ECE 301 - Lecture #4

Harmonically related set of complex exponentials

$x(t) = e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t$

Periodic

Fundamenta Period T_0

$\boxed{x(t) = x(t+T_0)}$, ~~on~~ for all values of t,
and T_0 is the smallest number such that this holds.

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T_0)} \quad \text{for all } t$$

$$= e^{j\omega_0 t}\, e^{j\omega_0 T_0}$$

$\Rightarrow e^{j\omega_0 T_0} = 1 \Rightarrow \omega_0 T_0$ is a multiple of 2π

$\cos \omega_0 T_0 + j \sin \omega_0 T_0$

$\omega_0 T_0$ is a multiple of 2π

but T_0 is the smallest number such that this holds

$$\Rightarrow T_0 = \frac{2\pi}{|\omega_0|} \qquad \omega_0 : \text{fundamental frequency}$$

$\{ e^{j\omega_k t} \}$ set of harmonically related complex exponentials

k is an index

Assume positive

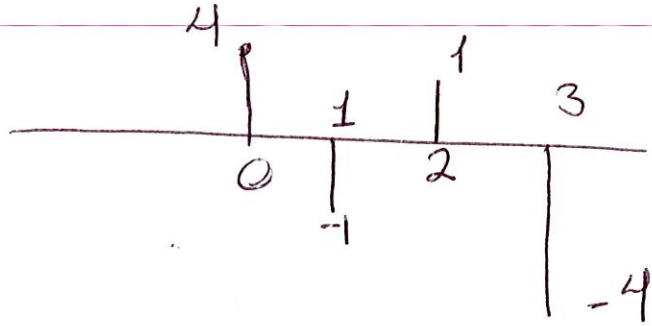
$$\omega_k = k\,\omega_0$$

fundamental Period

$$\phi_k(t) = e^{j\omega_k t} \Rightarrow T_k = \frac{2\pi}{|k|\,\omega_0} = \frac{T_0}{|k|}$$

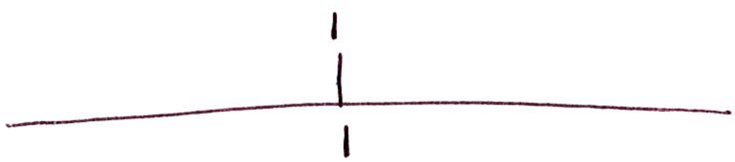
$\phi_2(t)$ is twice as fast as $\phi_1(t)$

