Convolution Property of DTFT

$$y[n] = x[n] * h[n]$$

$$Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$$

Example 5.11

$$h[n] = \delta[n-n_0]$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n-n_0]\, e^{-j\omega n}$$

$$= e^{-j\omega n_0}$$

$$y[n] = x[n] * h[n] = x[n] * \delta[n-n_0]$$

$$= x[n-n_0]$$

$$Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega}) = X(e^{j\omega})\, e^{-j\omega n_0}$$

Example 5.12

$$H(e^{j\omega}) = \begin{cases} 1, & -\omega_c \le \omega \le \omega_c \\ 0, & \cancel{\text{otherwise}} \quad \omega_c < |\omega| < \pi \end{cases}$$

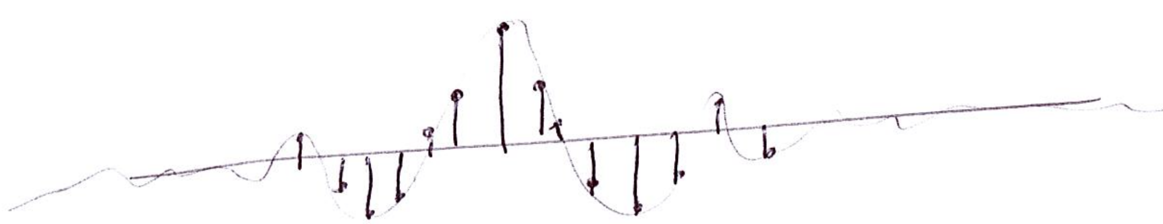
Freq. response of an ideal LPF

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} \, d\omega$$

$$= \frac{\sin \omega_c n}{\pi n}$$

DT-sinc



Non-Causal

Oscillatory behavior

Approximation of LPF

$$h[n] = \alpha^n u[n] \quad , \quad |\alpha| < 1$$

Example 5.13

$$x[n] = \beta^n u[n] \quad , \quad |\beta| < 1$$

What is $y[n]$?

$$H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{1}{1-\beta e^{-j\omega}}$$

$$Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}$$

$$= \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}$$

$$A = \frac{\alpha}{\alpha - \beta} \qquad B = \frac{-\beta}{\alpha - \beta}$$

$$Y(e^{j\omega}) = \frac{A}{1 - \alpha e^{-j\omega}} + \frac{B}{1 - \beta e^{-j\omega}}$$

$$\boxed{\begin{aligned} y[n] &= A\,\alpha^n u[n] + B\,\beta^n u[n] \\ &= \frac{1}{\alpha - \beta}\left[\alpha^{n+1} u[n] - \beta^{n+1} u[n]\right] \\ &\qquad\qquad \text{If } \alpha \neq \beta \end{aligned}}$$

If $\alpha = \beta$

$$Y(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})^2}$$

$$\frac{d}{d\omega}\left(\frac{1}{1 - \alpha e^{-j\omega}}\right) = -1 \left(\frac{1}{1 - \alpha e^{-j\omega}}\right)^2 \cdot -\alpha \cdot -j\, e^{-j\omega}$$

$$\frac{1}{(1-\alpha e^{-j\omega})^2} = \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right) \frac{j}{\alpha} e^{j\omega}$$

$$\alpha^n u[n] \xleftrightarrow{\mathcal{F}} \frac{1}{1-\alpha e^{-j\omega}}$$

$$-jn\,\alpha^n u[n] \xleftrightarrow{\mathcal{F}} \frac{d}{d\omega} \frac{1}{1-\alpha e^{-j\omega}}$$

$$\boxed{\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \\ \frac{d}{d\omega} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} -jn\, x[n]\, e^{-j\omega n} \end{aligned}}$$

$$\frac{j}{\alpha} \cdot -jn\,\alpha^n u[n] \xleftrightarrow{\mathcal{F}} \frac{j}{\alpha} \frac{d}{d\omega} \frac{1}{1-\alpha e^{-j\omega}}$$

$$n\,\alpha^{n-1} u[n] \xleftrightarrow{\mathcal{F}} \frac{j}{\alpha} \frac{d}{d\omega} \frac{1}{1-\alpha e^{-j\omega}}$$

