ECE 301: Signals and Systems Homework Assignment #3

Due on October 14, 2015

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Consider a causal LTI system S whose input x[n] and output y[n] are related by the difference equation

$$2y[n] - y[n-1] + y[n-3] = x[n] - 5x[n-4].$$

(a) Verify that S may be considered a cascade connection of two causal LTI systems S_1 and S_2 with the following input-output relationship:

$$S_1 : 2y_1[n] = x_1[n] - 5x_1[n-4],$$

$$S_2 : y_2[n] = \frac{1}{2}y_2[n-1] - \frac{1}{2}y_2[n-3] + x_2[n]$$

- (b) Draw a block diagram representation of S_1 .
- (c) Draw a block diagram representation of S_2 .
- (d) Draw a block diagram representation of S as a cascade connection of the block diagram representation of S_1 followed by the block diagram representation of S_2 .
- (e) Draw a block diagram representation of S as a cascade connection of the block diagram representation of S_2 followed by the block diagram representation of S_1 .
- (f) Show that the four delay elements in the block diagram representation of S obtained in part (e) may be collapsed to three. The resulting block diagram is referred to as a *Direct Form II* realization of S, while the block diagrams obtained in parts (d) and (e) are referred to as *Direct Form I* realizations of S.

Solution

(a) Realizing that $x_2[n] = y_1[n]$, we may eliminate these from the two given difference equations. This would give us

$$2y_2[n] - y_2[n-1] + y_2[n-3] = x_1[n] - 5x_1[n-4].$$

This is the same as the overall difference equation.

(b) The figures corresponding to the remaining parts of this problem are shown in Figure 1.



Figure 1: The block diagram representations in sub-questions (b), (c), (d), (e), (f).

Problem 2

Determine the Fourier series representations for the following signals. You shall not only give the Fourier series coefficients, but also give the Fourier series expression of the signals.

- (a) Each x[n] or x(t) illustrated in Figure 2.
- (b) x(t) periodic with period 2 and

$$x(t) = e^{-t}$$
 for $-1 \le t \le 1$.

(c) x(t) periodic with period 4 and

$$x(t) = \begin{cases} \sin(\pi t), & 0 \le t \le 2, \\ 0, & 2 < t \le 4. \end{cases}$$



Figure 2: The signal of x[n] or x(t) in sub-question (a).

Solution

(a) (i) It is known that N = 2. So that

$$a_{0} = \frac{1}{N} \sum_{N} x[n] e^{-jk(2\pi/N)n}$$
$$= \frac{1}{2} \sum_{n=0}^{1} x[n] e^{-j \cdot 0 \cdot \pi n}$$
$$= \frac{1}{2} \sum_{n=0}^{1} x[n]$$
$$= -\frac{1}{2}$$

Meanwhile,

$$a_{k} = \frac{1}{N} \sum_{N} x[n] e^{-jk(2\pi/N)r}$$
$$= \frac{1}{2} \sum_{n=0}^{1} x[n] e^{-jk\pi n}$$
$$= \frac{1}{2} - (-1)^{k}$$

The Fourier series representation is

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/N)n} = \sum_{k=-\infty}^{\infty} (\frac{1}{2} - (-1)^k) e^{jk\pi n}$$

(ii) It is known that T = 6. So that

$$a_0 = \frac{1}{N} \int_T x(t) e^{-jk(2\pi/T)t} dt$$
$$= \frac{1}{6} \int_0^6 x(t) e^{-j \cdot 0 \cdot (\pi/3)t} dt$$
$$= \frac{1}{6} \int_0^6 x(t) dt$$
$$= 0$$

Meanwhile,

$$\begin{aligned} a_k &= \frac{1}{T} \int_T x(t) e^{-jk(2\pi/T)t} dt \\ &= \frac{1}{6} \int_0^6 x(t) e^{-jk(\pi/3)t} dt \\ &= \frac{1}{6} \{ -(\int_1^2 e^{-jk(\pi/3)t} dt) + (\int_4^5 e^{-jk(\pi/3)t} dt) \} \\ &= \frac{1}{jk\pi} (\cos(\frac{2k\pi}{3}) - \cos(\frac{k\pi}{3})) \end{aligned}$$

