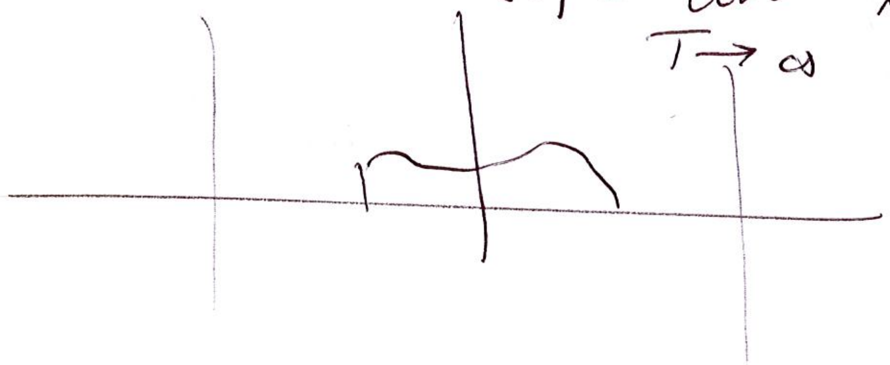
ECE 301 - Lecture #21

Fourier Transform of Aperiodic Signals

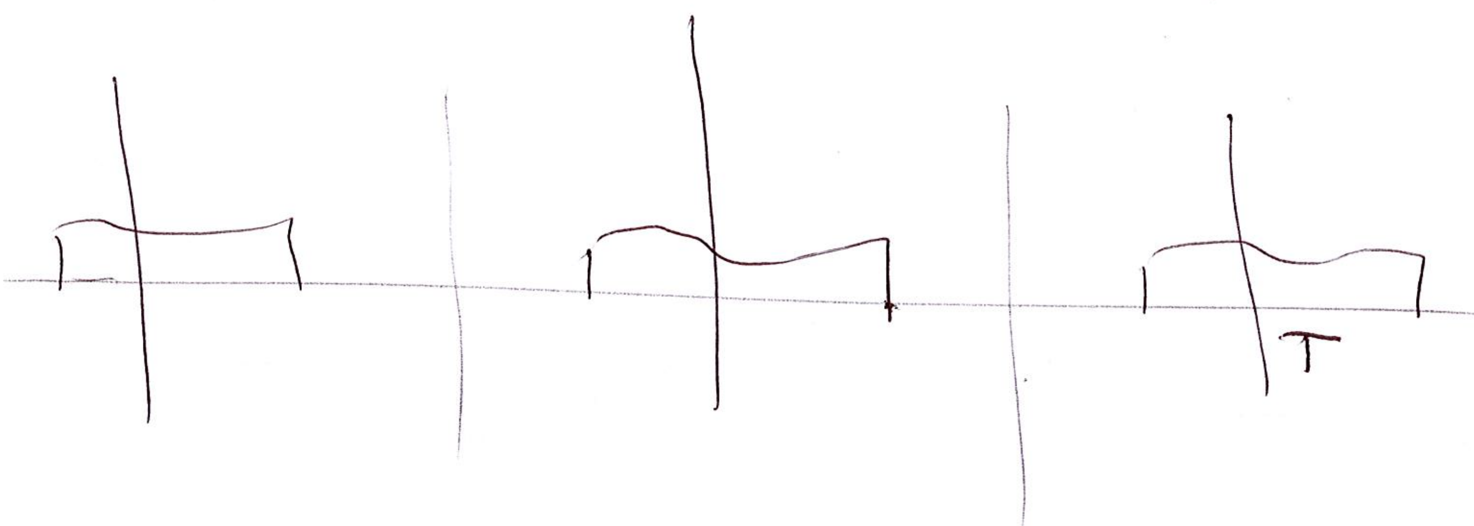
Continuous-Time

Finite Duration

$$x(t) = \lim_{T \to \infty} \tilde{x}_T(t)$$



Create a virtual signal $\tilde{x}_T(t)$



$$\tilde{x}_T(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_{T} \tilde{x}_T(t) \, e^{-jk\omega_0 t} \, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}_T(t) \, e^{-jk\omega_0 t} \, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} x(t) \, e^{-jk\omega_0 t} \, dt$$

