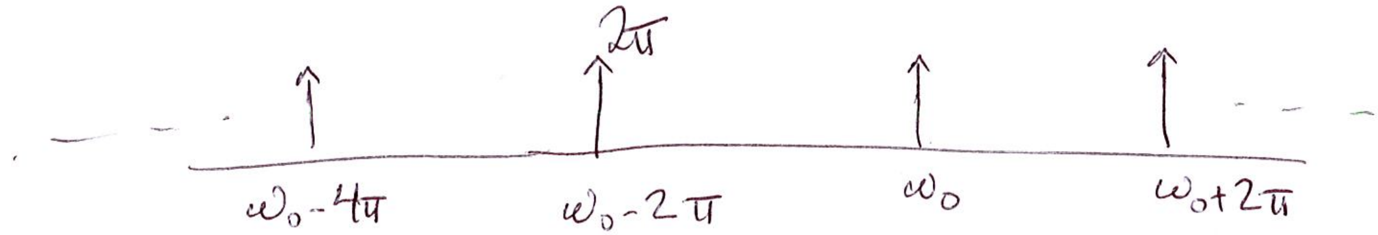
# ECE 301- Lecture # 27

## Fourier Transform for Periodic Signals
### (Discrete - Time)

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 - 2\pi l)$$




$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

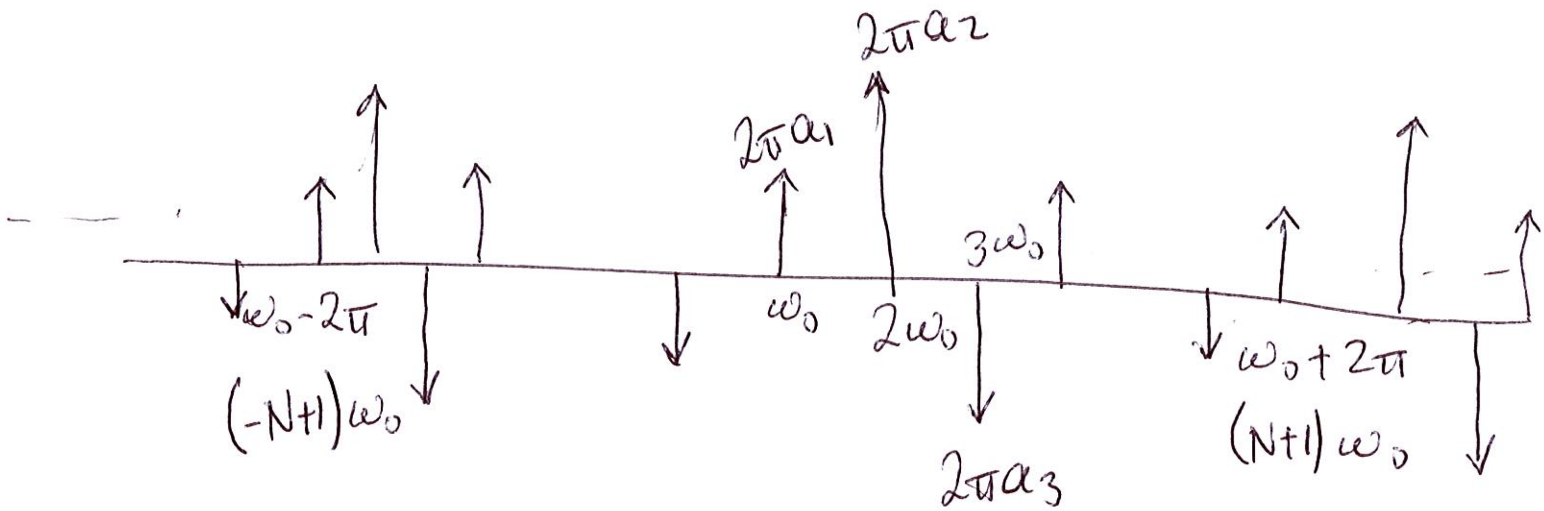
$$= \frac{1}{2\pi} \int_{2\pi} \left( \sum_{l=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 - 2\pi l) \right) e^{j\omega n}\, d\omega$$

