ECE 301 - Lecture #31

· DTFT Multiplication Property

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x_1[n]\, x_2[n]\, e^{-j\omega n}$$

$$x_1[n] = \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})\, e^{j\theta n}\, d\theta$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_2[n] \left\{ \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta})\, e^{j\theta n}\, d\theta \right\} e^{-j\omega n}$$

$$= \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) \underbrace{\left[\sum_{n=-\infty}^{\infty} x_2[n]\, \underbrace{e^{j(\theta-\omega)n}}_{e^{-j(\omega-\theta)n}} \right]}_{X_2(e^{j(\omega-\theta)})} d\theta$$

(2)

$$Y(e^{j\omega}) = \frac{1}{2\pi} \underbrace{\int_{2\pi} X_1(e^{j\theta})\, X_2\left(e^{j(\omega-\theta)}\right) d\theta}_{\text{Periodic Convolution}}$$

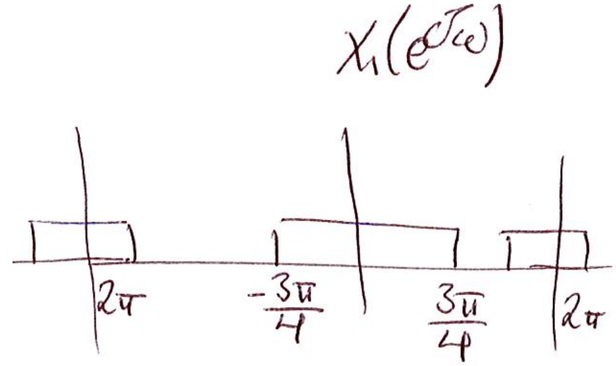
$$\hat{X}_1(e^{j\theta}) = \begin{cases} X_1(e^{j\theta}) & , \ -\pi \leq \theta \leq \pi \\ 0 & , \ \text{otherwise} \end{cases}$$

$$Y(e^{j\omega}) = \frac{1}{2\pi} \underbrace{\int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta})\, X_2\left(e^{j(\omega-\theta)}\right) d\theta}_{\text{Convolution}}$$

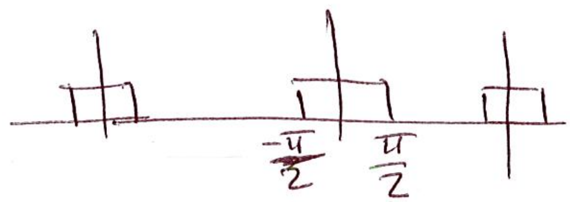
Example 5.15

$$x[n] = x_1[n]\, x_2[n]$$

$$x_1[n] = \frac{\sin\left(\frac{3\pi n}{4}\right)}{\pi n}$$



$$x_2[n] = \frac{\sin\left(\frac{\pi n}{2}\right)}{\pi n}$$



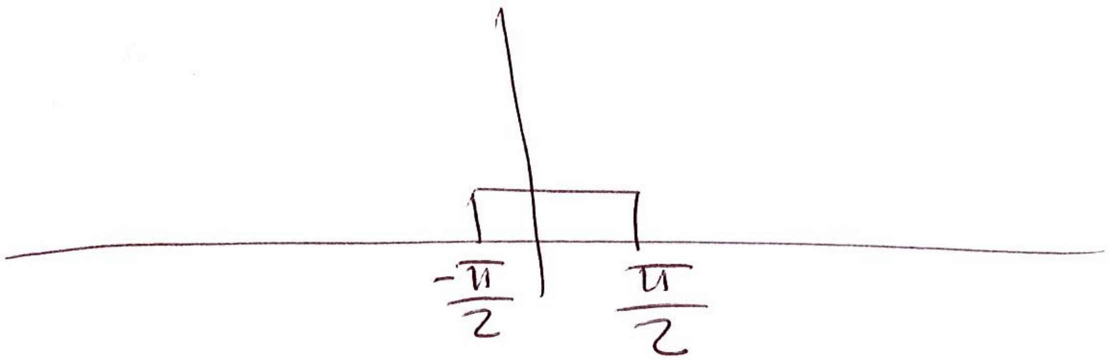
$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})\, X_2(e^{j(\omega-\theta)})\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(e^{j\theta})\, X_1(e^{j(\omega-\theta)})\, d\theta$$

