ECE 301 - Lecture #9

Properties of LTI Systems

Properties of the convolution operation

Commutative

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \qquad (1)$$

$$h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

Let $r = n-k$

Then $$h[n] * x[n] = \sum_{r=-\infty}^{\infty} h[n-r]\, x[r]$$

Same as (1)

Associative

$$x[n] * \left(h_1[n] * h_2[n] \right)$$

$$= \left(x[n] * h_1[n] \right) * h_2[n]$$

By combining commutativity and associativity, we know that the impulse response of a cascade of LTI systems is the convolution of the individual responses

"Taken in any order"

This cascade property does not hold in general, if the system is not LTI

for example

$$y(t) = 4x^2(t) = (2x(t))^2 \quad (2)$$

If we switch order

$y(t) = 2x^2(t)$ is not the same as (2)

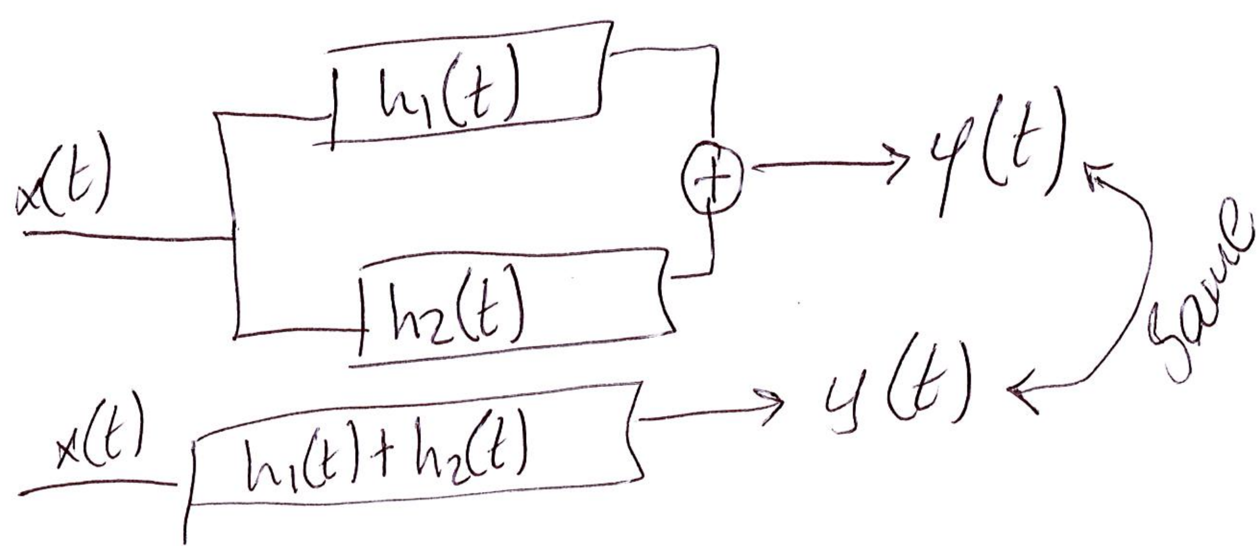
Distributive

$$x[n] * \left(h_1[n] + h_2[n] \right)$$

$$= x[n] * h_1[n] \quad + \quad x[n] * h_2[n]$$

$$\sum_{k=-\infty}^{\infty} x[k] \left(h_1[n-k] + h_2[n-k] \right)$$

$$\sum_{k=-\infty}^{\infty} x[k] h_1[n-k] \quad + \quad \sum_{k=-\infty}^{\infty} x[k] h_2[n-k]$$



when is an LTI System memoryless?

$$h[n] = 0 \quad \text{for } n \neq 0$$

If an LTI system is memoryless,

then $h[n] = k\,\delta[n]$

$$y[n] = k\,x[n]$$

Example

$$h[n] = \begin{cases} 1 & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = x[n] + x[n-1]$$

Not memoryless
but causal

when is an LTI system causal?

