ECE 301 - Lecture #30

Duality between Time and Frequency domains

A signal that is discrete in one domain is periodic in the other domain

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DTFS

has period N

$$x[n] \xrightarrow{FS} a_k$$

$$f[n] = a_n \xrightarrow{FS} \frac{1}{N} x[-k]$$

Example

$$x[n] = \begin{cases} \frac{1}{9} \frac{\sin(\pi n/9)}{\sin(\pi n/9)} & , n \text{ is not multiple of } 9 \\ 5/9 & , n \text{ is a multiple of } 9 \end{cases}$$

Example 3-12

Let $g[n] = \begin{cases} 1 & , |n| \leq 2 \\ 0 & , \cancel{2} \; 2 < |n| \leq 4 \end{cases}$

$\Big\downarrow \mathcal{FS}$

$$b_k = \begin{cases} \dfrac{1}{9} \dfrac{\sin(5\pi k/9)}{\sin(\pi k/9)} & , \text{k is not a multiple of 9} \\ 5/9 & , \text{k is a multiple of 9} \end{cases}$$

$$b_k = \frac{1}{9} \sum_{n=-2}^{2} e^{-j2\pi nk/9}$$

$b_n = x[n]$

$$x[n] = \frac{1}{9} \sum_{k=-2}^{2} e^{-j2\pi nk/9}$$

Let $k' = -k$

$$x[n] = \frac{1}{9} \sum_{k'=-2}^{2} e^{j2\pi n k'/9}$$

$$= \sum_{k'=-2}^{2} \frac{1}{9} e^{j2\pi n k'/9}$$

$$= \sum_{k'=-4}^{4} a_k e^{j n k' \frac{2\pi}{9}}$$

$$\Rightarrow \quad a_k = \begin{cases} \frac{1}{9}, & |k| \leq 2 \\ 0, & 2 < |k| \leq 4 \end{cases}$$

Duality between DTFT and CTFS

$$\text{DTFT}\begin{cases} x[n] = \dfrac{1}{2\pi}\displaystyle\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega & (1) \\[2ex] X(e^{j\omega}) = \displaystyle\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} & (2) \end{cases}$$

$$\text{CTFS}\begin{cases} x(t) = \displaystyle\sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega_0 t} \qquad \frac{2\pi}{T} & (3) \\[2ex] a_k = \dfrac{1}{T}\displaystyle\int_{T} x(t)\, e^{-jk\omega_0 t}\, dt & (4) \end{cases}$$

If we ~~stat~~ start with (1) and (2) and then think of $X(e^{j\omega})$ as a CT periodic signal, what would be its FS coeffs. ? $x[-k]$

Duality between (1) and (4), i.e., (5)

think of $x[n]$ as FS coeffs. for a CT periodic signal. $\boxed{\text{only when } T = 2\pi}$

Example 5.17

$$x[n] = \frac{\sin(\pi n/2)}{\pi n}$$

$$\frac{\text{DTFT?}}{X(e^{j\omega})}$$

$$g(t) = \begin{cases} 1, & |t| \leq T_1 \\ 0, & T_1 \leq |t| \leq \pi \end{cases}$$

$$a_k = \frac{\sin(kT_1)}{k\pi} \qquad T_1 = \frac{\pi}{2}$$

$$\frac{\sin(\pi k/2)}{\pi k} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-j\omega k}\, d\omega$$

- Switch k and n
- Replacing n by -n

$$\frac{\sin(\pi \cancel{k} n/2)}{\pi \cancel{k} n} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{j\omega n}\, d\omega$$

$$\Longrightarrow \quad X(e^{j\omega}) = \begin{cases} 1 , & |\omega| \leq \frac{\pi}{2} \\ 0 , & \frac{\pi}{2} \leq |\omega| \leq \pi \end{cases}$$

Systems Characterized by Linear Constant Coeff. Difference Equations

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$\sum_{k=0}^{N} a_k\, e^{-jk\omega}\, Y(e^{j\omega}) = \sum_{k=0}^{M} b_k\, e^{-jk\omega}\, X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} b_k\, e^{-jk\omega}}{\sum_{k=0}^{N} a_k\, e^{-jk\omega}}$$

