ECE 301 - Lecture #20

Fourier Series and LTI Systems

$$x(t) \longrightarrow \boxed{LTI} \longrightarrow y(t)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

$$H(s) = \int_{\tau=-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

Example 3.16

$$x(t) = \sum_{k=-3}^{3} a_k e^{jk\overbrace{2\pi}^{\omega_0} t}$$

$$a_0 = 1 \qquad a_1 = a_{-1} = \frac{1}{4} \qquad a_2 = a_{-2} = \frac{1}{2}$$

$$a_3 = a_{-3} = \frac{1}{3}$$

$$h(t) = e^{-t}\, u(t)$$

$$H(j\omega) = \int_{\tau=-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{\tau=0}^{\infty} e^{-\tau}\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{\tau=0}^{\infty} e^{-(1+j\omega)\tau}\, d\tau$$

$$= -\frac{1}{1+j\omega}\, e^{-(1+j\omega)\tau}\Big|_{0}^{\infty}$$

$$= -\frac{1}{1+j\omega}\left[0 - 1\right] = \frac{1}{1+j\omega}$$

(3)

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow \ ?\ y(t)$$

$$\sum_{k=-3}^{3} a_k e^{jk\omega_0 t} \qquad \omega_0 = 2\pi$$

$a_0 = 1$

$a_1 = a_{-1} = \frac{1}{4}$

$a_2 = a_{-2} = \frac{1}{2}$

$a_3 = a_{-3} = \frac{1}{3}$

$$h(t) = e^{-t} u(t)$$

$$H(j\omega) = \frac{1}{1 + j\omega}$$

$$y(t) = \sum_{k=-3}^{3} a_k H(e^{jk\omega_0}) e^{jk\omega_0 t}$$

$$= \sum_{k=-3}^{3} b_k e^{jk\omega_0 t}$$

$$b_1 = a_1 H(j\omega_0) = a_1 H(j2\pi)$$

$$= \frac{1}{4}\left(\frac{1}{1 + j2\pi}\right)$$

$$b_{-1} = a_{-1} H(-j\omega_0) = \frac{1}{4}\left(\frac{1}{1 - j2\pi}\right)$$

$$b_2 = a_2 H(j4\pi) = \frac{1}{2}\left(\frac{1}{1 + j4\pi}\right), \quad b_{-2} = \frac{1}{2}\left(\frac{1}{1 - j4\pi}\right)$$

$$b_3 = \frac{1}{3}\left(\frac{1}{1 + j6\pi}\right) \quad b_{-3} = \frac{1}{3}\left(\frac{1}{1 - j6\pi}\right) \quad b_0 = 1$$

Note that $x(t)$ is real because

$$a_k = a_{-k}^* \text{ for all } k$$

Also, $y(t)$ is real because

$$b_k = b_{-k}^* \text{ for all } k$$

which makes sense because $h(t)$ is real as well and $y(t) = x(t) * h(t)$

Example 3.17

$$x[n] = \cos\left(\frac{2\pi n}{N}\right)$$

$$= \frac{1}{2} e^{j\left(\frac{2\pi}{N}\right)n} + \frac{1}{2} e^{-j\left(\frac{2\pi}{N}\right)n}$$

$$h[n] = \alpha^n u[n], \quad |\alpha| < 1$$

Discrete-Time Fourier Series and LTI Systems

If $x[n] = z^n$

Impulse response

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$$

then $y[n] = H(z)\, z^n$

If we let $z = e^{j\omega}$

$$\boxed{H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega k}}$$

$h[n] = \alpha^n u[n]$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n u[n]\, e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \frac{1}{1-\alpha e^{-j\omega}}$$

$$y[n] = \sum_{k=-\infty}^{\infty} \underbrace{a_k}_{\text{FS coeff. of } x[n]} H(e^{jk\omega_0})\, e^{jk\underbrace{\omega_0}_{\frac{2\pi}{N}} n}$$

$$x[n] = \cos\left(\frac{2\pi}{N}\right)n \qquad h[n] = \alpha^n u[n]$$

$$a_1 = a_{-1} = \frac{1}{2}$$

$$y[n] = \frac{1}{2} H\left(e^{j\frac{2\pi}{N}}\right) e^{j\left(\frac{2\pi}{N}\right)n} + \frac{1}{2} H\left(e^{-j\frac{2\pi}{N}}\right) e^{-j\left(\frac{2\pi}{N}\right)n}$$

$$= \overbrace{\frac{1}{2}\left(\frac{1}{1-\alpha e^{-j\frac{2\pi}{N}}}\right) e^{j\left(\frac{2\pi}{N}\right)n}}^{re^{j\theta}\cdot e^{j\left(\frac{2\pi}{N}\right)n}}$$

$$+ \frac{1}{2}\left(\frac{1}{1-\alpha e^{j\frac{2\pi}{N}}}\right) e^{-j\left(\frac{2\pi}{N}\right)n}$$

If we write

$$\frac{1}{1-\alpha e^{-j\frac{2\pi}{N}}} = r e^{j\theta}$$

then

$$\boxed{y[n] = r\cos\left(\frac{2\pi}{N}n + \theta\right)}$$

For example, if $N=4$

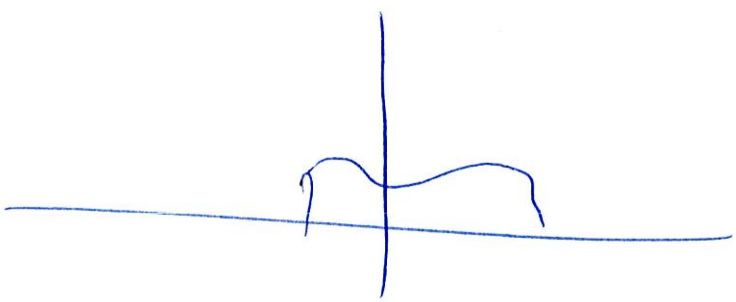
$$\frac{1}{1-\alpha \underbrace{e^{-j\frac{\pi}{2}}}} = \frac{1}{1+\alpha j} = \frac{1}{\sqrt{1+\alpha^2}}\, e^{j\left(-\tan^{-1}(\alpha)\right)}$$

$$\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} - j\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} = -j$$

$$y[n] = \frac{1}{\sqrt{1+\alpha^2}}\cos\left(\frac{\pi}{2}n - \tan^{-1}(\alpha)\right)$$

Next Lecture

Fourier Transform = Generalization of the Fourier Series that applies to Aperiodic Signals



$T \longrightarrow \infty \,,\ \omega_0 \longrightarrow 0$