ECE 301 - Lecture #15

Fourier Series Representation of <u>real</u> periodic signals

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \boxed{1}$$

a_k: Complex coefficient

Let's say $x(t)$ is a real signal

then $x(t) = x^*(t)$

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t} \qquad \boxed{2}$$

Comparing $\boxed{1}$ and $\boxed{2}$,

$$\boxed{x(t) \text{ is real} \iff a_k = a_{-k}^*}$$

Deriving an alternative Form of the Fourier (2)
Series, that applies to real periodic signals

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t} \right]$$

If $x(t)$ is real, then $a_{-k} = a_k^*$ for all k

$$= a_0 + \sum_{k=1}^{\infty} \left[\underbrace{a_k e^{jk\omega_0 t}} + \underbrace{a_k^* e^{-jk\omega_0 t}} \right]$$

Conjugates

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\, \mathrm{Re}\{ a_k e^{jk\omega_0 t} \}$$

Real part of

a_k in polar form

$$a_k = \underbrace{A_k}_{\text{Amplitude/Magnitude}} e^{j\overbrace{\theta_k}^{\text{Angle}}}$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{A_k e^{j(k\omega_0 t + \theta_k)}\right\}$$

$$= a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

Determination of the Fourier Series Coefficients

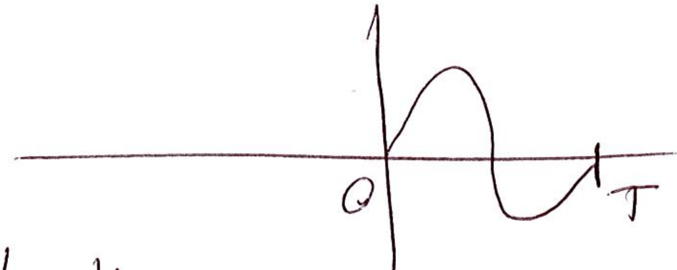
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Fixed

$$x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}$$

$$\int_{t=0}^{T} x(t)\, e^{-jn\omega_0 t}\, dt = \int_{t=0}^{T} \left(\sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t} \right) dt$$

$$= \sum_{k=-\infty}^{\infty} \int_{t=0}^{T} a_k e^{j(k-n)\omega_0 t}\, dt$$



If $k-n \neq 0$

then $\int_0^T e^{j(k-n)\omega_0 t}\, dt = 0$

If $k-n=0$

$$\int_0^T a_k e^{j(k-n)\omega_0 t}\, dt$$

$$= \int_0^T a_k\, dt = a_k T = a_n T$$

then

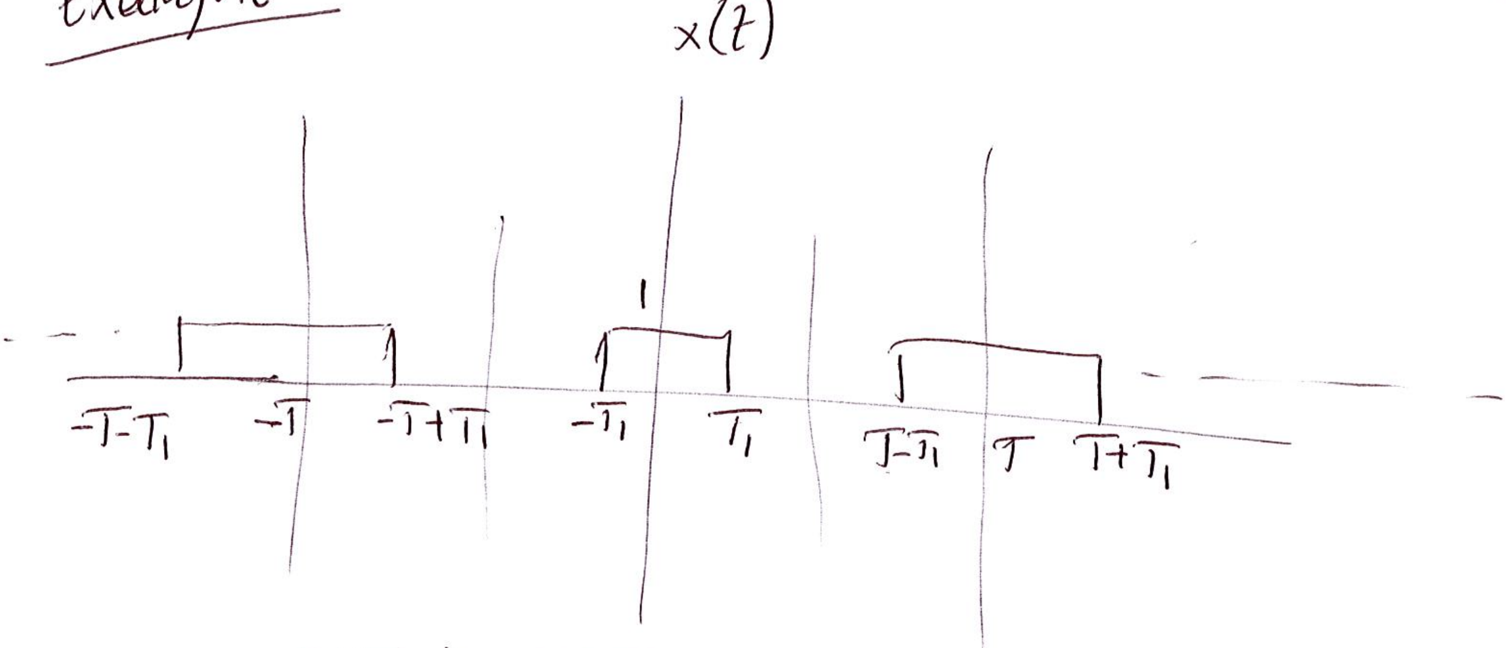
$$\sum_{k=-\infty}^{\infty} \int_0^T a_k e^{j(k-n)\omega_0 t}\, dt = a_n T$$

$$\int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$\frac{1}{T} \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = a_n$$

$$\boxed{\frac{1}{T} \int_T x(t)\, e^{-jn\omega_0 t}\, dt = a_n}$$

Example



$$T \geq 2T_1$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$a_0 = \frac{1}{T}\int_T x(t)\, dt = \frac{2T_1}{T}$$

$k \neq 0$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}\, dt$$

$$a_k = \frac{1}{T} \cdot -\frac{1}{jk\omega_0} e^{-jk\omega_0 t} \Bigg|_{-T_1}^{T_1}$$

$$= -\frac{1}{jk\omega_0 T}\left[e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1} \right]$$

$$= \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j} \right]$$

$$= \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}$$

Assume $T = 4T_1$, $\omega_0 T_1 = \frac{\pi}{2}$

$$a_k = \frac{\sin(k\pi/2)}{k\pi}, \quad k \neq 0$$

$$a_1 = \frac{1}{\pi} = a_{-1} \qquad a_3 = -\frac{1}{3\pi} = a_{-3}$$

$a_2 = 0 = a_{-2}$
$a_k = 0$
for k even

$$a_5 = a_{-5} = \frac{1}{5\pi} \qquad a_7 = a_{-7} = -\frac{1}{7\pi}$$