Note that not only $a_0 = 0$, but also $a_k even = 0$. The Fourier series representation is

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{jk\pi} \left(\cos(\frac{2k\pi}{3}) - \cos(\frac{k\pi}{3})\right)\right) e^{jk(\pi/3)t} \ (k \neq 0)$$

(b) It is known that T = 2. So that

$$a_{k} = \frac{1}{T} \int_{T} x(t) e^{-jk(2\pi/T)t} dt$$
$$= \frac{1}{2} \int_{-1}^{1} e^{-t} e^{-jk\pi t} dt$$
$$= \frac{1}{2} \int_{-1}^{1} e^{(-jk\pi-1)t} dt$$
$$= \frac{1}{2(1+jk\pi)} [e^{1+j\pi k} - e^{-1-j\pi k}]$$

for all k.

The Fourier series representation is

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2(1+jk\pi)} \left[e^{1+j\pi k} - e^{-1-j\pi k}\right]\right) e^{jk\pi t}$$

(c) It is known that T = 4. So that

$$a_{0} = \frac{1}{T} \int_{T} x(t) e^{-jk2\pi/Tt} dt$$

= $\frac{1}{4} \int_{0}^{2} sin(\pi t) e^{-j \cdot 0 \cdot (2\pi/4)t} dt$
= $\frac{1}{4} \int_{0}^{2} sin(\pi t) dt$
= 0

$$\begin{aligned} a_k &= \frac{1}{T} \int_T x(t) e^{-jk2\pi/Tt} dt \\ &= \frac{1}{4} \int_0^2 \sin(\pi t) e^{-jk(\pi/2)t} dt \\ &= \frac{1}{8j} \int_0^2 (e^{j\pi t} - e^{-j\pi t}) e^{-jk(\pi/2)t} dt \\ &= \frac{1}{8j} [\frac{e^{j\pi(2-k)-1}}{j\pi(1-k/2)} + \frac{e^{j\pi(2+k)-1}}{j\pi(1+k/2)}] \\ &= \frac{e^{-j\pi k}}{\pi(k^2 - 4)} = \frac{(-1)^k}{\pi(k^2 - 4)} \end{aligned}$$

The Fourier series representation is

$$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t} = \sum_{k=-\infty}^{\infty} \left(\frac{(-1)^k}{\pi(k^2 - 4)}\right) e^{jk(\pi/2)t} \ (k \neq 0)$$

Consider the following three continuous-time signals with a fundamental period of T = 1/2:

$$\begin{aligned} x(t) &= \cos(4\pi t), \\ y(t) &= \sin(4\pi t), \\ z(t) &= x(t)y(t). \end{aligned}$$

- (a) Determine the Fourier series coefficients of x(t).
- (b) Determine the Fourier series coefficients of y(t).
- (c) Use the results of parts (a) and (b), along with the *multiplication* property of the continuous-time Fourier series, to determine the Fourier series coefficients of z(t) = x(t)y(t).
- (d) Determine the Fourier series coefficients of z(t) through direct expansion of z(t) in trigonometric form, and compare your result with that of part(c).

Note that you shall only give the nonzero Fourier series coefficients as answers.

Solution

(a)

$$x(t) = \cos(4\pi t) = \frac{1}{2}e^{4\pi t} + \frac{1}{2}e^{-4\pi t}$$

So that the nonzero FS coefficients of x(t) are $a_1 = a_{-1} = 1/2$.

(b)

$$y(t) = \sin(4\pi t) = \frac{1}{2j}e^{4\pi t} - \frac{1}{2j}e^{-4\pi t}$$

So that the nonzero FS coefficients of x(t) are $b_1 = 1/2j$, $b_{-1} = -1/2j$.

