HW-6.

1. a). DTFT of $(1/4)^{n-1} u[n]$.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} (1/4)^{n-1} e^{-j\omega n} u[n]$$

$$= \sum_{n=0}^{\infty} (1/4)^{n-1} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (1/4 e^{-j\omega})^{n} \cdot 4.$$

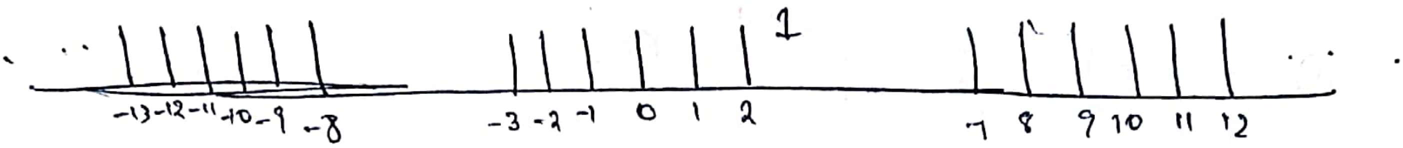
$$= 4 \cdot \left(\frac{1}{1-1/4 e^{-j\omega}} \right).$$

$$\boxed{X(e^{j\omega}) = \frac{4}{1-1/4 e^{-j\omega}}}.$$

b) DTFT of $\delta[-n] + 5\delta[n+2] + 3\delta[n-6]$

$$= 1 + 5e^{j(2\omega)} + 3e^{-j(6\omega)}.$$

c.



$$a_k = \frac{1}{10} \sum_{n=-3}^{2} 1 \cdot e^{jk\frac{2\pi}{10} \cdot n}.$$

$$\boxed{X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} a_k \, \delta\left(\omega - \frac{2\pi k}{10}\right)}.$$

2. $$x(t) = \frac{d^2(y(t))}{dt^2} + \frac{4\,dy(t)}{dt} + 3y(t).$$

$$\Rightarrow \quad X(j\omega) = (j\omega)^2 Y(j\omega) + 4 j\omega\, Y(j\omega) + 3Y(j\omega)$$

$$\Rightarrow \frac{Y(j\omega)}{X(j\omega)} = H(j\omega) = \frac{1}{(j\omega)^2 + 4j\omega + 3} = \frac{1}{(j\omega+1)(j\omega+3)}.$$

$$x(t) = e^{-t}u(t) \xleftrightarrow{FT} X(j\omega) = \frac{1}{1+j\omega}$$

$$Y(j\omega) = X(j\omega) \cdot H(j\omega) = \frac{1}{(j\omega+3)\cdot(j\omega+1)^2}.$$

Now, partial fractions.

$$\frac{1}{(j\omega+3)(j\omega+1)^2} = \frac{A}{(j\omega+1)^2} + \frac{B}{(j\omega+1)} + \frac{C}{j\omega+3}.$$

$$\Rightarrow 1 = A(j\omega+3) + B(j\omega+1)(j\omega+3) + C(j\omega+1)^2$$

$$= A(j\omega+3) + B((j\omega)^2+4j\omega+3) + C((j\omega)^2+2j\omega+1)$$

$$1. = j\omega(A+4B+2C)$$

$$+(j\omega)^2(B+C)$$

$$+ 3A+3B+C$$

$$\Rightarrow \begin{aligned} A+4B+2C &= 0 \\ B+C &= 0 \\ 3A+3B+C &= 1. \end{aligned} \quad \Rightarrow \quad \begin{aligned} A+2B &= 0. \\ 3A+2B &= 1. \end{aligned}$$

$$\Rightarrow 2A = 1 \Rightarrow A = 1/2.$$

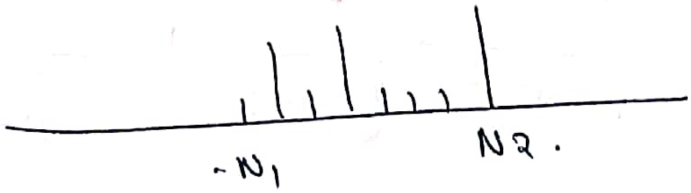
$$\Rightarrow B = -1/4.$$

$$\Rightarrow C = 1/4.$$

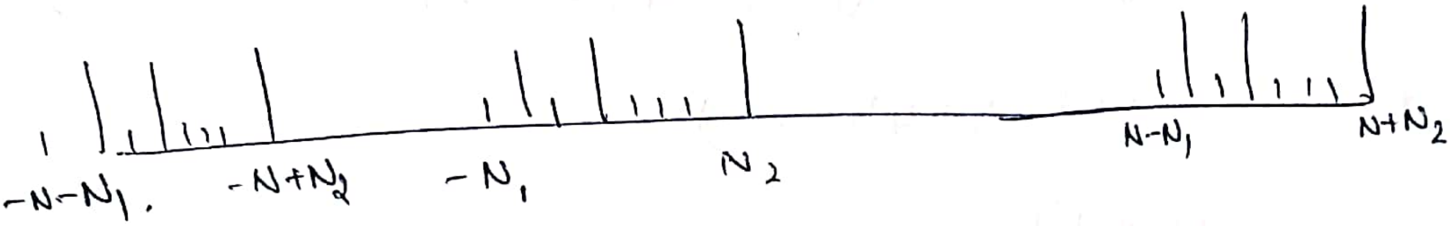
$$\Rightarrow Y(j\omega) = \frac{1}{2}\cdot\frac{1}{(j\omega+1)^2} + \left(-\frac{1}{4}\right)\cdot\frac{1}{j\omega+1} + \frac{1}{4}\left(\frac{1}{j\omega+3}\right)$$

$$\boxed{y(t) = \frac{1}{2}\cdot t\cdot e^{-t}u(t) + \left(-\frac{1}{4}\right)e^{-t}u(t) + \frac{1}{4}e^{-3t}u(t).}$$