Let's define

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t) \, e^{-j\omega t} \, dt}$$

$$a_k = \frac{1}{T} \int_{-\infty}^{\infty} x(t) \, e^{-jk\omega_0 t} \, dt = \frac{1}{T} X(jk\omega_0)$$

$$\tilde{x}_T(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0) \, e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) \, e^{jk\omega_0 t} \, \omega_0$$

$$x(t) = \lim_{T \to \infty} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) \, e^{jk\omega_0 t} \, \omega_0$$

$$\lim_{\omega_0 \to 0}$$

$$\boxed{x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X(j\omega) \, e^{j\omega t} \, d\omega}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Inverse Fourier Transform of $X(j\omega)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Fourier Transform of $x(t)$

Applies to General Aperiodic Signals

Under Certain Conditions

Signals with Finite Energy

$$\int_{-\infty}^{\infty} |x(t)|^2 < \infty$$

then if $e(t) = x(t) - \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$

then $\int_{-\infty}^{\infty} |e(t)|^2 = 0$

Dirichlet Conditions

- Guarantee that $x(t)$ equals its Fourier Transform except at points of discontinuity.

1. $\int_{-\infty}^{\infty} |x(t)| < \infty$

If $x(t)$ is the impulse response of an LTI system,

$\boxed{\text{Condition } 1 \Longleftrightarrow \text{System is stable}}$

2. Finite number of maxima and minima over any finite interval

3. Finite number of discontinuities over any finite interval. Each of these discontinuities is finite.

Example 4.1

$$x(t) = e^{-at} u(t) \quad , \quad a > 0$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

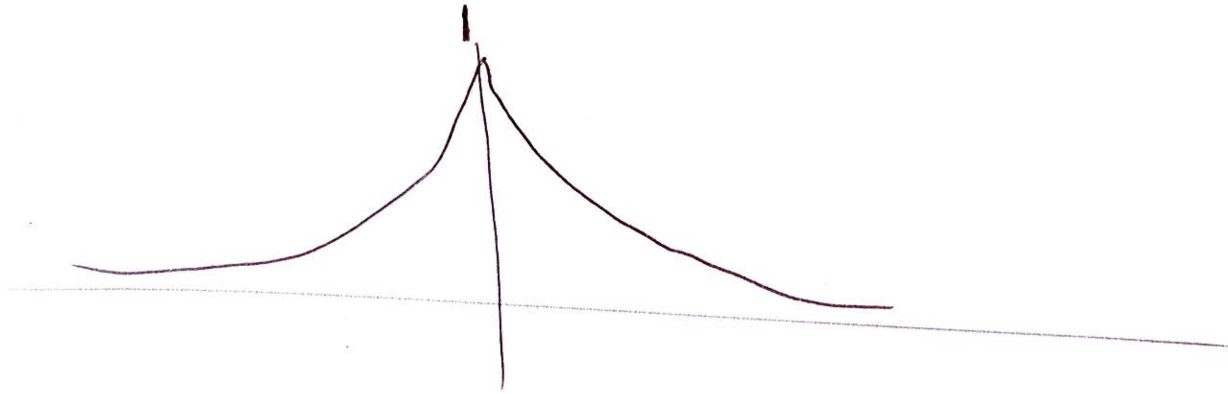
$$= -\frac{1}{a+j\omega}\, e^{-(a+j\omega)t} \Big|_{0}^{\infty}$$

$$= -\frac{1}{a+j\omega}\left[0 - 1\right] = \frac{1}{a+j\omega}$$

$$\mathcal{F}\left\{e^{-at} u(t)\right\} = \frac{1}{a+j\omega} \quad \text{when } a > 0$$

