ECE 301 - Lecture #8

Why LTI Systems are easier to analyze

Consider any DT signal $x[n]$:

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[-1]\delta[n+1] + x[2]\delta[n-2] + x[-2]\delta[n+2] + \cdots$$

$$= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

If we input $x[n]$ into an LTI System:

The output $y[n] = \sum_{k=-\infty}^{\infty} x[k] \underbrace{h_k[n]}_{\text{Output of } \delta[n-k]}$ (By Linearity)

Impulse Response

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k],$$

where $\boxed{h[n] \text{ is the output of } \delta[n]}$

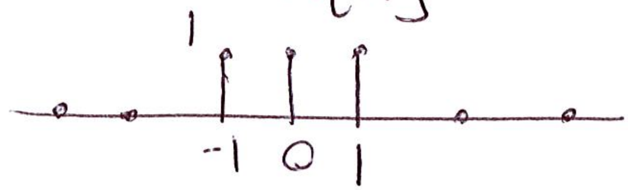
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$\triangleq x[n] * h[n]$$

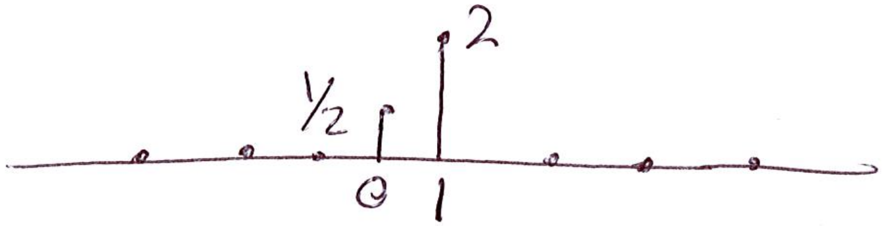
Convolution Sum

Example

h[n]



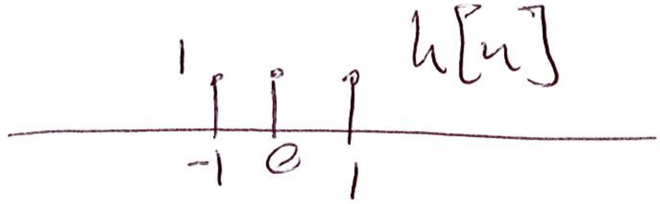
x[n]



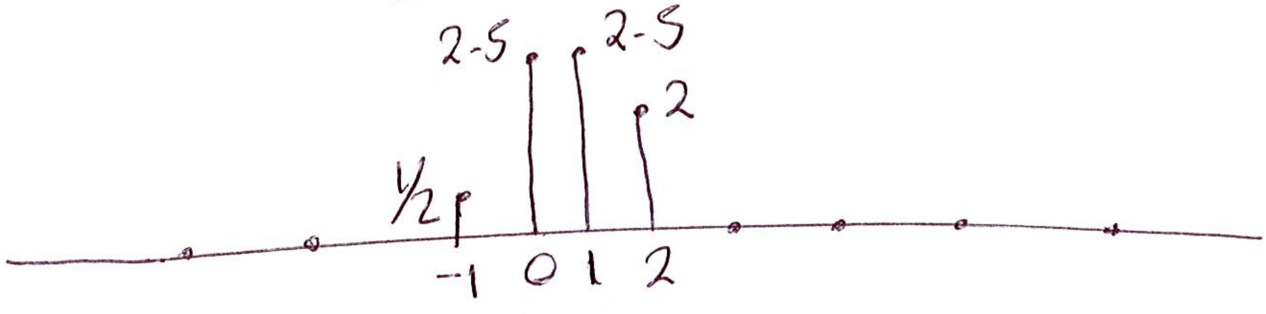
$$x[n] * h[n] = \sum_{k=0}^{1} x[k]\, h[n-k]$$

