ECE 301 - Lecture #17

Dirichlet Conditions

Condition 1

$$\int_{T} |x(t)| < \infty$$

Counter Example

$T = 1 \qquad x(t) = \frac{1}{t}, \quad 0 < t \leq 1$

$$\int_0^1 \frac{1}{t}\, dt = \ln t \Big|_0^1 = \ln 1 - \underbrace{\ln 0}_{-\infty}$$

Condition 2

In any finite interval of time, the signal has to have a finite number of maxima and minima

Counter Example

$T = 1$

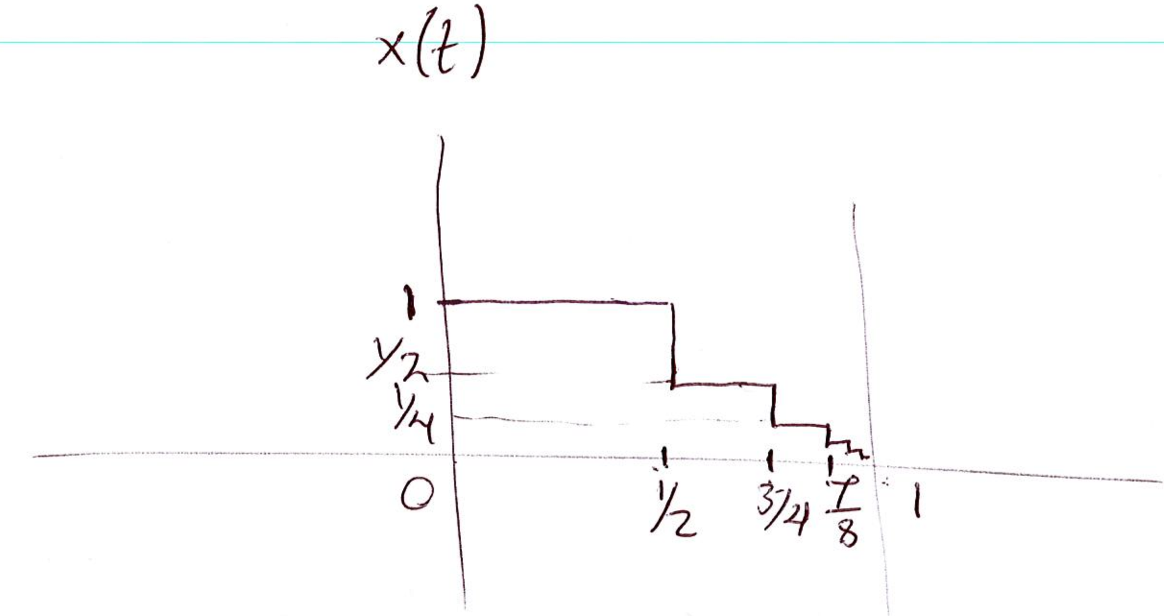
$$x(t) = \sin\left(\frac{2\pi}{t}\right) \qquad 0 < t \leq 1$$

Does not satisfy condition 2

Condition 3

In any finite interval of time, the signal has to have a finite number of discontinuities

$T = 1$



Properties of Continuous-Time Fourier Series

Linearity

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(t) \xleftrightarrow{FS} a_k$$

$$y(t) \xleftrightarrow{FS} b_k$$

Assuming that $x(t)$ and $y(t)$ have the same fundamental period

$$A\,x(t) + B\,y(t) \xleftrightarrow{FS} A a_k + B b_k$$

Time Shifting

$$x(t) \xleftrightarrow{FS} a_k$$

$$x(t-t_0) \xleftrightarrow{FS} ?\ e^{-jk\omega_0 t_0} a_k$$

$$b_k = \frac{1}{T} \int_T x(t-t_0)\, e^{-jk\omega_0 t}\, dt$$

Let $\tau = t - t_0$

$$b_k = \frac{1}{T} \int_T x(\tau)\, e^{-jk\omega_0 (\tau + t_0)}\, d\tau$$

$$= e^{-jk\omega_0 t_0} \underbrace{\left[\frac{1}{T} \int_T x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau \right]}_{a_k}$$

Time Reversal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x(t) \xleftrightarrow{FS} a_k$$

$$x(-t) \xleftrightarrow{FS} \; ? \quad a_{-k}$$

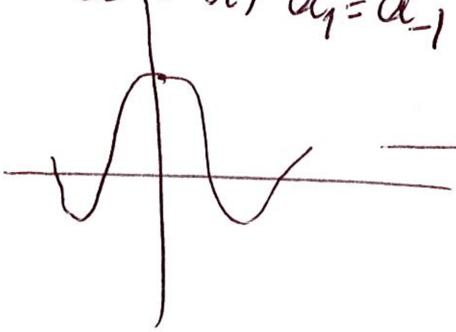
$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\omega_0 t}$$

Let $m = -k$

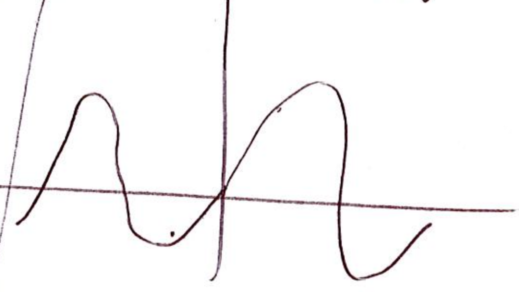
$$x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm\omega_0 t}$$

If $x(t)$ is $\frac{\text{odd}}{\text{even}}$, then the FS coeffs. are $\frac{\text{odd}}{\text{even}}$

$\cos(\omega_0 t)$ $a_1 = a_{-1}$



$\sin(\omega_0 t)$ odd $a_1 = -a_{-1}$



Multiplication

Have the same fundamental period

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

$$y(t) \overset{FS}{\longleftrightarrow} b_k$$

Convolution

$$x(t)\,y(t) \overset{FS}{\longleftrightarrow} \; ? \; c_k = \overbrace{\sum_{\ell=-\infty}^{\infty} a_\ell \, b_{k-\ell}}$$

$$x(t)\,y(t) = \left(\sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}\right)\left(\sum_{k=-\infty}^{\infty} b_k \, e^{jk\omega_0 t}\right)$$

Assume $x(t) = a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t}$

$y(t) = b_0 + b_1 e^{j\omega_0 t} + b_2 e^{j2\omega_0 t}$

$$x(t)\,y(t) = \underbrace{a_0 b_0}_{c_0} + \overbrace{(a_0 b_1 + b_0 a_1)}^{c_1} e^{j\omega_0 t}$$

$$+ \overbrace{(a_0 b_2 + b_0 a_2 + a_1 b_1)}^{c_2} e^{j2\omega_0 t}$$

$$+ \underbrace{(a_1 b_2 + a_2 b_1)}_{c_3} e^{j3\omega_0 t} + \underbrace{a_2 b_2}_{c_4} e^{j4\omega_0 t}$$

Conjugation

$$x(t) \xleftrightarrow{FS} a_k$$

$$x^*(t) \xleftrightarrow{FS} ? \quad a_{-k}^*$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

Let $m = -k$

$$x^*(t) = \sum_{m=-\infty}^{\infty} a_{-m}^* e^{jm\omega_0 t}$$

If $x(t)$ is real, then $a_k = a_{-k}^*$ for all k

If $x(t)$ is purely imaginary, then

$a_k = -a_{-k}^*$ for all k

Parserval's Relation

$$\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Proof

$$\frac{1}{T}\int_T \left|\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\right|^2 dt$$