ECE 301 - Lecture #19

Fourier Series Representation of Discrete-Time Period Signals

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

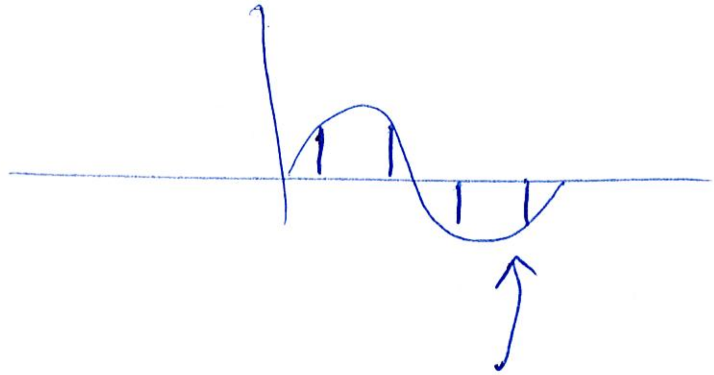
Determination of the FS coefficients

$$\sum_{n=\langle N \rangle} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)rn}$$

$$= \sum_{n=\langle N \rangle} \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \; e^{-j\left(\frac{2\pi}{N}\right)rn}$$

$$= \sum_{n=\langle N \rangle} \sum_{k=\langle N \rangle} a_k\, e^{j(k-r)\left(\frac{2\pi}{N}\right)n}$$

$$= \sum_{k=\langle N \rangle} a_k \underbrace{\sum_{n=\langle N \rangle} e^{j(k-r)\left(\frac{2\pi}{N}\right)n}}$$



If we sum these equi-spaced samples over one period, we get zero

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Fact

$$\sum_{n=\langle N\rangle} e^{j(k-r)\left(\frac{2\pi}{N}\right)n} = \begin{cases} 0 & \text{if } k \neq r \\ N & \text{if } k = r \end{cases}$$

$$\sum_{n=\langle N\rangle} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)rn}$$

$$= \sum_{k=\langle N\rangle} a_k \underbrace{\sum_{n=\langle N\rangle} e^{j(k-r)\left(\frac{2\pi}{N}\right)n}}$$

$$= N a_r$$

$$\Rightarrow a_r = \frac{1}{N} \sum_{n=\langle N\rangle} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)rn}$$

Properties of Discrete-Time Fourier Series

Linearity

have same N $\begin{cases} x[n] \xleftrightarrow{FS} a_k \\ y[n] \xleftrightarrow{FS} b_k \end{cases}$

$$\Rightarrow \quad A\,x[n] + B\,y[n] \xleftrightarrow{FS} A\,a_k + B\,b_k$$

Time Shifting

$$x[n] \xleftrightarrow{FS} a_k$$

$$x[n-n_0] \xleftrightarrow{FS} a_k\, e^{-jk\left(\frac{2\pi}{N}\right)n_0}$$

Frequency Shifting

$$x[n] \xleftrightarrow{FS} a_k$$

$$e^{jM\left(\frac{2\pi}{N}\right)n}\, x[n] = e^{jM\left(\frac{2\pi}{N}\right)n} \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$= \sum_{k=\langle N \rangle} a_k \, e^{j(M+k)\left(\frac{2\pi}{N}\right)n}$$

Let $r = k + M$

$$= \sum_{r=\langle N \rangle} a_{r-M} \, e^{jr\left(\frac{2\pi}{N}\right)n}$$

$$x[n] \xleftrightarrow{FS} a_k$$

$$\underbrace{e^{jM\left(\frac{2\pi}{N}\right)n} \, x[n]}_{\text{Frequency Shifting}} \xleftrightarrow{FS} a_{k-M}$$

Conjugation

$$x[n] \xleftrightarrow{FS} a_k$$

$$x^*[n] \xleftrightarrow{FS} a^*_{-k}$$

Time Reversal

$$x[n] \xleftrightarrow{FS} a_k$$

$$x[-n] \xleftrightarrow{FS} a_{-k}$$

First Difference

$$x[n] - x[n-1] = \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$- \sum_{k=\langle N \rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)(n-1)}$$

$$= \sum_{k=\langle N \rangle} \underbrace{\left(1 - e^{-jk\left(\frac{2\pi}{N}\right)}\right) a_k} e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$x[n] - x[n-1] \xleftrightarrow{FS} a_k \left(1 - e^{-jk\frac{2\pi}{N}}\right)$$

Running Sum

$$\sum_{k=-\infty}^{n} x[k] \xleftrightarrow{FS} \frac{1}{1-e^{-jk\frac{2\pi}{N}}}\, a_k$$

Multiplication

have same N
$$\begin{cases} x[n] \xleftrightarrow{FS} a_k \\ y[n] \xleftrightarrow{FS} b_k \end{cases}$$

$$x[n]\,y[n] = \left(\sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}\right)\left(\sum_{k=\langle N\rangle} b_k e^{jk\left(\frac{2\pi}{N}\right)n}\right)$$

$$x[n]\,y[n] \xleftrightarrow{FS} \underbrace{\sum_{l=\langle N\rangle} a_l\, b_{k-l}}_{\text{Periodic Convolution}}$$

Periodic Convolution

$$z[n] = \sum_{r=\langle N\rangle} x[r]\, y[n-r] \quad \xleftrightarrow{FS} \quad N a_k b_k$$

$$x[n] = \sum_{k=\langle N\rangle} a_k\, e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n]\, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

Fourier Series and LTI Systems

If $x(t) = e^{st} \longrightarrow \boxed{LTI} \longrightarrow e^{st} H(s)$

$$H(s) = \int_{\tau=-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

If $x(t)$ is period and has a FS representation

then $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$.

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} H(jk\omega_0)$$

$$\begin{array}{ccc} x(t) & \longrightarrow \boxed{LTI} \longrightarrow & y(t) \\ \updownarrow FS & & \updownarrow FS \\ a_k & & a_k H(jk\omega_0) \end{array}$$

Frequency Response

$$\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau}$$

Example 3.16

$$x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$$

Given

$$\boxed{a_0 = 1 \qquad a_1 = a_{-1} = \frac{1}{4} \qquad a_2 = a_{-2} = \frac{1}{2} \qquad a_3 = a_{-3} = \frac{1}{3}}$$

$$h(t) = e^{-t} u(t)$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} \, d\tau$$

$$b_1 = a_1 H(j2\pi) \qquad b_2 = a_2 H(j4\pi)$$

$$b_3 = a_3 H(j6\pi) \qquad b_{-1} = a_{-1} H(-j2\pi)$$

$$b_{-2} = a_{-2} H(-j4\pi) \qquad b_3 = a_{-3} H(-j6\pi)$$