$$= e^{j(\omega_0 + 2\pi \tilde{r})n} = e^{j\omega_0 n}\, \underbrace{\cancel{e^{j2\pi \tilde{r} n}}}_{1}$$

$$= e^{j\omega_0 n}$$

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} a_k 2\pi \, \delta\left(\omega - \frac{2\pi k}{N}\right)$$



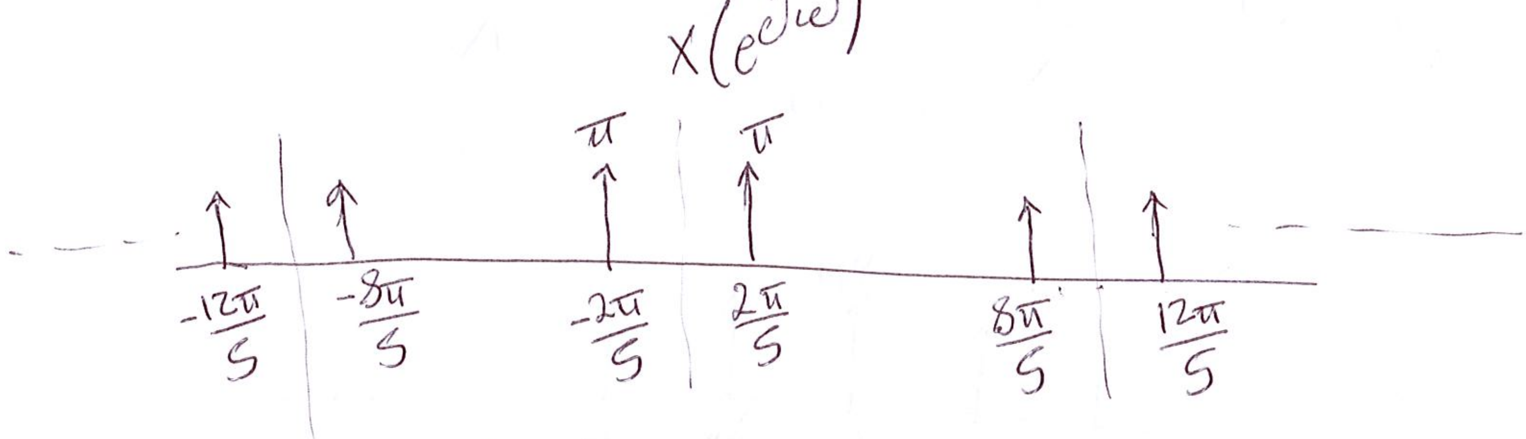

---

Example 5.5

$$x[n] = \cos \omega_0 n = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$

$$\omega_0 = \frac{2\pi}{5} \qquad N = 5$$

$a_1 = \frac{1}{2} \quad a_{-1} = \frac{1}{2}$




$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \pi\, \delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right)$$

$$+ \sum_{l=-\infty}^{\infty} \pi\, \delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right)$$