Example 5.18

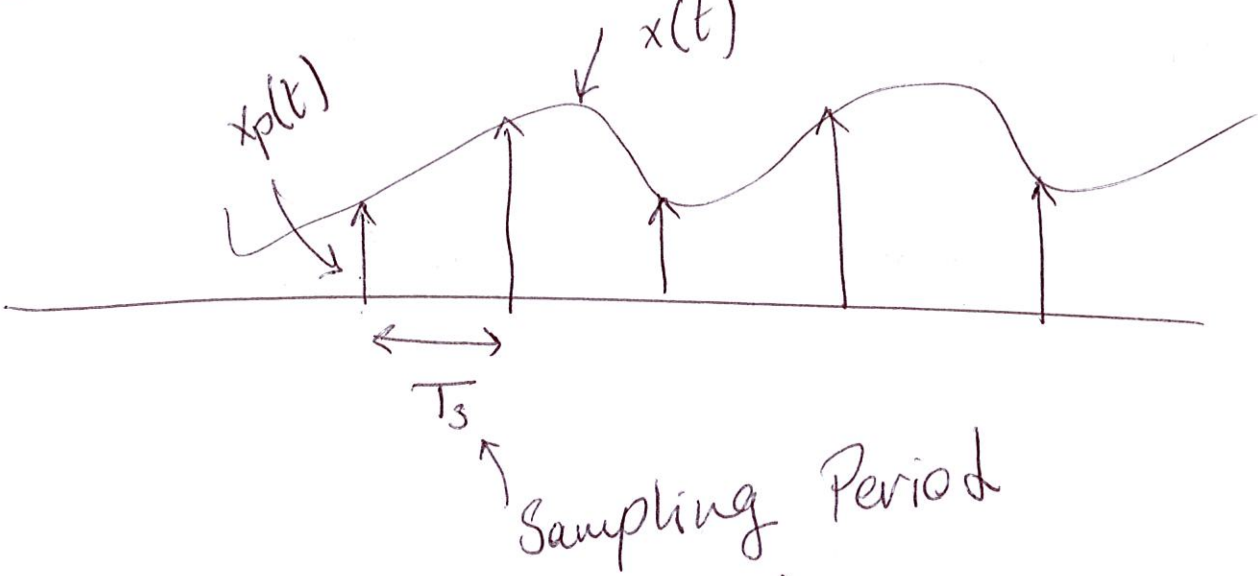
$$y[n] - a\,y[n-1] = x[n] \quad , |a| < 1$$

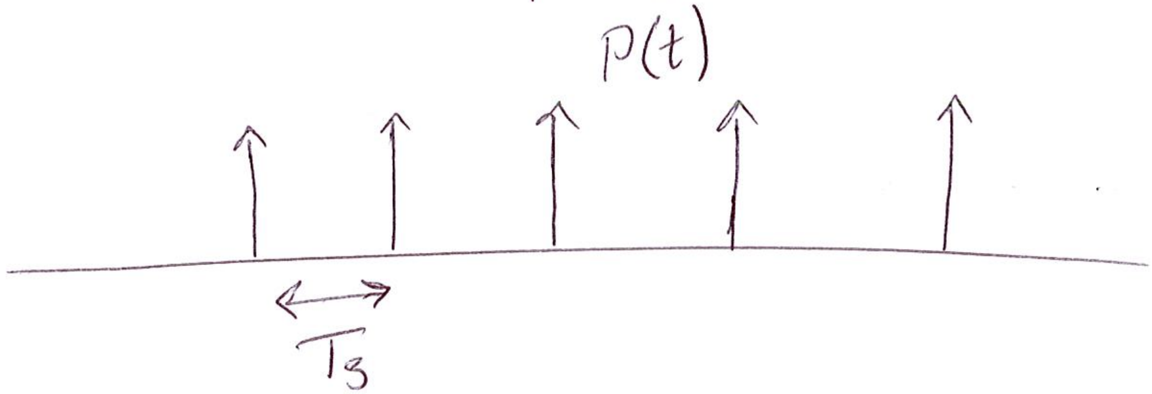
$$H(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

$$h[n] = a^n u[n]$$

LPF Approximation

Impulse Train Sampling





$$x_p(t) = x(t)\, p(t)$$

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

When is it possible to reconstruct $x(t)$ from $x_p(t)$? $\boxed{\text{Sampling Theory}}$