ECE 301 - Lecture #28

Side Comment

$$x(t) = e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a+j\omega} = (a+j\omega)^{-1}$$

Duality to Differentiation Property

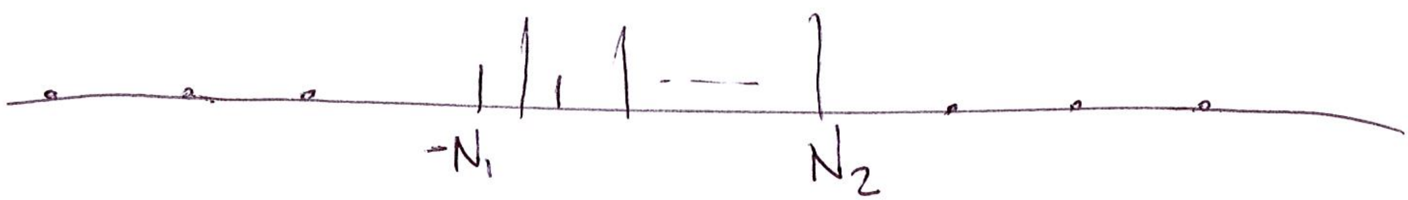
$$-jt\, x(t) \xleftrightarrow{\mathcal{F}} \frac{d}{d\omega} X(j\omega)$$

$$-jt\, e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \frac{-j}{(a+j\omega)^2}$$

$$t\, e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{(a+j\omega)^2}$$

Discrete-Time Fourier Transform (DTFT)

Finite Duration signal $x[n]$



We construct a periodic signal $\tilde{x}_N[n]$ with large N

$$x[n] = \lim_{N \to \infty} \tilde{x}_N[n]$$

$$\tilde{x}_N[n] = \sum_{k=\langle N\rangle} a_k \, e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$a_k = \frac{1}{N} \sum_{\cancel{k}\, n=\langle N\rangle} \tilde{x}_N[n] \, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$= \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}_N[n] \, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$= \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] \, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$= \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] \, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

Define

$$\boxed{X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] \, e^{-j\omega n}}$$

$$a_k = \frac{1}{N} X\left(e^{jk\frac{2\pi}{N}}\right)$$

Let $\omega_0 = \dfrac{2\pi}{N}$

$$a_k = \frac{1}{N} X\left(e^{jk\omega_0}\right)$$

$$\Rightarrow \tilde{x}_N[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n}$$

$$= \frac{1}{2\pi} \sum_{k=\langle N \rangle} X\left(e^{jk\omega_0}\right) e^{jk\omega_0 n} \, \omega_0$$

$$x[n] = \lim_{N \to \infty} \tilde{x}_N[n] = \lim_{\omega_0 \to 0} \tilde{x}_N[n]$$

$$\boxed{\begin{aligned} x[n] &= \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} \, d\omega \\ X\left(e^{j\omega}\right) &= \sum_{n=-\infty}^{\infty} x[n] \, e^{-j\omega n} \end{aligned}}$$

Example 5.1

$$x[n] = a^n u[n] \quad , \quad |a| < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n$$

$$= \frac{1}{1 - a e^{-j\omega}}$$

Example 5.2

$$x[n] = a^{|n|} \quad , \quad |a| < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}$$

$$\sum_{n=0}^{\infty} a^{n} e^{-j\omega n} = \frac{1}{1-ae^{-j\omega}} \qquad (6)$$

$$\sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} \underset{m=-n}{=} \sum_{m=1}^{\infty} a^{m} e^{j\omega m}$$

$$= \sum_{m=1}^{\infty} \left(ae^{j\omega}\right)^{m}$$

$$= \sum_{m=0}^{\infty} \left(ae^{j\omega}\right)^{m} - 1$$

$$= \frac{1}{1-ae^{j\omega}} - 1$$

$$= \frac{ae^{j\omega}}{1-ae^{j\omega}}$$

$$a^{|n|} \xleftrightarrow{\mathcal{F}} \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}} = \frac{1-a^{2}}{1-2a\cos\omega+a^{2}}$$

Note

$x[n] = a^{|n|}$ real and even

$$X(e^{j\omega}) = \frac{1-a^2}{1-2a\cos\omega + a^2}$$ real and even

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Example 5.3

$$x[n] = \begin{cases} 1 , & |n| \leq N_1 \\ 0 , & |n| > N_1 \end{cases}$$

$X(e^{j\omega})$? (Bonus Exercise 0.3%)
(Due Fri. Apr. 12)

Convergence of DTFT

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

OR $$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Example 5.4

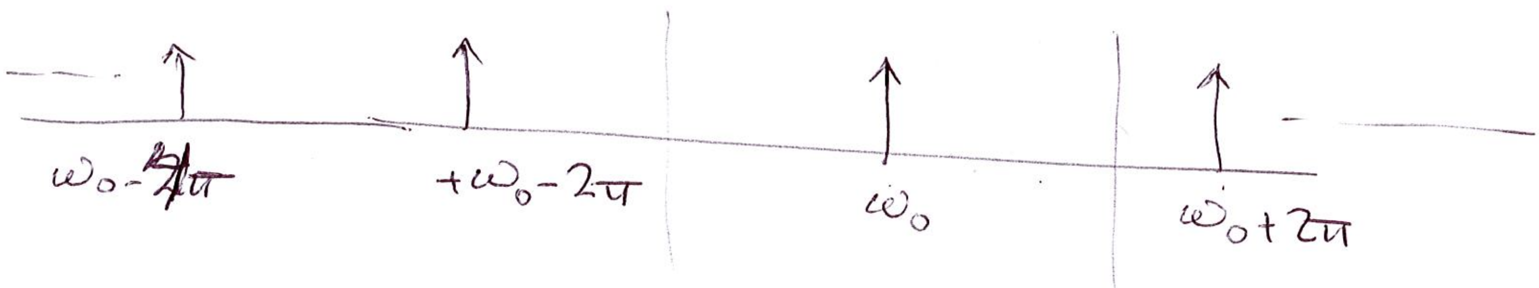
$$x[n] = \delta[n]$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1$$

Discrete-Time Fourier Transform for Periodic Signals

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 - 2\pi l)$$



$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \int_{2\pi} \left(\sum_{l=-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0 - 2\pi l) \right) e^{j\omega n}\, d\omega$$

$$= \frac{2\pi}{2\pi} \int \delta(\omega - \omega_0)\, e^{j\omega n}\, d\omega$$

$$= e^{j\omega_0 n}$$

For a DT Periodic Signal

$$x[n] = \sum_{k=\langle N \rangle} a_k \, e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} a_k \, 2\pi \, \delta\left(\omega - \frac{2\pi k}{N}\right)$$

Example 5.5

$N = 5$

$\omega_0 = \frac{2\pi}{5}$

$$x[n] = \cos \omega_0 n$$

$$= \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$

$$X(e^{j\omega})$$