theta)\, P(j(\omega - \theta))\, d\theta$$

$p(t) = \cos \omega_0 t$

$P(j\omega)$

$S(j\omega)$