ECE 301 - Lecture #4

Periodicity properties of discrete-time complex exponentials

Compare $e^{j\omega_0 t}$ to $e^{j\omega_0 n}$

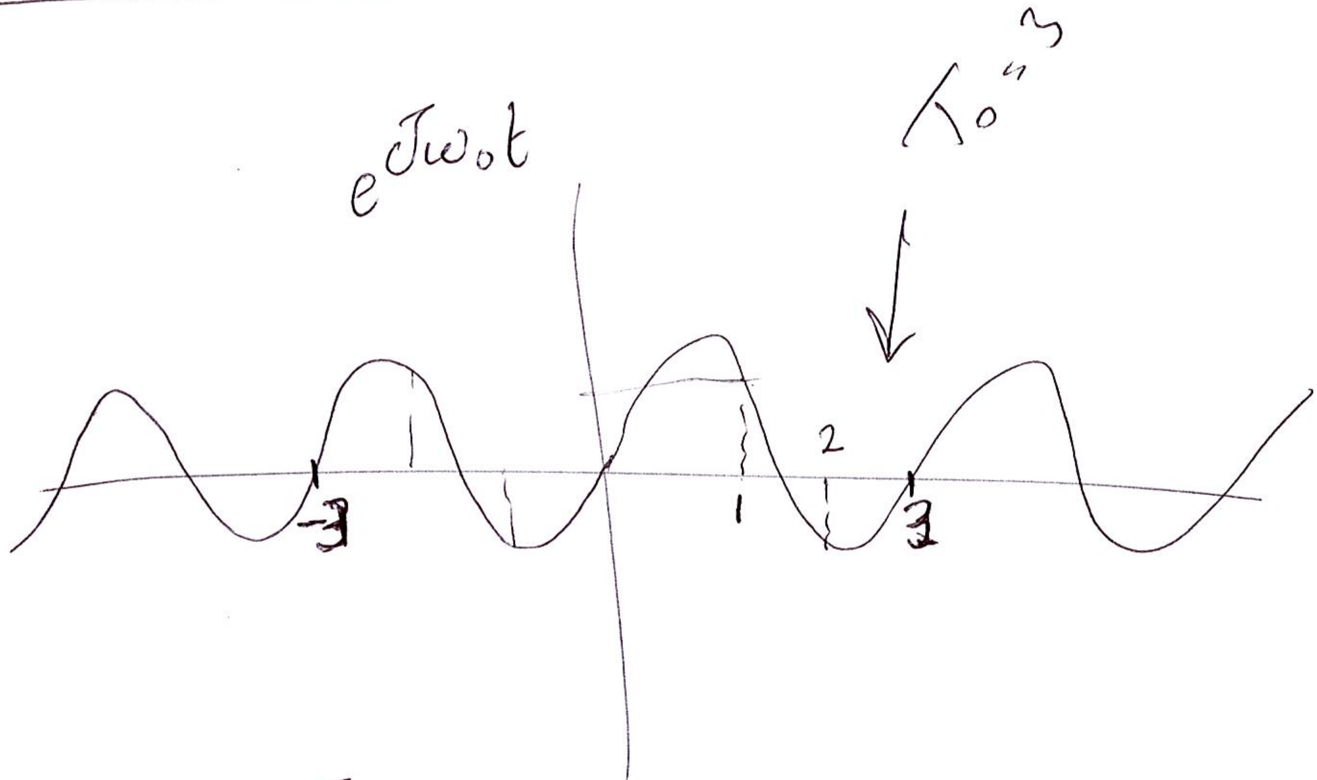
what are the possible values for ω_0, such that the complex exponential is periodic?

$e^{j\omega_0 t}$: Periodic for any ω_0
and the fundament period

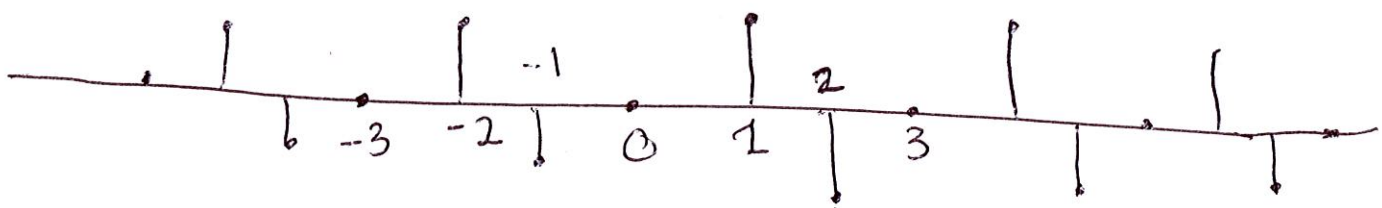
$$T_0 = \frac{2\pi}{|\omega_0|}$$

What is $e^{j\omega_0 n}$?