$x[n]$

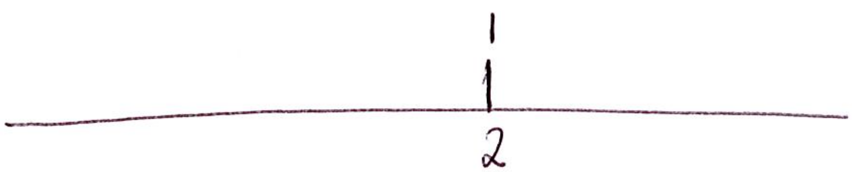


$$= 4\,x_0[n] - x_1[n] + x_2[n] - 4\,x_3[n]$$

$x_0[n]$



$x_1[n]$



$x_2[n]$

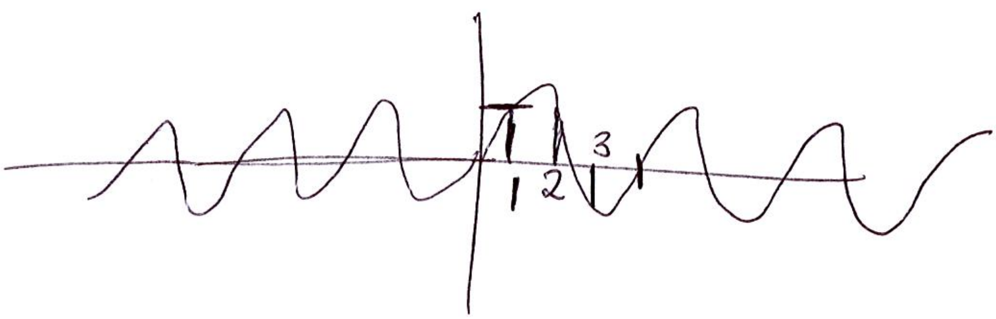
$x_3[n]$

Discrete-Time Complex Exponentials

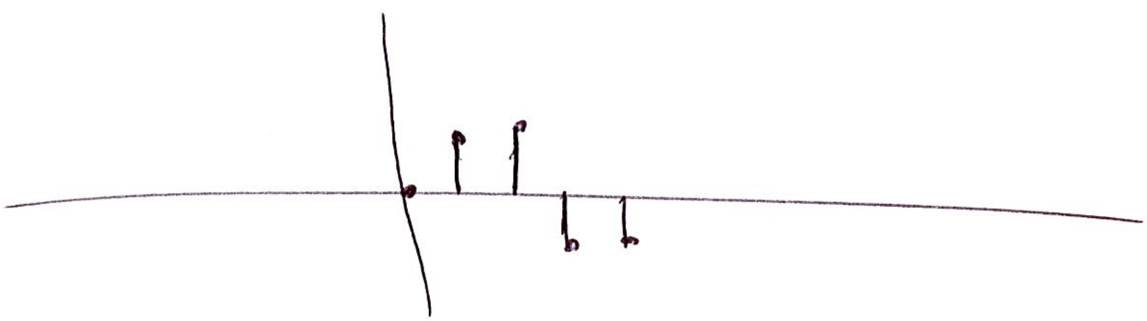
$e^{j\omega_0 n}$ Integer

$= \cos \omega_0 n + j \sin \omega_0 n$ → This signal may be periodic or not, depending on the value of ω_0

$\overset{\sin}{\cancel{\cos}}\ \underline{\omega_0}\, t$



$\overset{\sin}{\cancel{\cos}}\ \omega_0 n$



$x_1[n] = e^{j\omega_0 n}$

$x_2[n] = e^{j(\omega_0 + 2\pi) n}$

$= e^{j\omega_0 n} \underbrace{e^{j 2\pi n}}_{}$

$\cos 2\pi n + j \sin 2\pi n$

$= 1$ for all n

$\Rightarrow x_2[n] = x_1[n]$

In Discrete-Time, shifting the fundamental frequency by 2π results in the same frequency

$\Rightarrow$ All the frequency values in DT ~~are~~ are in the range $[0, 2\pi)$

what are the low frequencies?

$x[n] = e^{j\omega_0 n}$

constant, for $\boxed{\omega_0 = 0}$

Lowest frequency

What ~~is~~ are the high frequencies?

$|x[n]| = |e^{j\omega_0 n}| = 1$

$|\cos \omega_0 n + j \sin \omega_0 n|$

$= \sqrt{\cos^2 \omega_0 n + \sin^2 \omega_0 n} = 1$

Maximum change happens when $x[n]$ switches between 1 and -1 at every point in time

$e^{j\omega_0 n}$

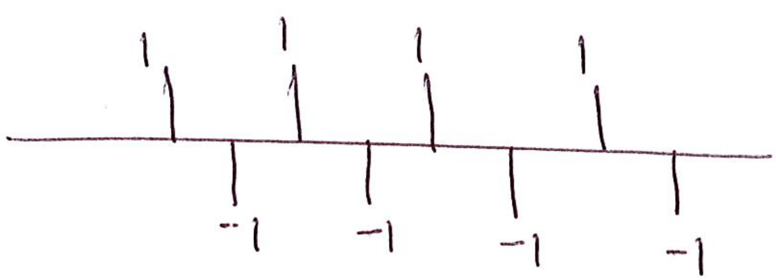
<u>when</u> does it switch between 1 and -1?

At which value of ω_0?

$\cos \omega_0 n + j \sin \omega_0 n$

$\boxed{\omega_0 = \pi}$ Highest DT frequency

$e^{j\pi n} = (-1)^n$



$e^{j\pi n} = \left(e^{j\pi}\right)^n = (-1)^n$

In DT, we go from the lowest to highest frequency as we increase ω_0 from 0 to π.

and then from the highest to lowest frequency as we decrease ω_0 from π to 0.

When is $e^{j\omega_0 n}$ periodic?

If it has a period N

$$x[n] = e^{j\omega_0 n}$$

$x[n] = x[n+N]$ for all values of n

$e^{j\omega_0 n} = e^{j\omega_0 (n+N)}$ for all values of n

$$= e^{j\omega_0 n}\, e^{j\omega_0 N}$$

$$\Rightarrow e^{j\omega_0 N} = 1$$

$\boxed{\cos \omega_0 N + j \sin \omega_0 N}$

Integer

$$\Rightarrow \omega_0 N = m\, 2\pi$$

$$\Rightarrow \frac{\omega_0}{2\pi} = \frac{m}{N}$$

rational number

$e^{j\omega_0 n}$ is periodic if and only if

$\frac{\omega_0}{2\pi}$ is a rational number

For example

$e^{j\pi n}$ $\quad \frac{\omega_0 = \pi}{2\pi} = \frac{1}{2} = \frac{m}{N}$

$\Rightarrow N = 2$

Periodic
with period 2

$e^{j\frac{3\pi}{2}n}$ $\quad \frac{\omega_0 = 3\pi/2}{2\pi} = \frac{3}{4}$

$\Rightarrow N = 4$

Periodic
with period 4