ECE 301 - Lecture #28

DTFT Convolution Property

$$y[n] = x[n] * h[n]$$

$$Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$$

Interpretation for LTI Systems

Example 5.11

$$h[n] = \delta[n-n_0]$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n-n_0] e^{-j\omega n}$$

$$= e^{-j\omega n_0}$$

$$Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$$

Time shifting property

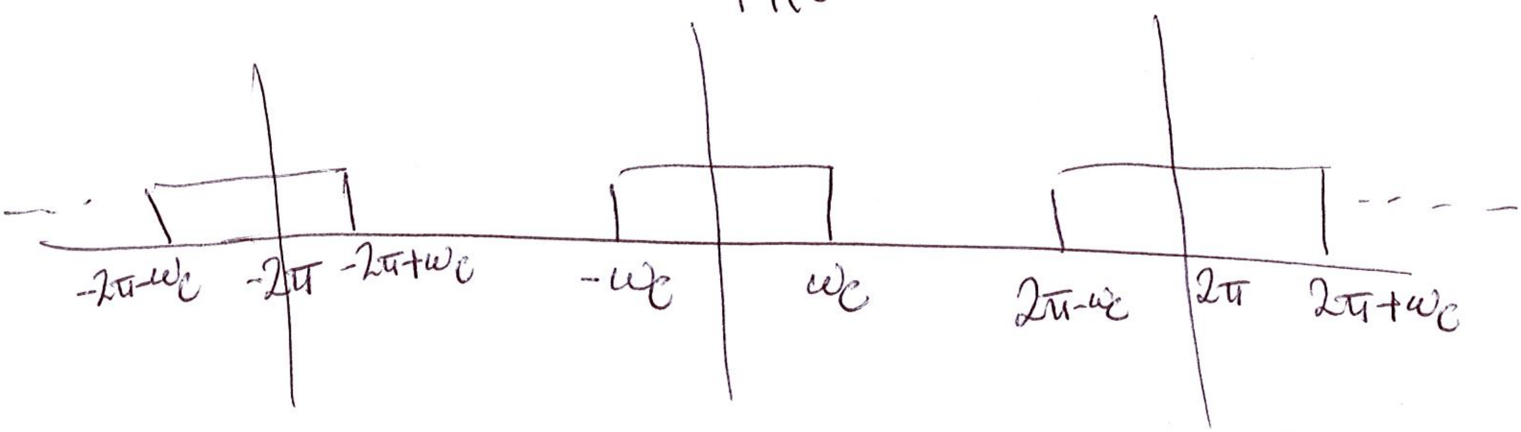
Example 5-12

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} \, d\omega$$

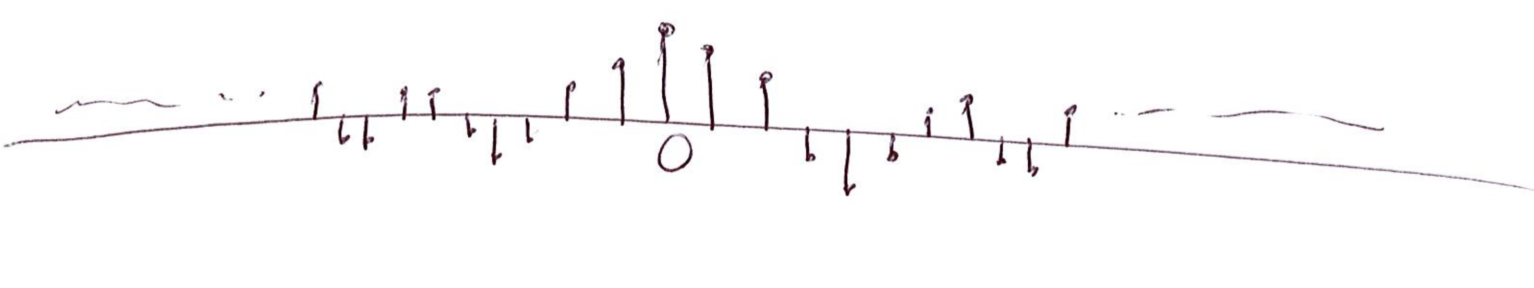
$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} \, d\omega$$

$$= \frac{\sin \omega_c n}{\pi n}$$ ← DT Sinc

$H(e^{j\omega})$



$h[n]$ Non Causal
oscillatory behavior



Example 5.13

LPF Approximation

$$h[n] = \alpha^n u[n], \quad |\alpha| < 1$$

$$x[n] = \beta^n u[n], \quad |\beta| < 1$$

$$H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{1}{1-\beta e^{-j\omega}}$$

$$Y(e^{j\omega}) = X(e^{j\omega})\,H(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}$$