Example 4.2

$$x(t) = e^{-a|t|} \quad , \quad a > 0$$

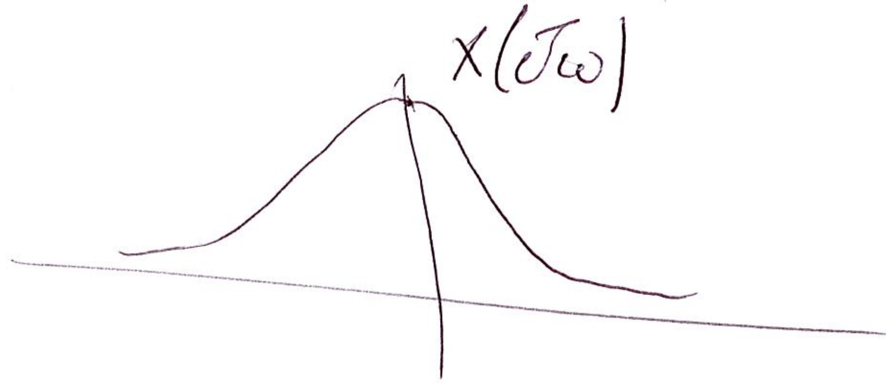


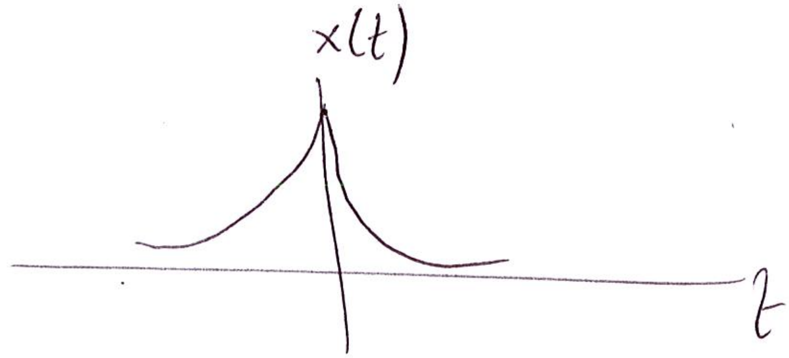
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

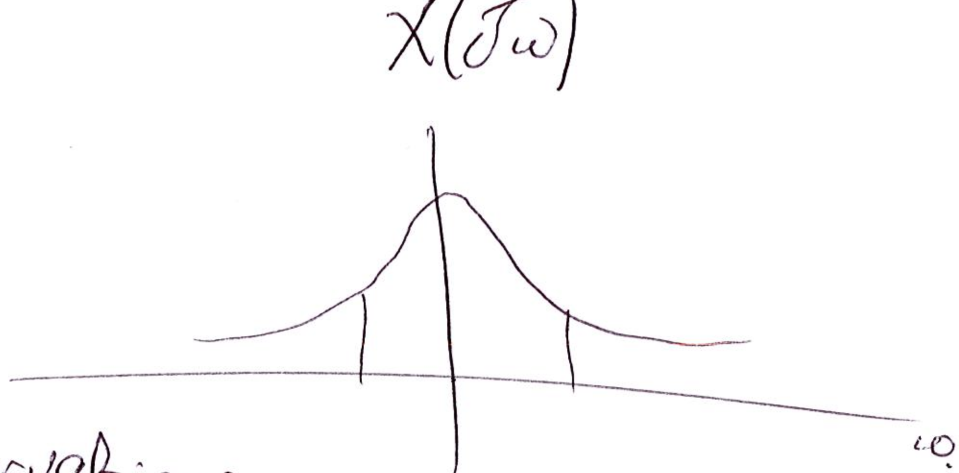
$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$= \frac{2a}{a^2+\omega^2}$$







<u>observations</u>

- Most of the energy is at lower values of ω (<u>fairly smooth</u>)

- $X(j\omega)$ <u>does not die out</u> as $\omega \to \infty$

  does not become 0