ECE 301 - Lecture #26

Example from last lecture

$$\frac{dy(t)}{dt} + y(t) = x(t)$$

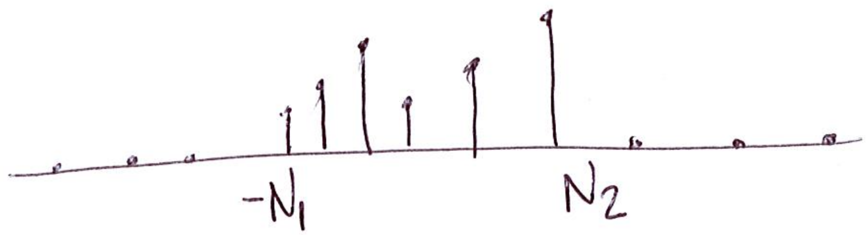
$$H(j\omega) = \frac{1}{1 + j\omega}$$

$$h(t) = e^{-t}u(t)$$

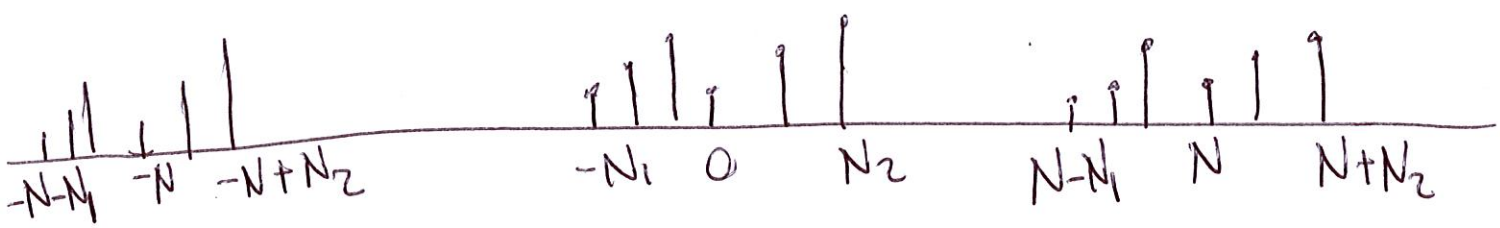
Approximation to Low Pass filter

Discrete-Time Fourier Transform

$x[n]$



$\tilde{x}_N[n]$



$$\lim_{N \to \infty} \tilde{x}_N[n] = x[n]$$

$$\tilde{x}_N[n] = \sum_{k=\langle N \rangle} a_k \, e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}_N[n] \, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

(3)

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}_N[n] \, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$= \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] \, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$\boxed{X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] \, e^{-j\omega n}}$$

$$a_k = \frac{1}{N} X\left(e^{jk\omega_0}\right)$$

$$\tilde{x}_N[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n}$$

$$= \frac{1}{2\pi} \sum_{k=\langle N \rangle} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n} \omega_0$$

$$\tilde{x}_N[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \, \omega_0 \qquad (4)$$

$$x[n] = \lim_{N \to \infty} \tilde{x}_N[n]$$

$$= \lim_{\omega_0 \to 0} \tilde{x}_N[n]$$

$$= \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

$$\boxed{\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \\ x[n] &= \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega \end{aligned}}$$

Example 5.1

$$x[n] = a^n u[n] \quad , \quad |a| < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

$$= \frac{1}{1 - a e^{-j\omega}}$$

$$\boxed{\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \qquad |r| < 1}$$

Example 5.2

$$x[n] = a^{|n|}, \quad |a| < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^{n} e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^{n} e^{-j\omega n} + \sum_{m=1}^{\infty} a^{m} e^{j\omega m}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^{n} + \sum_{m=1}^{\infty} (a e^{j\omega})^{m}$$

$$= \frac{1}{1 - a e^{-j\omega}} + \frac{a e^{j\omega}}{1 - a e^{j\omega}}$$

$$= \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$

Example 5.3

$$x[n] = \begin{cases} 1 & , |n| \le N_1 \\ 0 & , |n| > N_1 \end{cases}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=-N_1}^{N_1} e^{-j\omega n}$$

$$= \sum_{n=-N_1}^{\infty} e^{-j\omega n} - \sum_{n=N_1+1}^{\infty} e^{-j\omega n}$$

$$= e^{j\omega N_1} \sum_{n=0}^{\infty} e^{-j\omega n} - e^{-j\omega(N_1+1)} \sum_{n=0}^{\infty} e^{-j\omega n}$$

$$X(e^{j\omega}) = \frac{e^{j\omega N_1} - e^{-j\omega(N_1+1)}}{1 - e^{-j\omega}}$$

$$= \frac{e^{j\omega N_1} - e^{-j\omega(N_1+1)}}{1 - e^{-j\omega}} \quad \frac{e^{+j\omega \frac{1}{2}}}{e^{j\omega \frac{1}{2}}}$$

$$= \frac{e^{j\omega(N_1+\frac{1}{2})} - e^{-j\omega(N_1+\frac{1}{2})}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}$$

$$= \frac{\sin \omega(N_1+\frac{1}{2})}{\sin \frac{\omega}{2}}$$

Example 5.4

$x[n] = \delta[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = 1$$

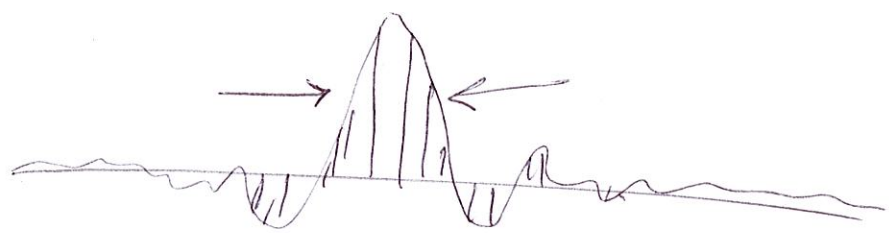
* Assume we pass $X(e^{j\omega})$ by an ideal LPF with cutoff frequency $\omega_c < \pi$

$$\hat{x}[n] = \frac{1}{2\pi} \int X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

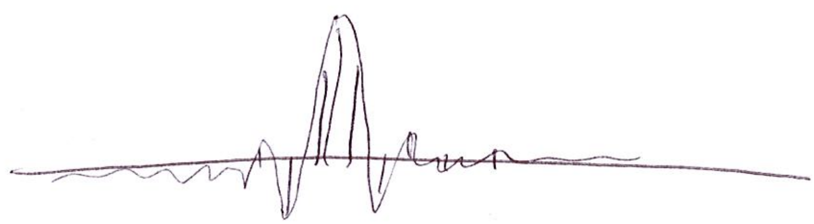
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi\ 2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega$$

$$= \frac{\sin \omega_c n}{\pi n}$$



As $\omega_c \uparrow$



becomes an impulse when $\omega_c = \pi$

Convergence Issues

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Only condition for existence of Fourier Transform

Next time: FT of DT Periodic Signals