If $\alpha \neq \beta$

Partial Fraction Expansion

$$Y(e^{j\omega}) = \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}$$

$$A = \frac{\alpha}{\alpha-\beta} \qquad B = \frac{-\beta}{\alpha-\beta}$$

$$y[n] = \frac{\alpha}{\alpha-\beta}\,\alpha^n u[n] - \frac{\beta}{\alpha-\beta}\,\beta^n u[n] \quad (4)$$

$$= \frac{1}{\alpha-\beta}\left[\alpha^{n+1} u[n] - \beta^{n+1} u[n]\right]$$

If $\alpha = \beta$

$$Y(e^{j\omega}) = \left(\frac{1}{1-\alpha e^{-j\omega}}\right)^2 \qquad \boxed{y[n] = (n+1)\alpha^n u[n]}$$

$$= \frac{j}{\alpha}\, e^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$\alpha^n u[n] \xleftrightarrow{\mathcal{F}} \frac{1}{1-\alpha e^{-j\omega}}$$

$$n\,\alpha^n u[n] \xleftrightarrow{\mathcal{F}} j\,\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$(n+1)\,\alpha^{n+1} u[n+1] \xleftrightarrow{\mathcal{F}} j\,e^{j\omega}\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$(n+1)\,\alpha^{n} u[n+1] \xleftrightarrow{\mathcal{F}} \frac{j}{\alpha}\,e^{j\omega}\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

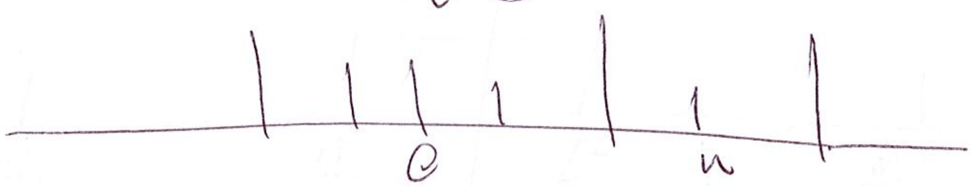
Why is it an LPF?

$$h[n] = \frac{1}{2}^n u[n]$$

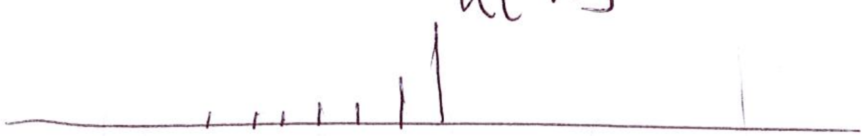
Any $x[n]$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