$$= x[0]\, h[n] + x[1]\, h[n-1]$$

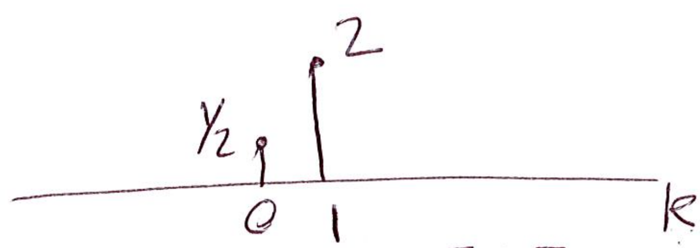
$$= \frac{1}{2} h[n] + 2\, h[n-1]$$

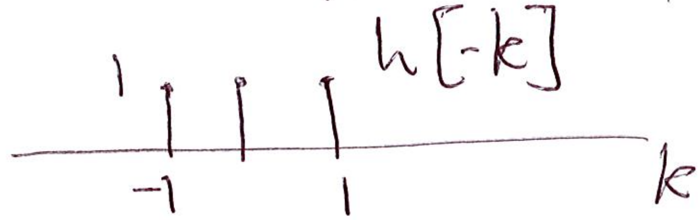


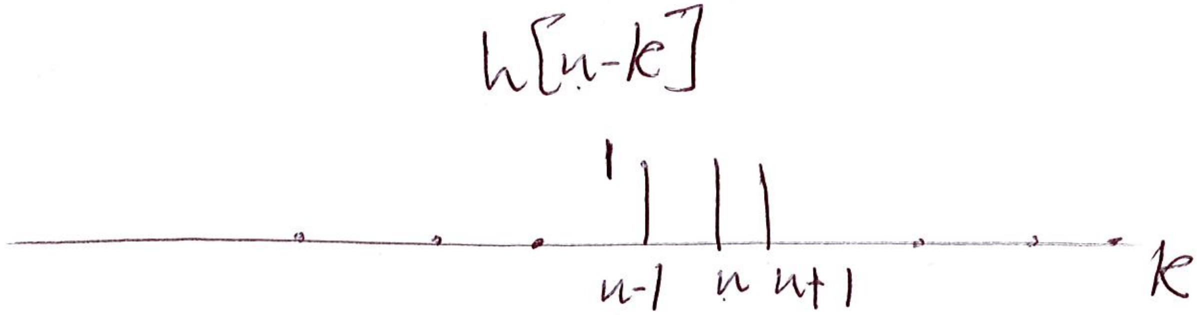
$$y[n] = \frac{1}{2} h[n] + 2h[n-1]$$

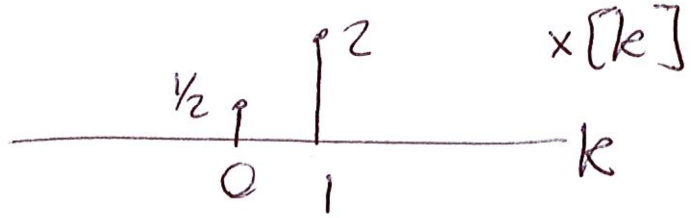


$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

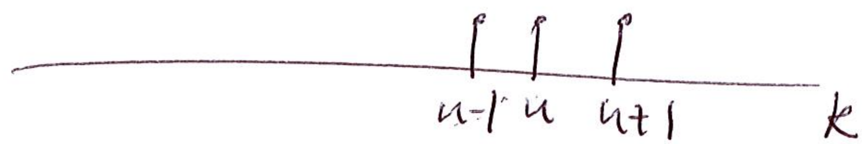








$h[n-k]$



$$\sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

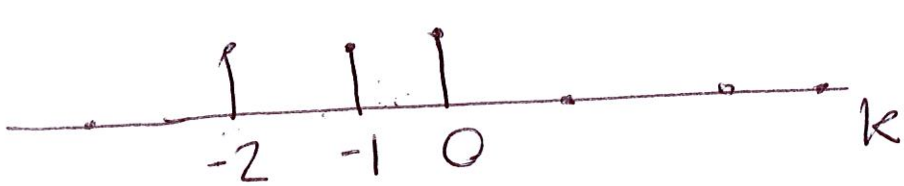
If $n-1 > 1$, then $y[n]=0$

If $n+1 < 0$, then $y[n]=0$

$y[n] \neq 0$ only for $n \in \{-1, 0, 1, 2\}$

<u>At $n=-1$</u>

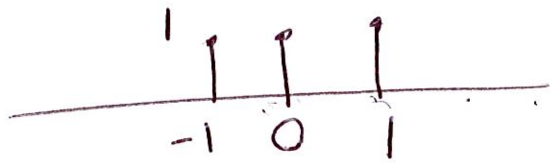
$h[-1-k]$



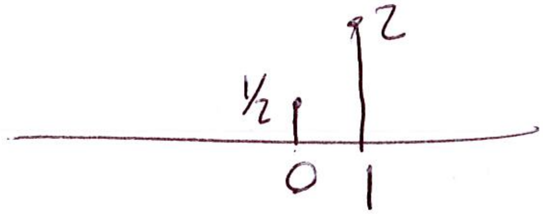
$$y[-1] = \sum_{k=-\infty}^{\infty} x[+k]\, h[-1-k] = \frac{1}{2}$$

At $n=0$

$h[0-k]$



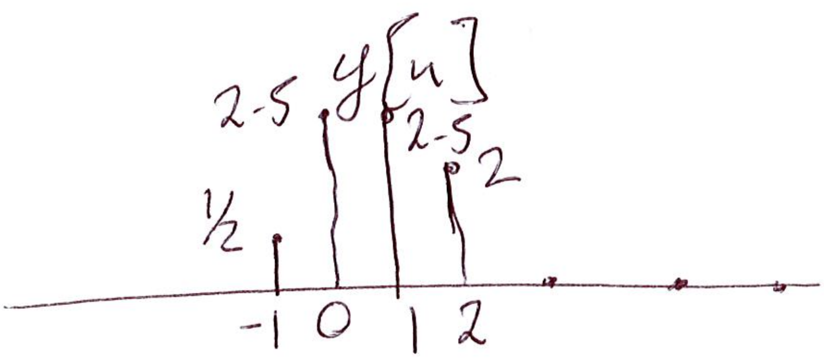
$x[k]$



$$y[0] = \sum_{k=-\infty}^{\infty} x[k]\, h[0-k] = 2.5$$

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]\, h[1-k] = 2.5$$