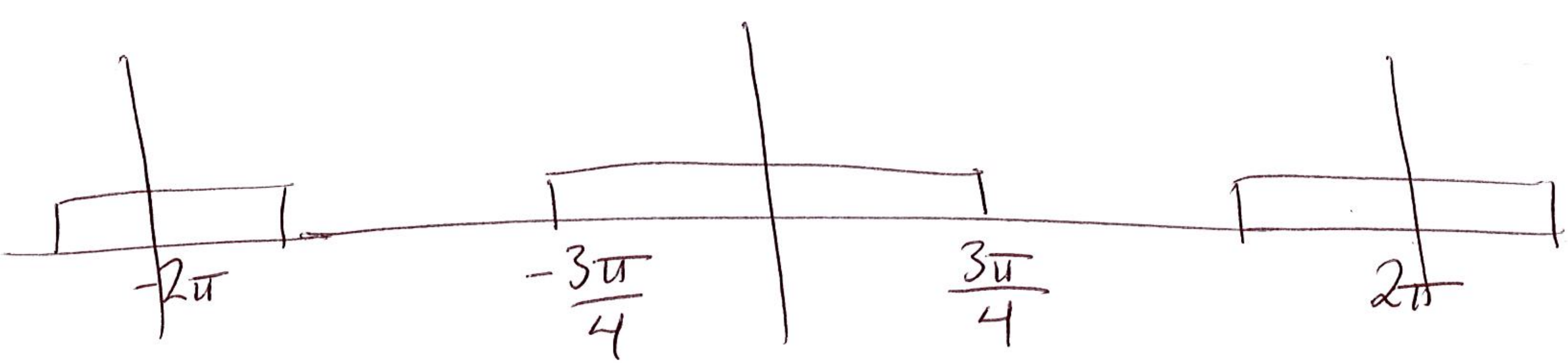
$$\hat{X}_2(e^{j\theta}) = \begin{cases} X_2(e^{j\theta}), & -\pi \le \theta \le \pi \\ 0 & \text{otherwise} \end{cases}$$

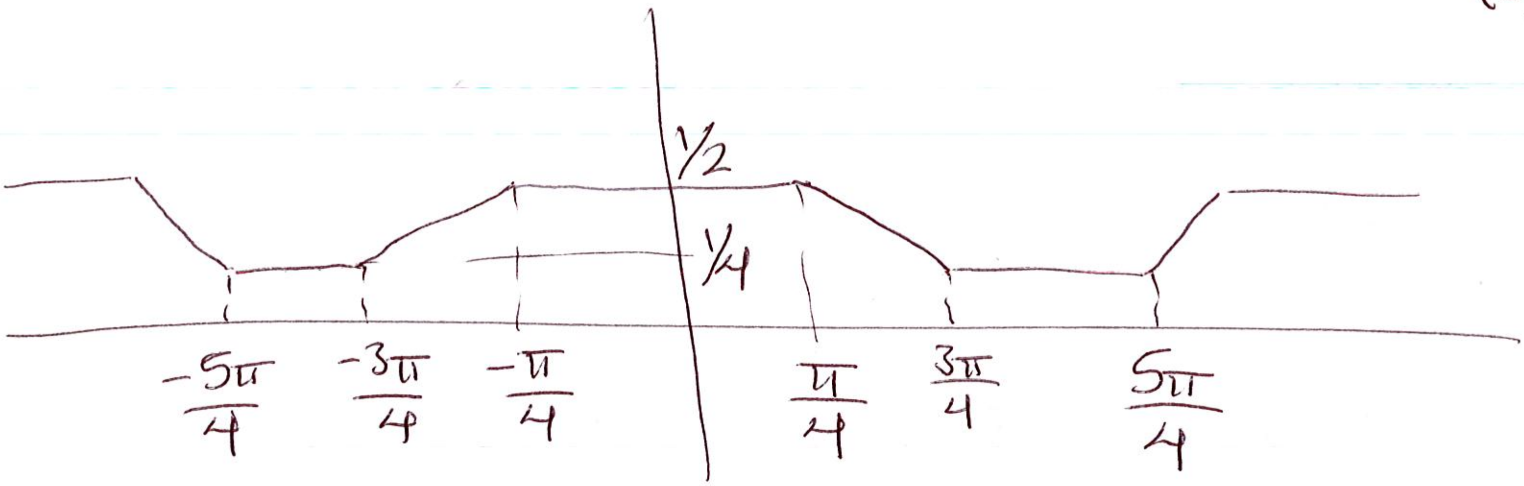
$$X(e^{j\omega}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{X}_2(e^{j\theta})\, X_1(e^{j(\omega-\theta)})\, d\theta$$