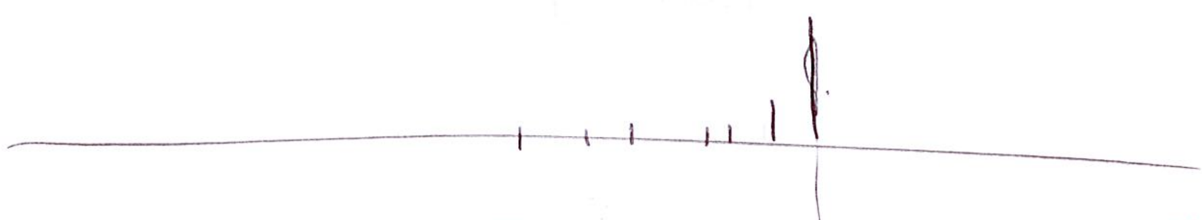
$x[k]$



$h[-k]$



$h[n-k]$

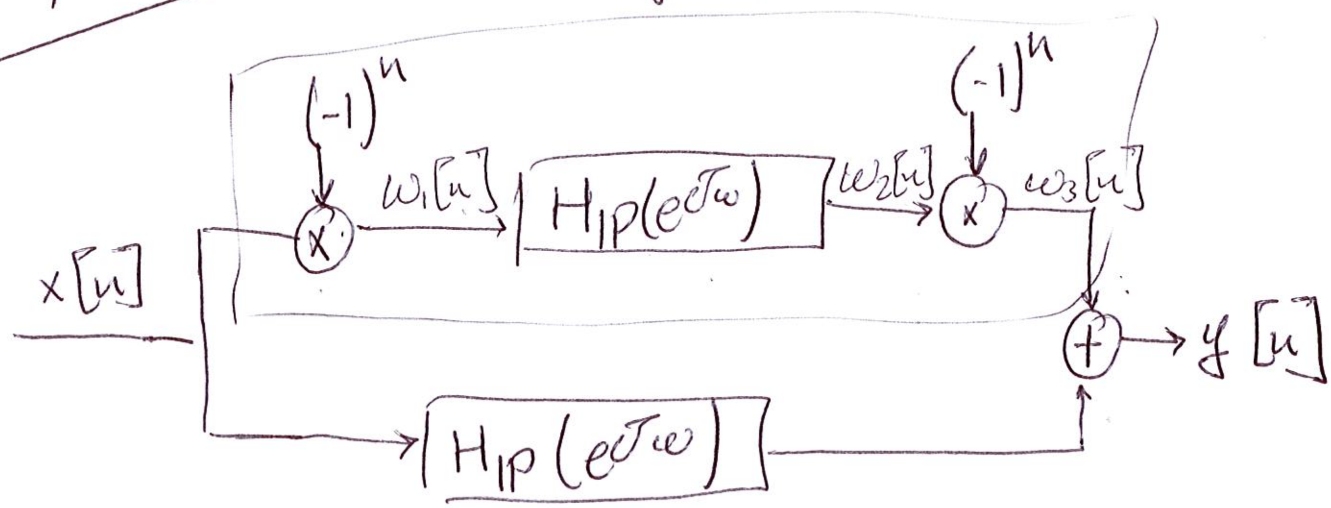


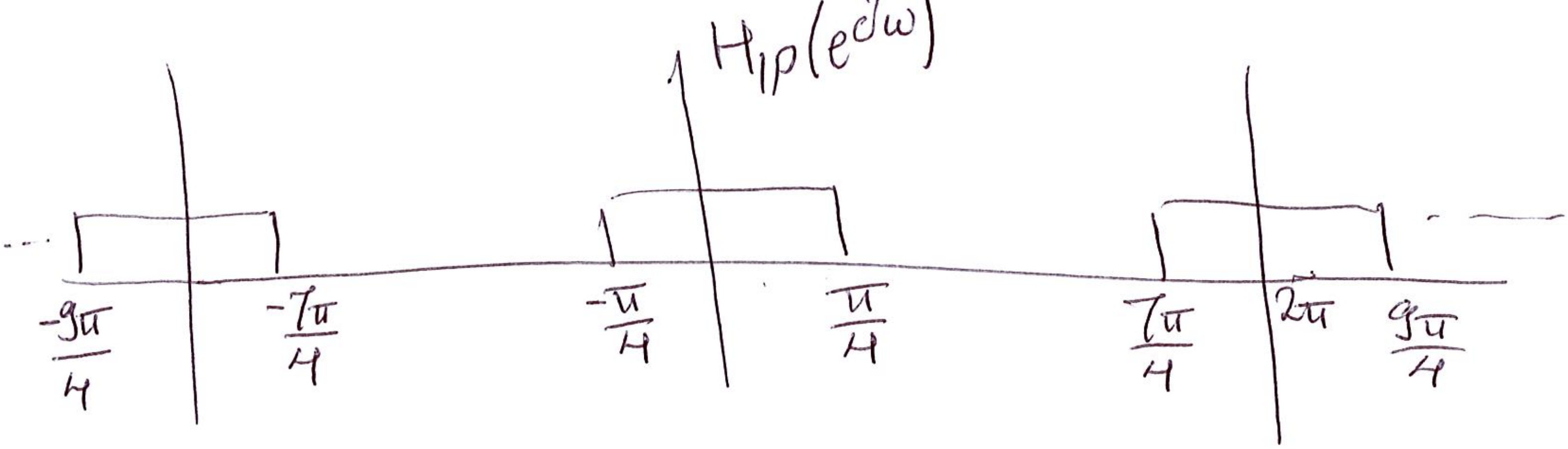
$$y[n] = x[n] + \frac{1}{2} x[n-1] + \frac{1}{4} x[n-2] + \frac{1}{8} x[n-3] + \frac{1}{16} x[n-4] + \cdots$$