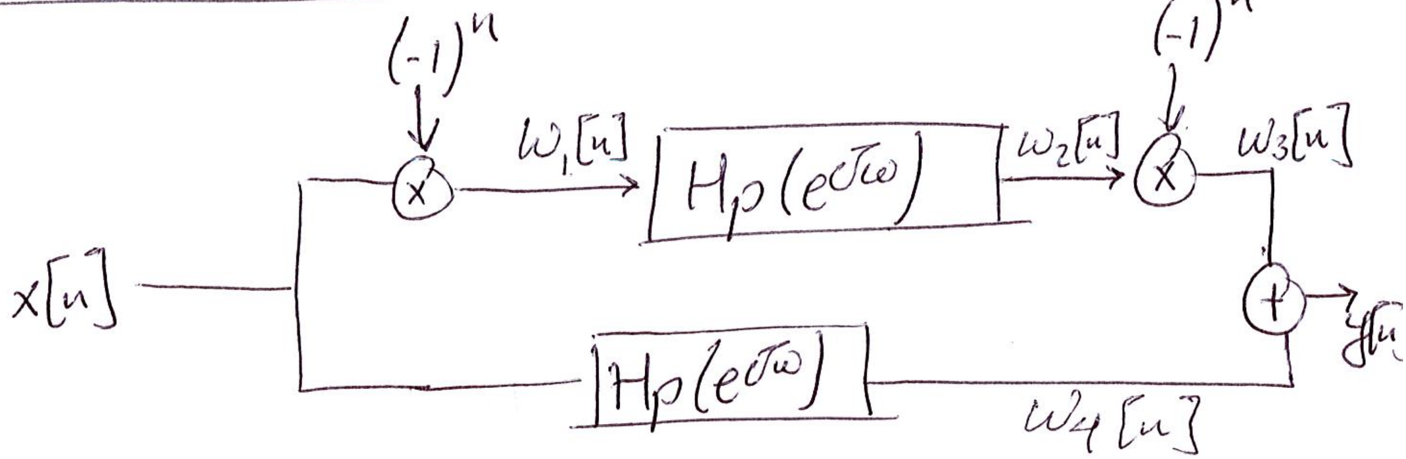
$$(n+1)\,\alpha^n u[n+1] \xleftrightarrow{\mathcal{F}} e^{j\omega} \frac{j}{\alpha} \frac{d}{d\omega} \frac{1}{1-\alpha e^{-j\omega}}$$

$$Y(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})^2}$$

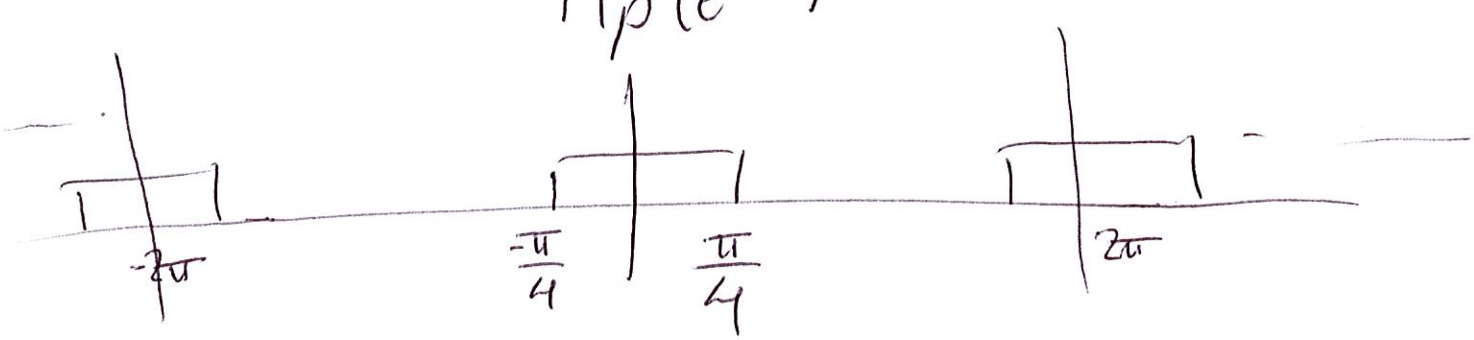
$$y[n] = (n+1)\,\alpha^{n}\,u[n+1]$$

$$= (n+1)\,\alpha^{n}\,u[n]$$

Example 5.14



$H_p(e^{j\omega})$



$$(-1)^n = e^{j\pi n}$$

$$w_1[n] = x[n]\, e^{j\pi n}$$

Frequency Shifting

$$W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$$

$$W_2(e^{j\omega}) = W_1(e^{j\omega})\, H_p(e^{j\omega})$$

$$W_3(e^{j\omega}) = W_2(e^{j(\omega-\pi)})$$

$$= W_1(e^{j(\omega-\pi)})\, H_p(e^{j(\omega-\pi)})$$

$$= X(e^{j(\omega-2\pi)})\, H_p(e^{j(\omega-\pi)})$$

$$= X(e^{j\omega})\, H_p(e^{j(\omega-\pi)})$$

HPF

$$W_4(e^{j\omega}) = X(e^{j\omega})\, H_p(e^{j\omega})$$

$$Y(e^{j\omega}) = W_3(e^{j\omega}) + W_4(e^{j\omega})$$

$$= X(e^{j\omega}) \underbrace{\left(H_p(e^{j\omega}) + H_p(e^{j(\omega-\pi)}) \right)}_{H(e^{j\omega})}$$

Band-stop Filter