$e^{j\omega_0 t}$



$e^{j\omega_0 n}$



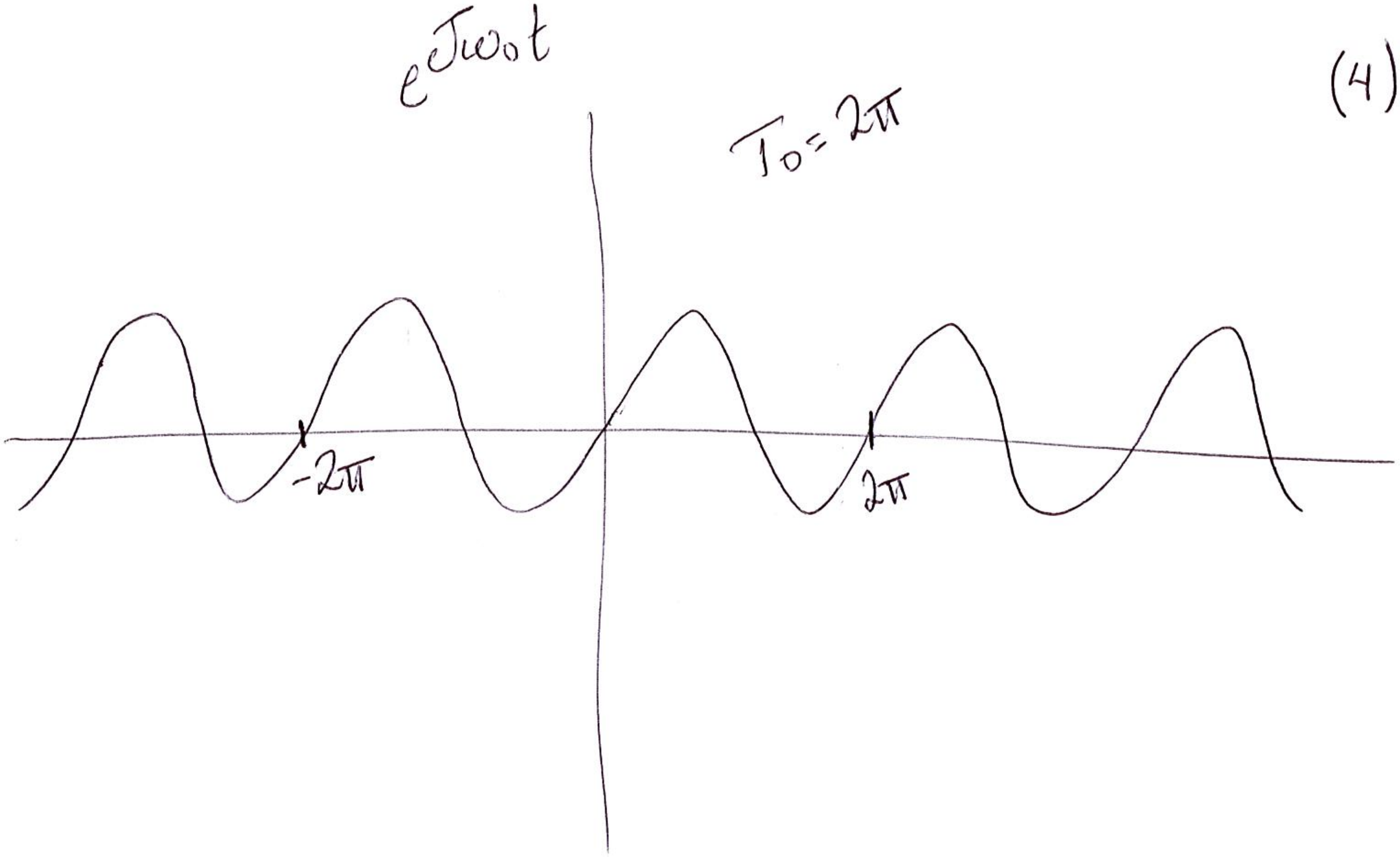
Periodic DT-signal with fundamental Period $N_0 = 3$