---

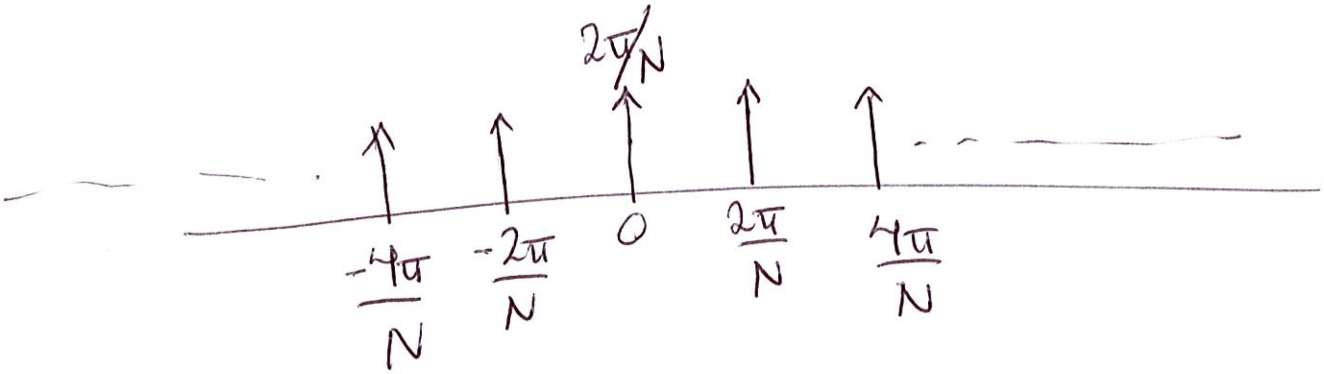
## Example 5.6

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$$


$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n]\, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$= \frac{1}{N} \text{ for every } k$$

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$




## Properties of DTFT

### Differencing

$$x[n] - x[n-1] \xleftrightarrow{\mathcal{F}} \left(1 - e^{-j\omega}\right) X(e^{j\omega})$$

### Accumulation

$$y[n] = \sum_{m=-\infty}^{n} x[m]$$

$$\xleftrightarrow{\mathcal{F}} \frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

## Example 5.8

$$g[n] = \delta[n] \xleftrightarrow{\mathcal{F}} G(e^{j\omega}) = 1$$

$$x[n] = \sum_{m=-\infty}^{n} g[m] = u[n]$$

$$X(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} G(e^{j\omega}) + \pi G(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$= \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

---

## Time Expansion

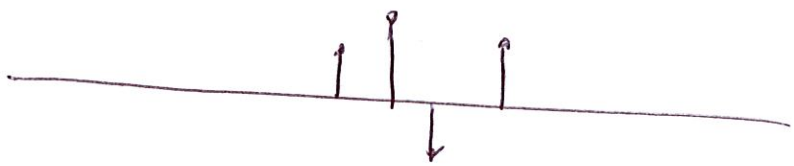
CT

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

(6)

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if } n \text{ is a multiple of } k \\ 0 & \text{otherwise} \end{cases}$$

$x[n]$



$x_{(3)}[n]$

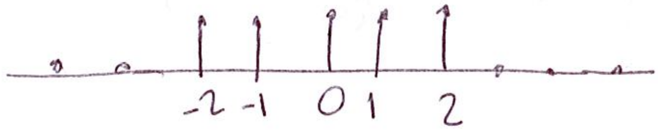
$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_{(k)}[n]\, e^{-j\omega n}$$

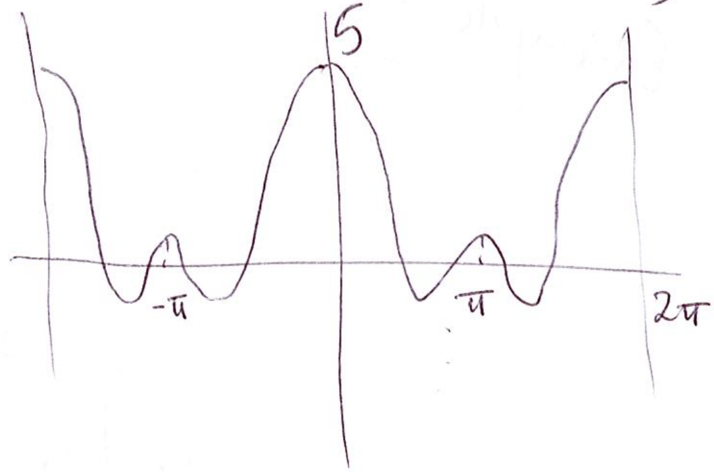
$$= \sum_{r=-\infty}^{\infty} x_{(k)}[rk]\, e^{-j\omega rk}$$