Example 5.14

High Pass filter

(6)





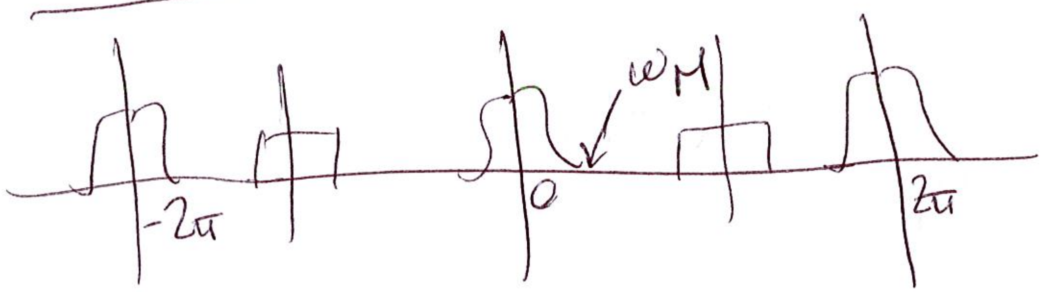
what is the functionality of this system?

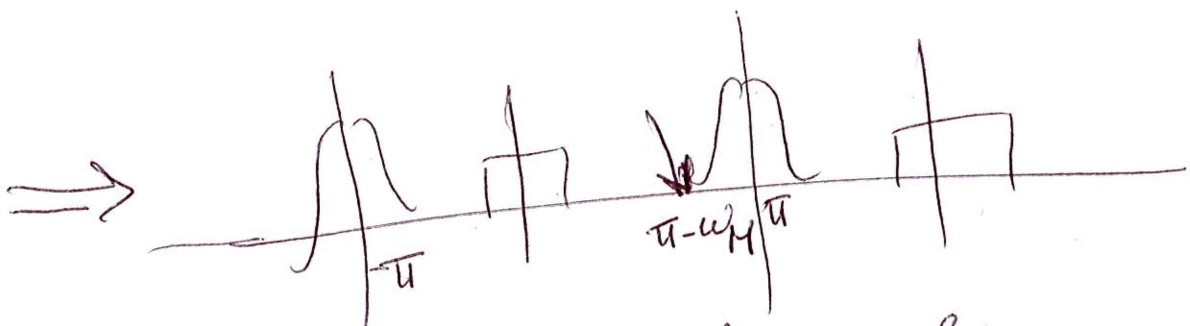
$$(-1)^n = e^{j\pi n}$$

$$w_1[n] = x[n]\, e^{j\pi n}$$

$$W_1(e^{j\omega}) = X\left(e^{j(\omega-\pi)}\right)$$

what's the meaning of this?