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]\, h[2-k] = 2$$

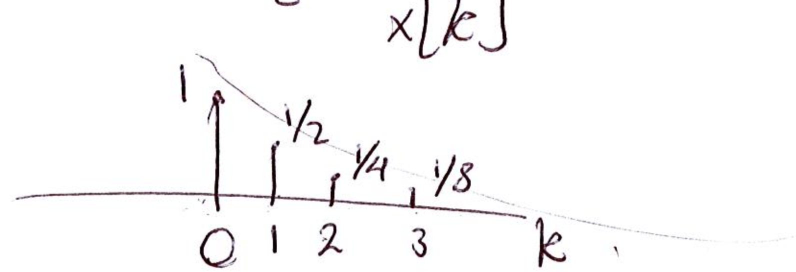


Example 2-3

Real number

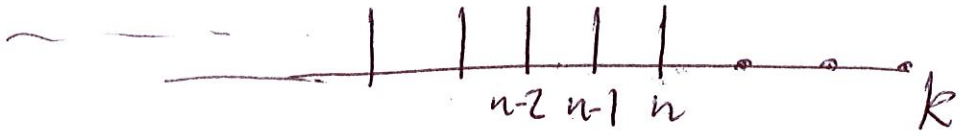
$$x[n] = \underbrace{\alpha^n u[n]}_{\text{Unit Step}}, \quad 0 < \alpha < 1$$

$\alpha = \frac{1}{2}$



$h[n] = u[n] \leftarrow$ Impulse Response

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$



If $n < 0$, $y[n] = 0$

If $n \geq 0$

$$y[n] = \sum_{k=0}^{n} \alpha^k$$

$$= \left(\frac{1-\alpha^{n+1}}{1-\alpha} \right) u[n] \quad (*)$$

Let's verify (*)

$n \longrightarrow \infty$

$$y[n] = \sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}$$

$$\alpha = \frac{1}{2} \Rightarrow 2$$

$$\sum_{k=0}^{n} \alpha^k = \sum_{k=0}^{\infty} \alpha^k - \sum_{k=n+1}^{\infty} \alpha^k$$

$$= \sum_{k=0}^{\infty} \alpha^k - \alpha^{n+1} \sum_{k=0}^{\infty} \alpha^k$$

$$h[n] = u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Output

Input

Accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$x[k] = \alpha^k$$