3.a). $x[n]$.



Construct a periodic signal $\tilde{x}_N[n]$ with large N.



$$x[n] = \lim_{N \to \infty} \tilde{x}_N[n]$$

$$\tilde{x}_N[n] = \sum_{k=\langle N \rangle} a_k \, e^{jk\left(\frac{2\pi}{N}\right)n}.$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}_N[n] \, e^{-jk\left(\frac{2\pi}{N}\right)n}.$$

$$= 1/N \sum_{n=-N_1}^{N_2} \tilde{x}_N[n] \, e^{-jk\left(\frac{2\pi}{N}\right)n}.$$

$$= \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] \, e^{-jk\left(\frac{2\pi}{N}\right)n}.$$

$$= \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] \, e^{-jk\left(\frac{2\pi}{N}\right)n}.$$

Define.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}.$$

$$a_k = \frac{1}{N} X\left(e^{jk\frac{2\pi}{N}}\right).$$

Let $\omega_0 = \frac{2\pi}{N}$.

$$a_k = \frac{1}{N} X(e^{jk\omega_0}).$$

$$\tilde{x}_N[n] = \sum_{k=\langle N\rangle} X(e^{jk\omega_0})\, e^{jk\omega_0 n}.$$

$$= \frac{1}{2\pi} \sum_{k=\langle N\rangle} X(e^{jk\omega_0})\, e^{jk\omega_0 n}\, \omega_0.$$

$$x[n] = \lim_{N\to\infty} \tilde{x}_N[n] = \lim_{\omega_0\to 0} \tilde{x}_N[n].$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega.$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}.$$

b) $x[n] = \delta[n+13] + \delta[n+11] + \delta[n+7] + \delta[n+5]$

$+ \delta[n+1] + \delta[n-1] + \delta[n-5] + \delta[n-7]$

$+ \delta[n-11] + \delta[n-13]$.

$\updownarrow$ FT

$X(e^{j\omega}) = e^{13j\omega} + e^{11j\omega} + e^{7j\omega} + e^{5j\omega} + e^{j\omega}$

$+ e^{-j\omega} + e^{-5j\omega} + e^{-7j\omega} + e^{-11j\omega} + e^{-13j\omega}$

$= 2\cos(13\omega) + 2\cos(11\omega) + 2\cos(7\omega) + 2\cos(5\omega)$

$+ 2\cos\omega$.

4. $h[n] = \left(\frac{1}{4}\right)^n u[n] \longleftrightarrow H(e^{j\omega}) = \dfrac{1}{1-\frac{1}{4}e^{-j\omega}}$.

$x[n] = \left(\frac{1}{3}\right)^n u[n] \longleftrightarrow X(e^{j\omega}) = \dfrac{1}{1-\frac{1}{3}e^{-j\omega}}$.

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega}) = \frac{1}{\left(1-\frac{1}{4}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)}.$$

Partial fractions:

$$\frac{1}{\left(1-\frac{1}{4}e^{-j\omega}\right)\left(1-\frac{1}{3}e^{-j\omega}\right)} = \frac{A}{1-\frac{1}{4}e^{-j\omega}} + \frac{B}{1-\frac{1}{3}e^{-j\omega}}$$

$$\Rightarrow 1 = A(1 - \tfrac{1}{3} e^{-j\omega}) + B(1 - \tfrac{1}{4} e^{-j\omega}).$$

$$1 = e^{-j\omega}\left(\frac{-A}{3} - B/4\right) + A + B.$$

$\Rightarrow$ $A+B = \cancel{0}\,1.$ $\Rightarrow$ $A+B = 1.$

$\frac{-A}{3} - B/4 = 0.$ $\cancel{4A+3B} = 0.$

$\Rightarrow$ $B = 4.$

$\cancel{A} = -3.$

So, $\boxed{y(t) = -3\left(\frac{1}{4}\right)^n u[n] + 4\left(\frac{1}{3}\right)^n u[n]}$.

5. $Y(e^{j\omega}) = \cos\omega$.

(i) (ii) $\}$ $\cos\omega$ is real and even
$\therefore$ $y[n]$ is also real and even.

(iii) $\sum_{n=-\infty}^{\infty} y[n] = Y(e^{j\omega})\big|_{\omega=0} = 1.$

(iv) $\sum_{n=-\infty}^{\infty} (-1)^n y[n] = Y(e^{j\omega})\big|_{\omega=\pi} = -1.$