$\hat{X}_2(e^{j\theta})$



$X_1(e^{j\theta})$





Duality between DT/CT FS/FT

Time		Frequency
Cont. & Periodic	$\xleftrightarrow{\text{CTFS}}$	Discrete & (Not nec.) Periodic
Disc. & Periodic	$\xleftrightarrow{\text{DTFS}}$	Discrete & Periodic
Cont. & (Not nec.) Periodic	$\xleftrightarrow{\text{CTFT}}$	Cont. & (Not nec.) Periodic
Disc. & (Not nec.) Periodic	$\xleftrightarrow{\text{DTFT}}$	Cont. & Periodic

Duality in the Discrete-Time Fourier Series

$$
\begin{array}{cl}
x[n] & \text{periodic with period } N \\
\Big\updownarrow \text{DTFS} & \\
\{a_k\} & \text{periodic with period } N
\end{array}
$$

Theorem

If we think of $\{a_k\}$ as a discrete-time sequence, then that sequence has DTFS coefficients given by $\frac{1}{N}x[-n]$

Proof

Let's take two discrete sequences $f[\cdot]$ and $g[\cdot]$ that ~~each~~ both have period N, and are related as follows,

$$f[m] = \frac{1}{N} \sum_{r=\langle N\rangle} g[r]\, e^{-jr\left(\frac{2\pi}{N}\right)m}$$

Let $m = k$, $r = n$

$$f[k] = \frac{1}{N} \sum_{n=\langle N\rangle} g[n]\, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$\boxed{g[n] \xleftrightarrow{FS} f[k]}$$

Let $m = n$, $r = -k$

$$f[n] = \sum_{k=\langle N\rangle} \underbrace{\frac{1}{N} g[-k]}\, e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$\boxed{f[n] \xleftrightarrow{FS} \frac{1}{N} g[-k]}$$

Theorem follows by letting
$g[n] = x[n]$ & $a_k = f[k]$

Example 5.16

$$x[n] = \begin{cases} \frac{1}{9} \dfrac{\sin\left(\frac{5\pi n}{9}\right)}{\sin\left(\frac{\pi n}{9}\right)} & \text{, if } n \text{ is not a multiple of } 9 \\ \frac{5}{9} & \text{, if } n \text{ is a multiple of } 9 \end{cases}$$

$$x[n] \xleftrightarrow{DTFS} \; ?$$

We know that!

Let $g[n] = \begin{cases} 1 & , -2 \leq n \leq 2 \\ 0 & , 2 < |n| \leq 4 \end{cases}$

$N = 9$

$g[n] \xleftrightarrow{DTFS} b_k$

then $b_k = \begin{cases} \frac{1}{9} \dfrac{\sin\left(\frac{5\pi k}{9}\right)}{\sin\left(\frac{\pi k}{9}\right)} & , k \text{ is not multiple of } 9 \\ \frac{5}{9} & k \text{ is multiple of } 9 \end{cases}$

$$b_k = \frac{1}{N} \sum_{n=\langle N \rangle} g[n] e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$= \frac{1}{9} \sum_{n=-2}^{2} e^{-j\frac{2\pi nk}{9}}$$

$$\boxed{b_n = x[n] \quad \text{Note}}$$

$$x[n] = \sum_{k=-2}^{2} \frac{1}{9} e^{-j\frac{2\pi nk}{9}}$$

Replace k by $-k$

$$= \sum_{k=-2}^{2} \frac{1}{9} e^{j\frac{2\pi nk}{9}}$$

DTFS

$$a_k = \begin{cases} \frac{1}{9} & , \quad -2 \leq k \leq 2 \\ 0 & , \quad 2 < |k| \leq 4 \end{cases}$$