Low frequency components shift to high frequencies and vice versa

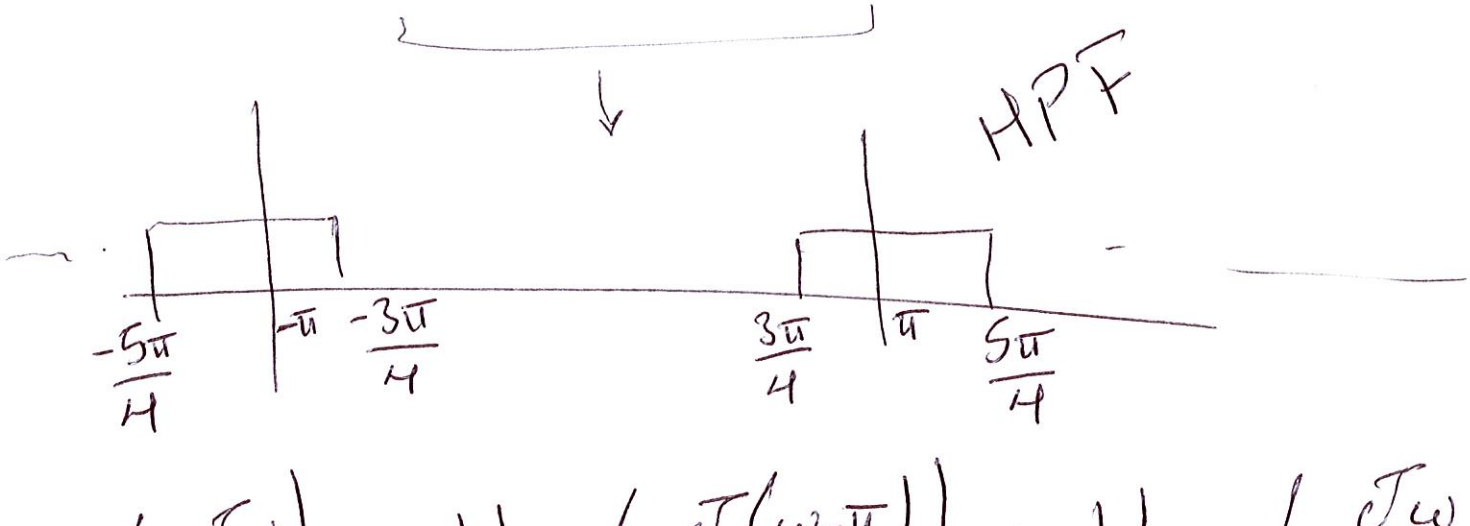
$$W_2(e^{j\omega}) = H_{lp}(e^{j\omega})\, X(e^{j(\omega-\pi)})$$

$$w_3[n] = e^{j\pi n}\, w_2[n]$$

$$W_3(e^{j\omega}) = W_2(e^{j(\omega-\pi)})$$

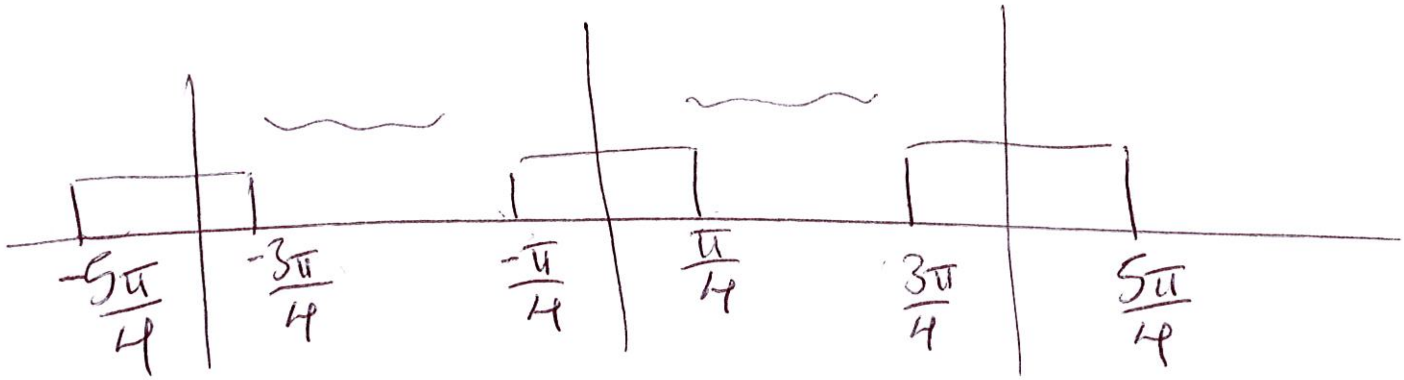
$$= H_{lp}(e^{j(\omega-\pi)})\, X(e^{j(\omega-2\pi)})$$

$$= \underbrace{H_{lp}(e^{j(\omega-\pi)})}\, X(e^{j\omega})$$



$$H(e^{j\omega}) = \underbrace{H_{lp}(e^{j(\omega-\pi)})}_{\text{HPF}} + \underbrace{H_{lp}(e^{j\omega})}_{\text{LPF}}$$