$e^{j\omega_0 t}$

$T_0 = 8/3$

$e^{j\omega_0 n}$

$N_0 = 8$



Is $e^{j\omega_0 n}$ periodic? No

Why? because T_0 is irrational

Let's say $e^{j\omega_0 n}$ is periodic

What ~~are~~ is the range of possible frequencies?

$$\tilde{\omega}_0 = \omega_0 + 2\pi$$

$$e^{j\tilde{\omega}_0 n} = e^{j(\omega_0 + 2\pi)n}$$

$$= e^{j\omega_0 n}\, e^{j2\pi n}$$

$$e^{j2\pi n} = \underbrace{\cos 2\pi n}_{1} + j\cancel{\sin 2\pi n}$$

$$e^{j\tilde{\omega}_0 n} = e^{j\omega_0 n}$$

All possible DT-frequencies range from 0 to 2π

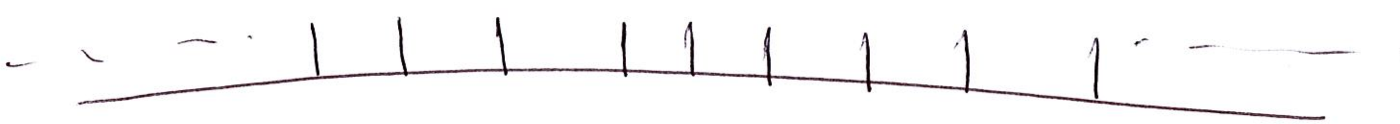
Which DT-frequencies are low

~~e~~ and which are high

$e^{j\omega_0 n}$ $\boxed{\omega_0 = 2\pi}$ ← Lowest frequency

$e^{j2\pi n} = \cos 2\pi n + j \sin 2\pi n$

Does not change



$\boxed{\omega_0 = \pi}$ ← Highest frequency

$e^{j\pi n} = \cos \pi n + j \sin \pi n$

The frequency ω_0 gets higher as we move from 0 to π and then becomes lower as we move from π to 2π

When is $e^{j\omega_0 n}$ periodic?

For a DT-signal to have period N,

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}$$

$$\Rightarrow e^{j\omega_0 n} e^{j\omega_0 N} = e^{j\omega_0 n}$$

$$\Rightarrow e^{j\omega_0 N} = 1$$

$$e^{j\omega_0 N} = 1$$

$$\cos \omega_0 N + j \sin \omega_0 N = 1$$

$$\Rightarrow \quad \sin \omega_0 N = 0$$

$$\cos \omega_0 N = 1$$

$\Rightarrow$ $\omega_0 N$ is a multiple of 2π.

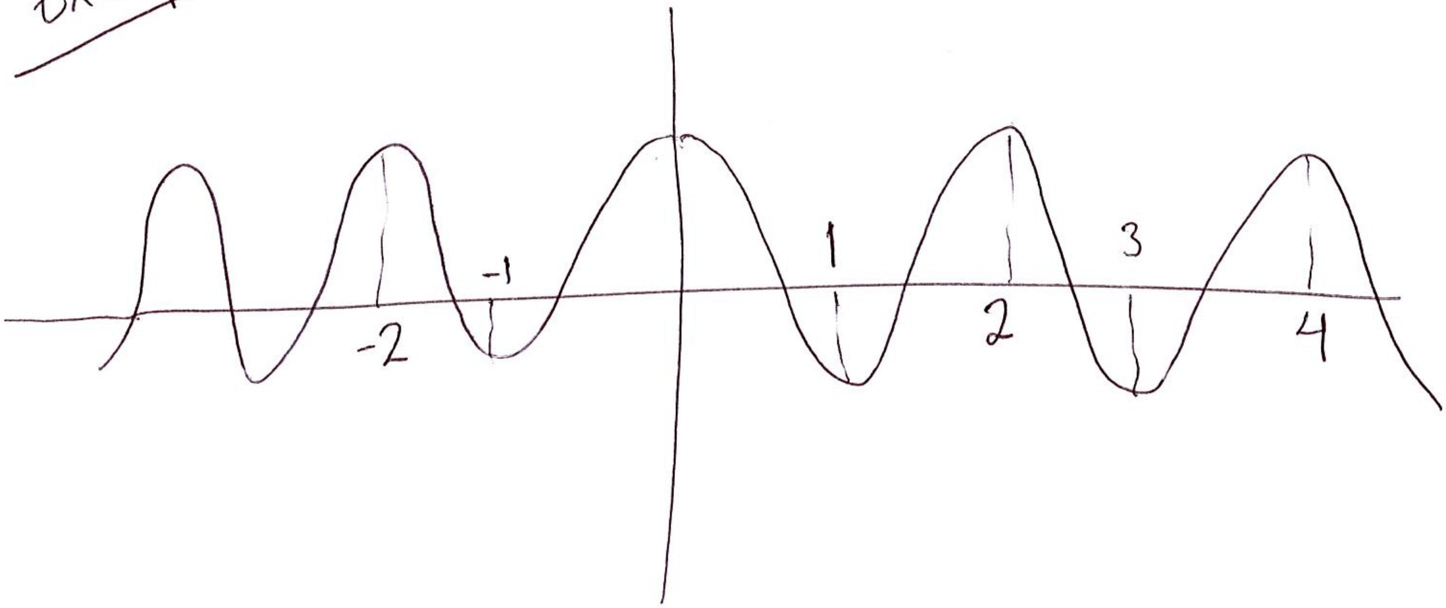
$\Rightarrow$ there exist an integer m, such that

