Parserval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

Proof

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x(t)\, x^*(t) \, dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega)\, e^{-j\omega t} \, d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \underbrace{\left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} \, dt \right]}_{X(j\omega)} d\omega$$

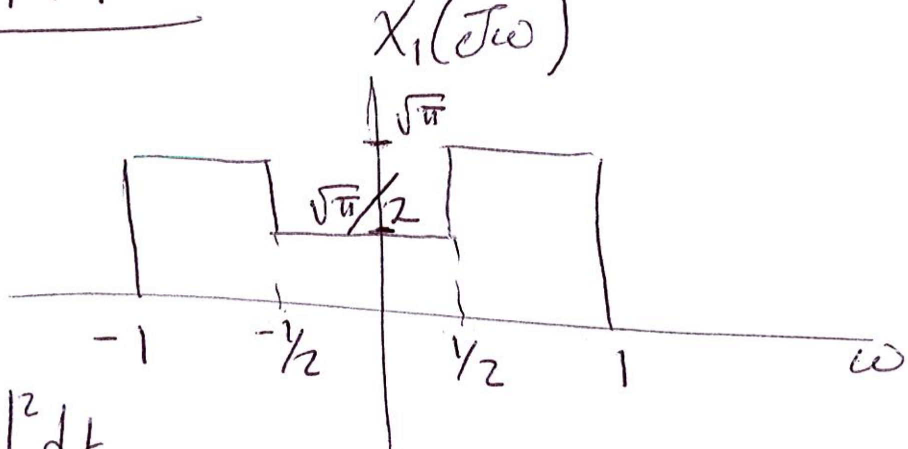
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

$$\frac{|X(j\omega)|^2}{2\pi}$$

Energy per unit frequency

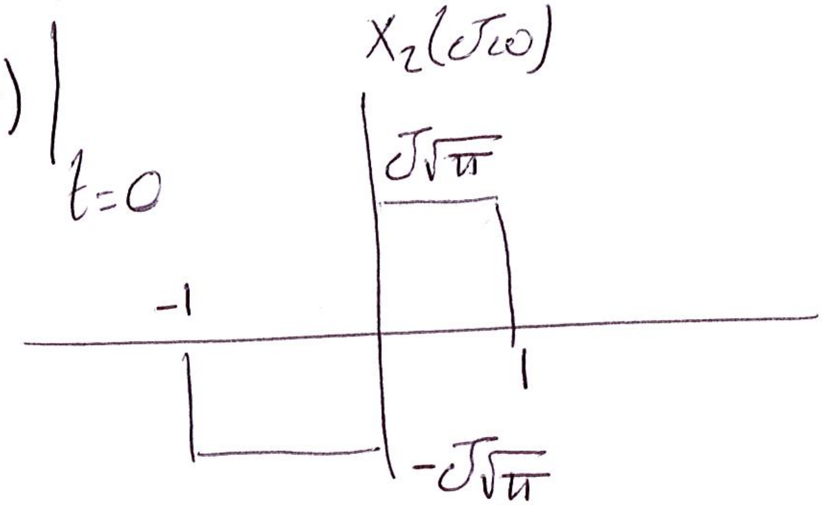
Energy-Density Spectrum

Example 4.14



a) $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$

b) $D = \left.\frac{d}{dt} x(t)\right|_{t=0}$



$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \qquad (3)$$

$$= \frac{1}{2\pi}\left[\pi + \frac{\pi}{4}\right] = \frac{5}{8}$$

$$D = \frac{d}{dt} x(t) \Big|_{t=0}$$

Let $g(t) = \frac{d}{dt} x(t)$

$$G(j\omega) = j\omega \, X(j\omega)$$

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega) e^{j\omega t} d\omega$$

$$g(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega \, X(j\omega) d\omega = 0$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int |X(j\omega)|^2 d\omega \qquad (4)$$

$$= \frac{1}{2\pi}\left[\pi + \pi\right] = 1$$

$$D = \frac{d}{dt} x(t) \Big|_{t=0}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega X(j\omega)\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_0^1 -\sqrt{\pi}\,\omega\, d\omega + \int_{-1}^{0} \sqrt{\pi}\,\omega\, d\omega \right]$$

$$= \frac{1}{2\sqrt{\pi}}\left[-\frac{\omega^2}{2}\Big|_0^1 + \frac{\omega^2}{2}\Big|_{-1}^{0} \right]$$

$$= \frac{1}{2\sqrt{\pi}}\left[-\frac{1}{2} - \frac{1}{2} \right] = -\frac{1}{2\sqrt{\pi}}$$

The Convolution Property

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0)\, e^{jk\omega_0 t}\, \omega_0$$

$$x(t) \longrightarrow \overset{h(t)}{\boxed{LTI}} \longrightarrow y(t)$$

Complex exponentials are Eigen Functions of LTI Systems

$$e^{jk\omega_0 t} \longrightarrow \overset{h(t)}{\boxed{LTI}} \longrightarrow e^{jk\omega_0 t} H(jk\omega_0)$$

$$H(jk\omega_0) = \int_{-\infty}^{\infty} h(\tau)\, e^{-jk\omega_0 \tau}\, d\tau$$

Frequency Response

$$y(t) = \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0)\, H(jk\omega_0)\, e^{jk\omega_0 t}\, \omega_0$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, H(j\omega)\, e^{j\omega t}\, d\omega$$

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\omega)\, e^{j\omega t}\, d\omega$$

By observation, $Y(j\omega) = X(j\omega) H(j\omega)$

$$y(t) = x(t) * h(t)$$

Formal Derivation

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \underbrace{\left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\, dt \right]}_{e^{-j\omega\tau} H(j\omega)} d\tau$$

$$= \int_{-\infty}^{\infty} H(j\omega)\, x(\tau)\, e^{-j\omega\tau}\, d\tau = H(j\omega) X(j\omega)$$

Comment

$$x(t) \longrightarrow \overset{h(t)}{\boxed{LTI}} \longrightarrow y(t)$$

$$X(j\omega) \underbrace{H(j\omega)}_{\text{when does this exist?}} = Y(j\omega)$$

1st Dirichlet Condition

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

Condition for Stability of LTI Systems

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

Same

$\Rightarrow$ Fourier Transform is useful only for analyzing Stable LTI Systems

Laplace Transform:

Generalization of Fourier Transform, that is used for analysis of unstable LTI Systems.

Example 4.15

$$h(t) = \delta(t - t_0)$$

$$H(j\omega) = e^{-j\omega t_0}$$

$$y(t) = x(t) * h(t) = x(t - t_0)$$

$$Y(j\omega) = X(j\omega)\, e^{-j\omega t_0} = X(j\omega)\, H(j\omega)$$