$H(e^{j\omega})$



Ideal bandstop filter

Multiplication Property

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x_1[n]\, x_2[n]\, e^{-j\omega n}$$

$$x_1[n] = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})\, e^{j\theta n}\, d\theta$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_2[n] \left\{ \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})\, e^{j\theta n}\, d\theta \right\} e^{-j\omega n}$$

$$Y(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) \left[\sum_{n=-\infty}^{\infty} x_2[n] e^{-j(\omega-\theta)n} \right] d\theta \qquad (10)$$

$$= \frac{1}{2\pi} \underbrace{\int_{2\pi} X_1(e^{j\theta}) X_2\left(e^{j(\omega-\theta)}\right) d\theta}_{\text{Periodic Convolution}}$$

Example 5.15

$$x[n] = x_1[n]\, x_2[n]$$

$$x_1[n] = \frac{\sin(3\pi n/4)}{\pi n}$$

$$x_2[n] = \frac{\sin(\pi n/2)}{\pi n}$$

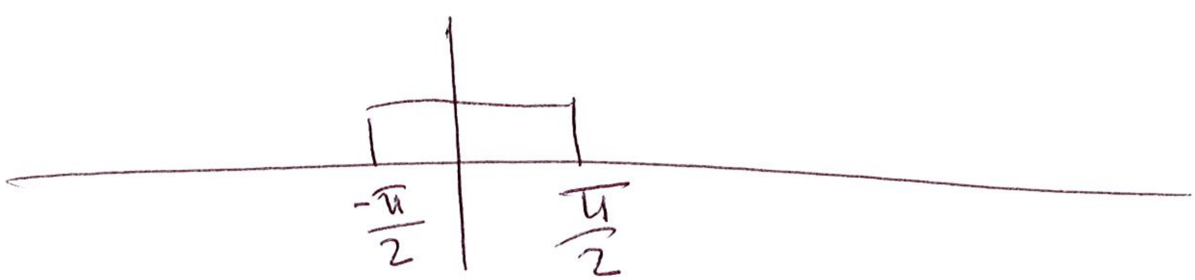
$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})\, X_2\left(e^{j(\omega-\theta)}\right) d\theta$$

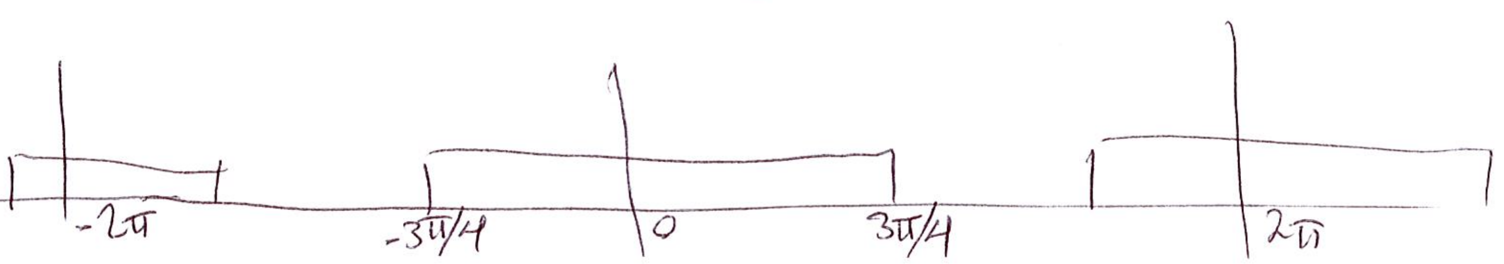
$$\hat{X}_1(e^{j\omega}) = \begin{cases} X_1(e^{j\omega}), & -\pi \le \omega \le \pi \\ 0 & \text{otherwise} \end{cases}$$