$$\omega_0 N = 2\pi m$$

$$\Rightarrow \quad \frac{\omega_0}{2\pi} = \frac{m}{N}$$

$\Rightarrow$ $\dfrac{\omega_0}{2\pi}$ is a rational number

Example



$T_0 = 2$

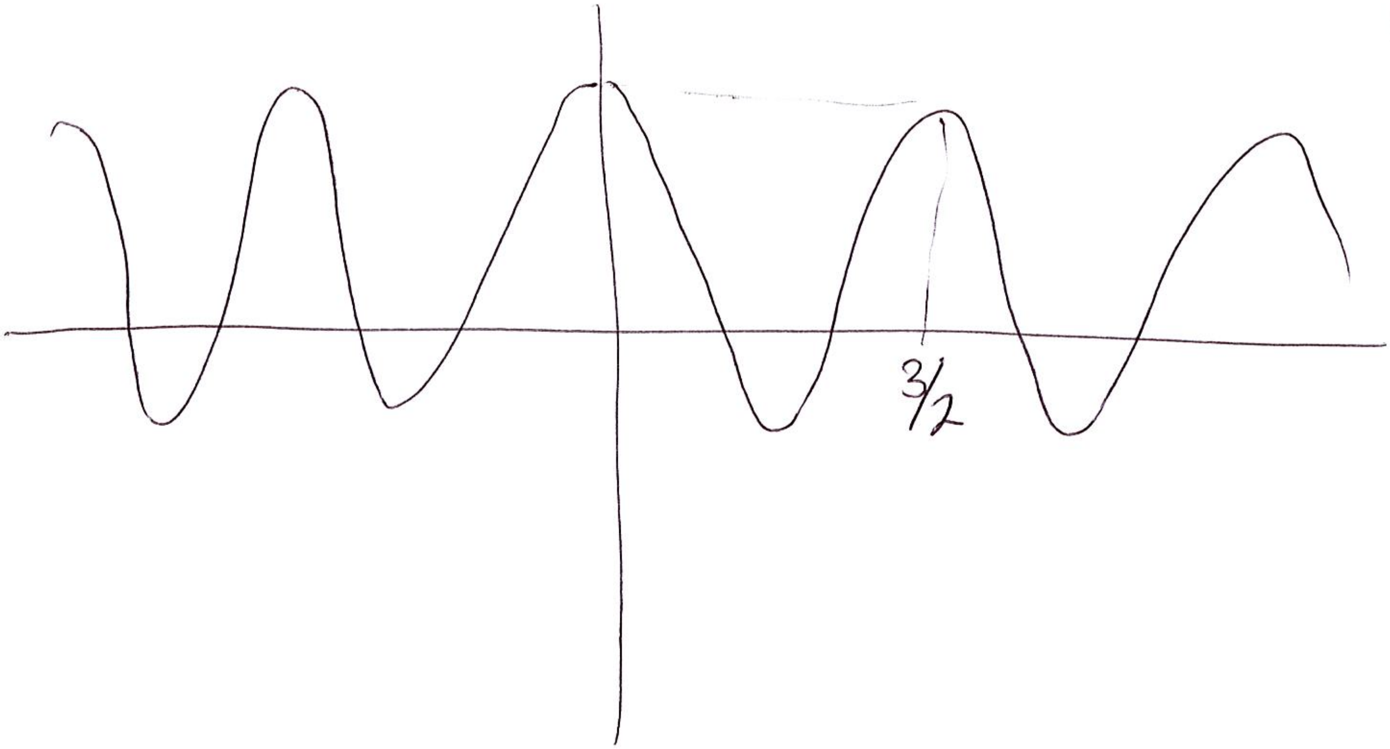
$\omega_0 = \frac{2\pi}{T_0} = \pi$

$\frac{\omega_0}{2\pi} = \frac{1}{2}$ rational

$e^{j\omega_0 n}$ is periodic

$\frac{\omega_0}{2\pi} = \frac{m}{N} = \frac{1}{2}$

$m = 1$ $\boxed{N = 2}$ Period



$T_0 = \frac{3}{2}$

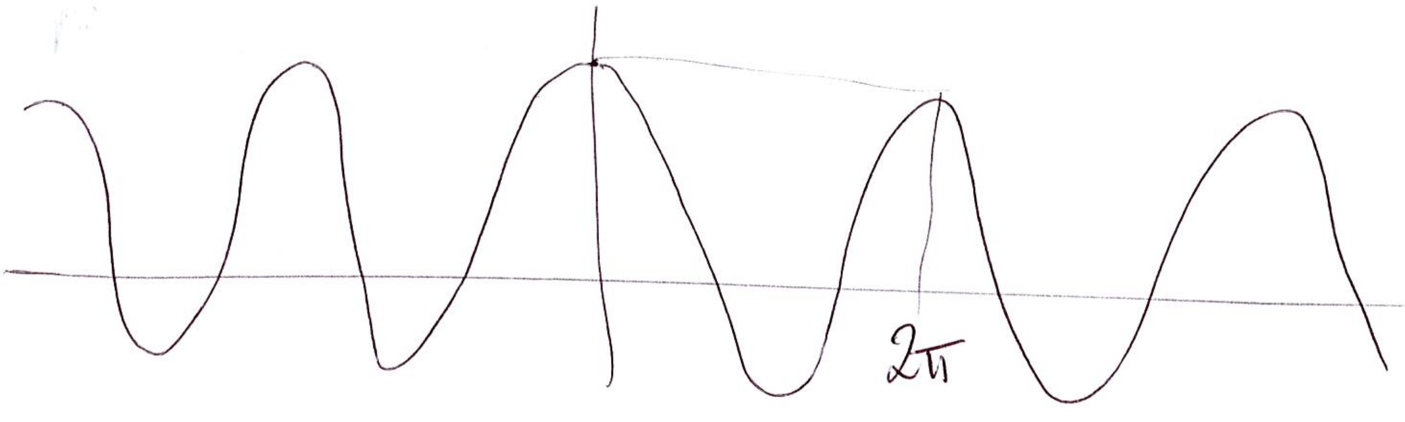
$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{3/2} = \frac{4\pi}{3}$

$\frac{\omega_0}{2\pi} = \frac{4\pi/3}{2\pi} = \frac{2}{3}$ rational

$e^{j\omega_0 n}$ is periodic

$\frac{\omega_0}{2\pi} = \frac{m}{N} = \frac{2}{3}$

$\Rightarrow m = 2,\ \boxed{N = 3}$ Period



$T_0 = 2\pi$

$\omega_0 = \frac{2\pi}{T_0} = 1$

