ECE 301 - Lecture #18

Parserval's Relation

$$\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Proof

$$\frac{1}{T}\int_T |x(t)|^2\,dt$$

$$= \frac{1}{T}\int_T \Big|\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\Big|^2\,dt$$

why the cross-terms are zero?

$$= \frac{1}{T}\int_T \sum_{k=-\infty}^{\infty} |a_k e^{jk\omega_0 t}|^2\,dt$$

$$\int_T |e^{j\omega_0 t} + e^{j2\omega_0 t}|^2\,dt$$

$$= \int_T |e^{j\omega_0 t}|^2 + |e^{j2\omega_0 t}|^2 + 2\underbrace{e^{j3\omega_0 t}}\,dt$$

$$\frac{1}{T}\int_T \sum_{k=-\infty}^{\infty} |a_k e^{jk\omega_0 t}|^2 \, dt$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T}\int_T |a_k e^{jk\omega_0 t}|^2 \, dt$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T}\int_T |a_k|^2 \, |e^{jk\omega_0 t}|^2 \, dt$$

$\cos(k\omega_0 t) + j\sin(k\omega_0 t)$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T}\int_T |a_k|^2 \, dt$$

$$= \sum_{k=-\infty}^{\infty} |a_k|^2 = \frac{1}{T}\int_T |x(t)|^2 \, dt$$

Differentiation

$$x(t) \xleftrightarrow{FS} a_k$$

$$\frac{dx(t)}{dt} \xleftrightarrow{FS} \ ? \ jk\omega_0 a_k$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\frac{dx(t)}{dt} \quad \sum_{k=-\infty}^{\infty} \underbrace{jk\omega_0 a_k} e^{jk\omega_0 t}$$

Integration

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{FS} \left(\frac{1}{jk\omega_0}\right) a_k$$

If $x(t)$ is real

$$Ev\{x(t)\} = \frac{1}{2}\left(x(t) + x(-t)\right)$$

$$\xleftrightarrow{FS} \frac{1}{2}\left(a_k + a_{-k}\right)$$

But if $x(t)$ is real, then $a_k = a_{-k}^*$ for all k

$$a_{-k} = a_k^*$$

$$Ev\{x(t)\} \xleftrightarrow{FS} \underbrace{\frac{1}{2}\left(a_k + a_k^*\right)}_{Re\{a_k\}}$$

$$Od\{x(t)\} = \frac{1}{2}\left(x(t) - x(-t)\right\}$$

$$\xleftrightarrow{FS} \frac{1}{2}\left(a_k - a_{-k}\right)$$

$$= \frac{1}{2}\left(a_k - a_k^*\right) = Im\{a_k\}$$

Fourier Series Representation of Discrete-Time Periodic Signals

$$x[n] = \sum_{k=\langle N \rangle} a_k \, \phi_k[n]$$

$$\phi_k[n] = \begin{cases} \cancel{e^{jk\omega_0 n}} \\ e^{jk\left(\frac{2\pi}{N}\right)n} \end{cases}$$

Because $\dfrac{\omega_0}{2\pi} = \dfrac{m}{N}$ and m is not necessarily 1

$\{\phi_k[n]\}_k$ has only $\overbrace{\text{a finite}}^{N}$ number of elements

$$\phi_{k+N}[n] = e^{j(k+N)\left(\frac{2\pi}{N}\right)n}$$

$$= e^{jk\left(\frac{2\pi}{N}\right)n} \underbrace{e^{j\cancel{N}\left(\frac{2\pi}{\cancel{N}}\right)n}}_{1}$$

$$= \phi_k[n]$$

Example

$$x[n] = \sin \omega_0 n$$

$$\omega_0 = \frac{2\pi}{N}$$

$$x[n] = \frac{1}{2j} e^{j\left(\frac{2\pi}{N}\right)n} - \frac{1}{2j} e^{-j\left(\frac{2\pi}{N}\right)n}$$

$$a_1 = \frac{1}{2j} \quad , \quad a_{-1} = -\frac{1}{2j}$$

Assume $N = 5$

$$a_6 = \frac{1}{2j} \qquad a_4 = -\frac{1}{2j}$$

$$a_{-4} = \frac{1}{2j} \qquad a_{-6} = -\frac{1}{2j}$$

$$x[n] = \sin \omega_0 n$$

$$\omega_0 = \frac{2\pi m}{N} \qquad m = 3 \qquad N = 5$$

$$x[n] = \frac{1}{2j} e^{j3\left(\frac{2\pi}{N}\right)n} - \frac{1}{2j} e^{-j\left(\frac{2\pi}{N}\right)3n}$$

$$a_3 = \frac{1}{2j} \qquad a_{-3} = -\frac{1}{2j}$$

$$\boxed{a_{-2} = \frac{1}{2j} \qquad a_2 = -\frac{1}{2j}}$$

$$a_{-7} = \frac{1}{2j} \qquad a_7 = -\frac{1}{2j}$$

$$\underbrace{a_{-2}\, e^{-j2\left(\frac{2\pi}{N}\right)n}}_{\frac{1}{2j} e^{\underbrace{-j2\left(\frac{2\pi}{N}\right)n}_{e^{j3\left(\frac{2\pi}{N}\right)n}}}} + \underbrace{a_2\, e^{j2\left(\frac{2\pi}{N}\right)n}}_{-\frac{1}{2j} \underbrace{e^{j2\left(\frac{2\pi}{N}\right)n}}_{e^{-j3\left(\frac{2\pi}{N}\right)n}}}$$