$h[n] = 0 \quad$ for $n < 0$

$$y[n] = x[n] * h[n]$$
$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

If the system is causal

then all the terms in the summation where $k > n$ have to be zeroed out

If $k > n$, then $n-k < 0$

Same thing as saying

$h[n] = 0$ for $n < 0$

Stability of LTI Systems

Consider a bounded input

$$|x[n]| < B \quad \text{for all } n$$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |h[k]|\, |x[n-k]|$$

$$\leq B \sum_{k=-\infty}^{\infty} |h[k]|$$

If this is finite
then the LTI system
is stable

Condition for stability of LTI Systems

Impulse response is absolutely summable

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

For Continuous-time

$$\int_{-\infty}^{\infty} |h(\tau)| \, d\tau < \infty$$

Impulse response absolutely integrable

Example 2-13

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\delta[n-n_0]| = 1$$

Stable

$$y[n] = x[n] * h[n] = x[n] * \delta[n-n_0]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-n_0-k]$$

$$= x[n-n_0]$$

$$h[n] = u[n]$$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} 1 = \infty$$

Unstable

$$y[n] = x[n] * h[n] = x[n] * u[n]$$

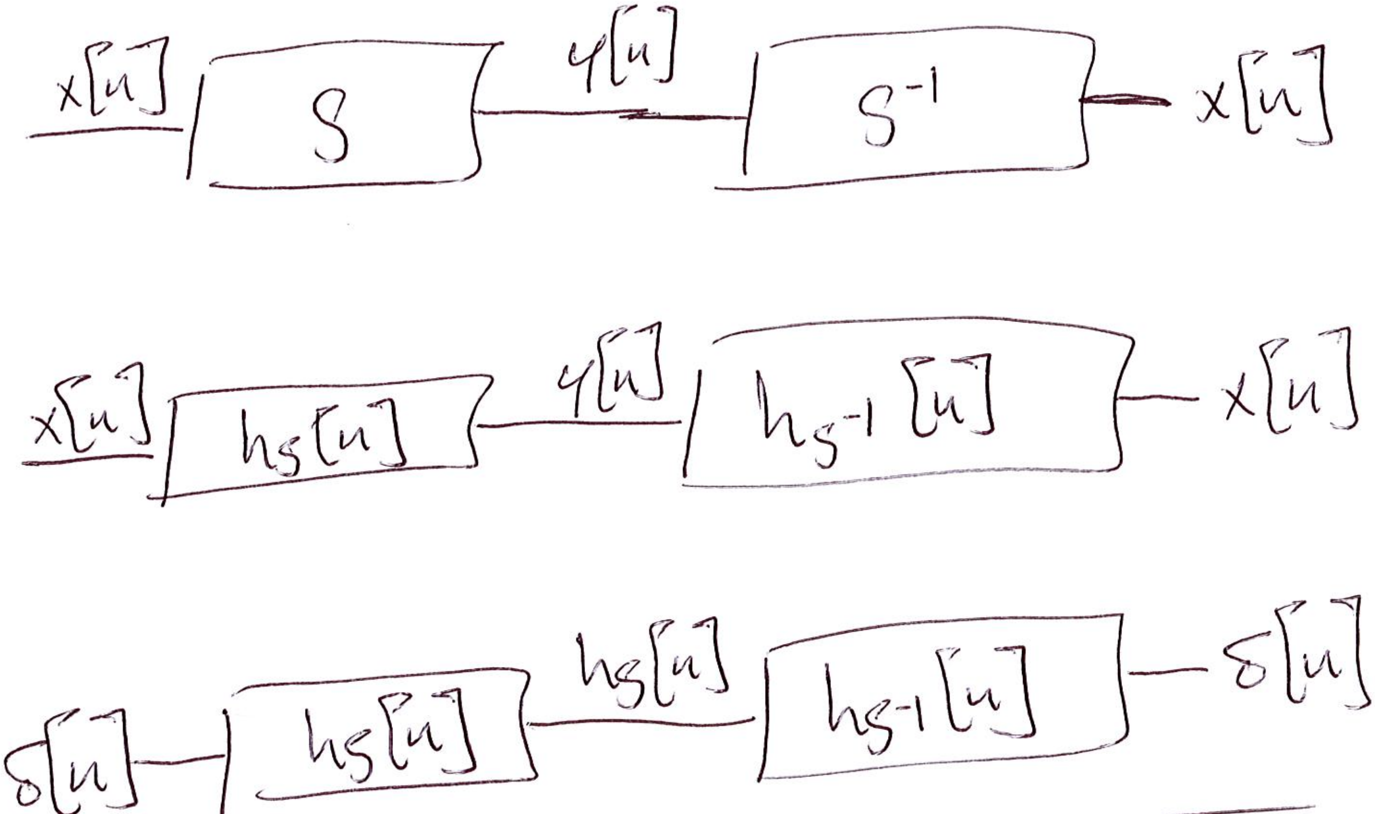
$$= \sum_{k=-\infty}^{\infty} x[k]\, u[n-k]$$

$$= \sum_{k=-\infty}^{n} x[k]$$

Accumulator

when is an LTI System invertible?

If there is an inverse system



$$\boxed{h_S[n] * h_{S^{-1}}[n] = \delta[n]}$$