$$= \sum_{r=-\infty}^{\infty} x[r]\, e^{-j\omega k r} = X(e^{jk\omega})$$

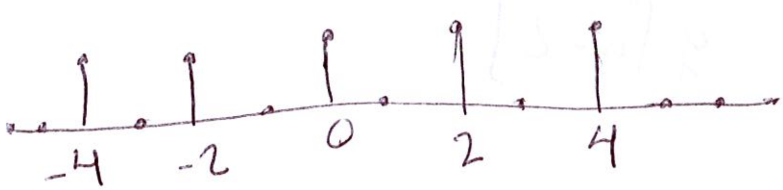
# Example

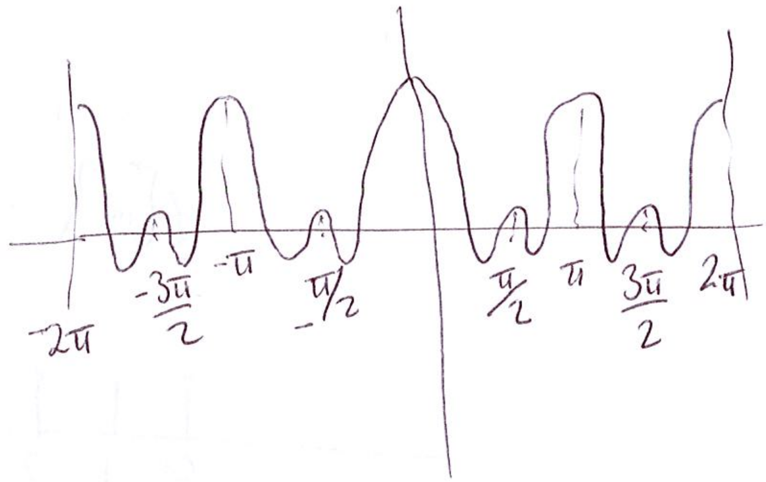
$x[n]$



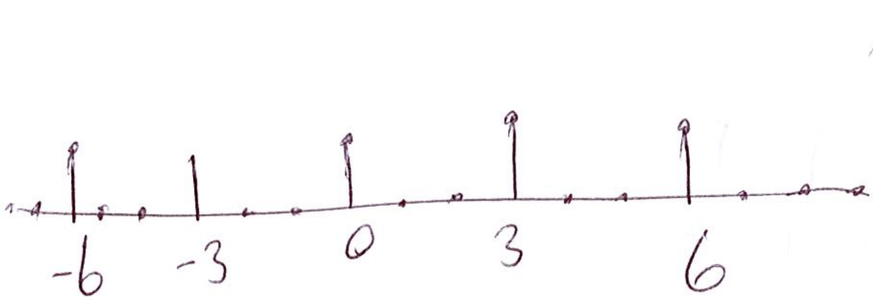



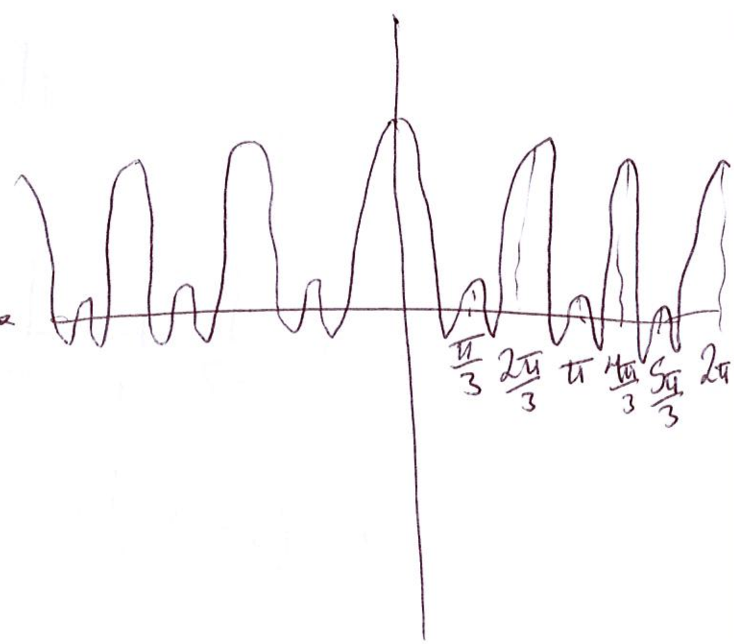

$x_{(2)}[n]$



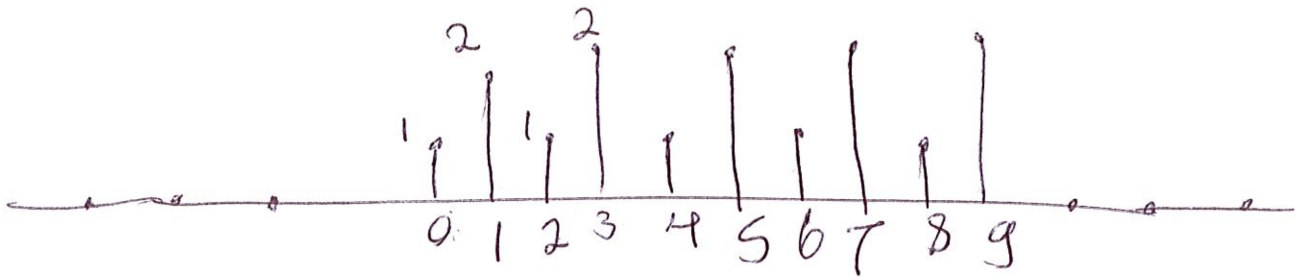

$x_{(3)}[n]$



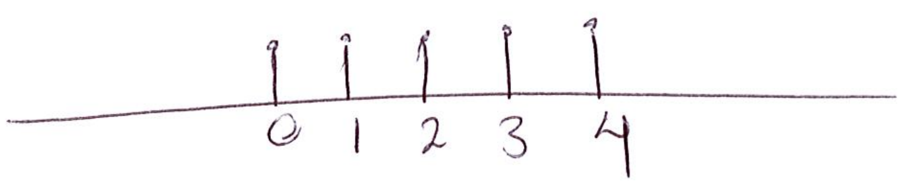



# Example 5.9

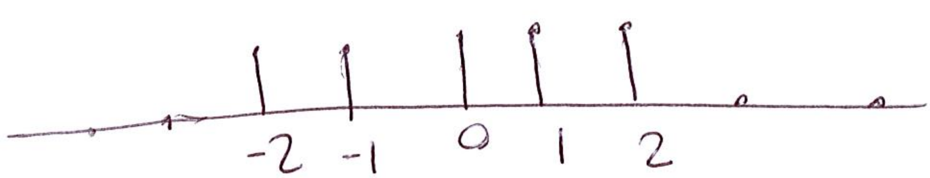
$$x[n] = y_{(2)}[n] + 2\, y_{(2)}[n-1]$$




$$y[n] = g[n-2]$$




$$g[n]$$




$$G(e^{j\omega}) = \frac{\sin\left(\frac{5\omega}{2}\right)}{\sin(\omega/2)}$$

(9)

$$Y(e^{j\omega}) = e^{-j2\omega} \frac{\sin\left(\frac{5\omega}{2}\right)}{\sin(\omega/2)}$$

$$Y_{(2)}(e^{j\omega}) = Y(e^{j2\omega}) = e^{-j4\omega} \frac{\sin(5\omega)}{\sin(\omega)}$$

$$X(e^{j\omega}) = \left(1+2e^{-j\omega}\right) e^{-j4\omega} \frac{\sin(5\omega)}{\sin\omega}$$