(c) Using the *multiplication* property, we know that

$$z(t) = x(t)y(t) \stackrel{FS}{\leftrightarrow} c_k = \sum_{t=-\infty}^{\infty} a_l b_{k-l}$$

Therefore,

$$c_k = a_k * b_k = \sum_{t=-\infty}^{\infty} a_l b_{k-l} = \frac{1}{4j} \delta[k-2] - \frac{1}{4j} \delta[k+2]$$

This implies that the nonzero Fourier series coefficients of z(t) are $c_2 = \frac{1}{4j}$, $c_{-2} = -\frac{1}{4j}$.

(d) We have

$$z(t) = \sin(4t)\cos(4t) = \frac{1}{2}\sin(8t) = \frac{1}{4j}e^{8\pi t} - \frac{1}{4j}e^{-8\pi t}$$

Therefore, the nonzero Fourier series coefficients of z(t) are $c_2 = \frac{1}{4j}$, $c_{-2} = -\frac{1}{4j}$, the same with part (c).

Let x(t) be a periodic signal whose Fourier series coefficients are

$$a_k = \begin{cases} 2, & k = 0\\ j(\frac{1}{2})^{|k|}, & \text{otherwise} \end{cases}$$

Known that the period of signal x(t) is T, use the Fourier series properties to answer the following questions:

- (a) Is x(t) real?
- (b) Is x(t) even?
- (c) Is $\frac{dx(t)}{dt}$ even?

Solution

(a) If x(t) is real, then $x(t) = x^*(t)$. This implies that for x(t), $a_k = a^*_{-k}$. Since this is not true in this case, for example,

$$a_1 = j(\frac{1}{2}) \neq -j(\frac{1}{2}) = a_{-1}^*.$$

Therefore, x(t) is not real.

(b) If x(t) is even, then x(t) = x(-t). This implies that for x(t), $a_k = a_{-k}$. Since this is true in this case, because,

$$a_k = j(\frac{1}{2})^{|k|} = j(\frac{1}{2})^{|k|} = a_{-k}.$$

Therefore, x(t) is even.

(c) We have

$$g(t) = \frac{dx(t)}{dt} \stackrel{FS}{\leftrightarrow} b_k = jk\frac{2\pi}{T}a_k,$$

where T refer to the period of signal x(t). Therefore,

$$b_k = \begin{cases} 0, & k = 0 \\ -k(1/2)^{|k|}(2\pi/T), & \text{otherwise} \end{cases}$$

Since b_k is not even, g(t) is not even.

Let

$$x[n] = \begin{cases} 1, & 0 \le n \le 7\\ 0, & 8 \le n \le 9 \end{cases}$$

be a periodic signal with fundamental period N = 10 and Fourier series coefficients a_k . Also, let

$$g[n] = x[n] - x[n-1]$$

- (a) Draw the graph of signal g[n], and determine the fundamental period of g[n].
- (b) Determine the Fourier series coefficients of g[n].
- (c) Using the Fourier series coefficients of g[n] and the First-Difference property, determine a_k for $k \neq 0$.

Solution

(a) For $0 \le n \le 9$, We have

$$g[n] = x[n] - x[n-1] = \begin{cases} 1, & n = 0\\ 0, & 1 \le n \le 7\\ -1, & n = 8\\ 0, & n = 9 \end{cases}$$

This period begin to show again in the following 10 points, it's clearly to draw the conclusion that g[n] is periodic with period of 10. The graph of signal g[n] is shown in Figure 3.



Figure 3: The graph of signal g[n].

(b) It is known that T = 10. So that the FS coefficients of g[n] is b_k , which is

$$b_k = \frac{1}{N} \sum_N g[n] e^{-jk(2\pi/N)n}$$
$$= \frac{1}{10} \sum_{n=0}^9 g[n] e^{-jk(\pi/5)n}$$
$$= \frac{1}{10} (1 - e^{-j8k(\pi/5)})$$

(c) Since g[n] = x[n] - x[n-1], the FS coefficients a_k and b_k must be related as

$$b_k = a_k - e^{-jk(\pi/5)}a_k$$

Therefore,

$$a_k = \frac{b_k}{1 - e^{-jk(\pi/5)}} = \frac{1}{10} \frac{1 - e^{-j8k(\pi/5)}}{1 - e^{-jk(\pi/5)}}$$