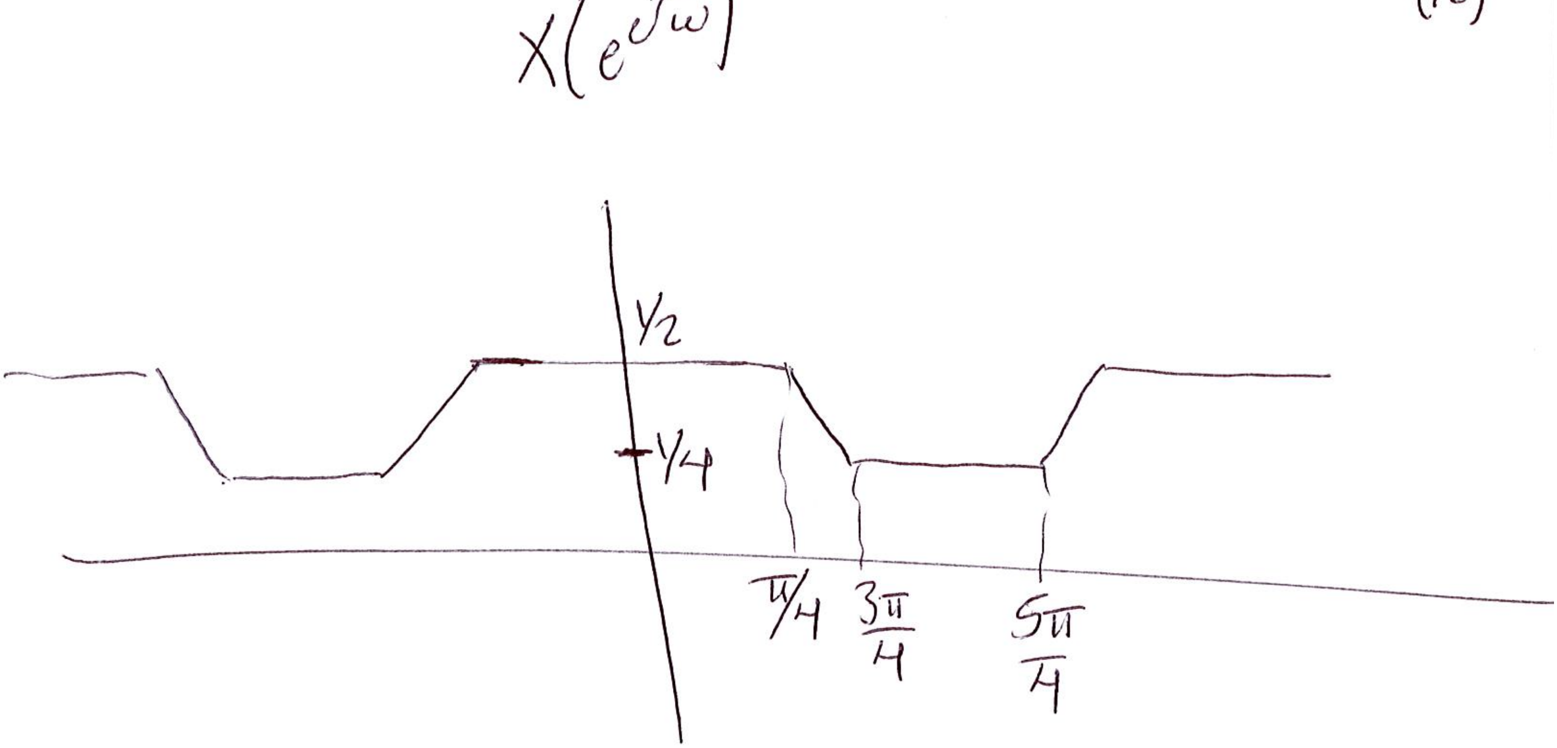
$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{X}_1(e^{j\theta})\, X_2\left(e^{j(\omega-\theta)}\right) d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta})\, X_2\left(e^{j(\omega-\theta)}\right) d\theta$$

$\hat{X}_1(e^{j\omega})$







X(e^{jω})
1/2
1/4
π/4
3π/4
5π/4