$\frac{\omega_0}{2\pi} = \frac{1}{2\pi}$ Irrational number

$e^{j\omega_0 n}$ is $\boxed{\text{not periodic}}$

Example 1.6

Periodic with Fundamental Period $\boxed{N = 24}$ (12)

$$x[n] = \underbrace{e^{j(2\pi/3)n}}_{x_1[n]} + \underbrace{e^{j(3\pi/4)n}}_{x_2[n]}$$

$\mathrm{lcm}(N_1, N_2)$

$$x_1[n] = e^{j\omega_{0,1} n} = e^{j(2\pi/3)n}$$

$$\omega_{0,1} = \frac{2\pi}{3} \qquad \frac{\omega_{0,1}}{2\pi} = \frac{1}{3} = \frac{m_1}{N_1}$$

$$\Rightarrow m_1 = 1, \ \boxed{N_1 = 3}$$

Fundamental period of $x_1[n]$

$$x_2[n] = e^{j\omega_{0,2} n} = e^{j\left(\frac{3\pi}{4}\right)n}$$

$$\omega_{0,2} = \frac{3\pi}{4} \qquad \frac{\omega_{0,2}}{2\pi} = \frac{3\pi/4}{2\pi} = \frac{3}{8} = \frac{m_2}{N_2}$$

$$\Rightarrow m_2 = 3, \ \boxed{N_2 = 8}$$

Fundamental Period of $x_2[n]$