# ECE 301: Signals and Systems Homework Assignment \#3 

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Professor: Aly El Gamal
TA: Xianglun Mao

## Problem 1

Consider a causal LTI system $S$ whose input $x[n]$ and output $y[n]$ are related by the difference equation

$$
2 y[n]-y[n-1]+y[n-3]=x[n]-5 x[n-4] .
$$

(a) Verify that $S$ may be considered a cascade connection of two causal LTI systems $S_{1}$ and $S_{2}$ with the following input-output relationship:

$$
\begin{aligned}
& S_{1}: 2 y_{1}[n]=x_{1}[n]-5 x_{1}[n-4], \\
& S_{2}: y_{2}[n]=\frac{1}{2} y_{2}[n-1]-\frac{1}{2} y_{2}[n-3]+x_{2}[n]
\end{aligned}
$$

(b) Draw a block diagram representation of $S_{1}$.
(c) Draw a block diagram representation of $S_{2}$.
(d) Draw a block diagram representation of $S$ as a cascade connection of the block diagram representation of $S_{1}$ followed by the block diagram representation of $S_{2}$.
(e) Draw a block diagram representation of $S$ as a cascade connection of the block diagram representation of $S_{2}$ followed by the block diagram representation of $S_{1}$.
(f) Show that the four delay elements in the block diagram representation of $S$ obtained in part (e) may be collapsed to three. The resulting block diagram is referred to as a Direct Form II realization of $S$, while the block diagrams obtained in parts (d) and (e) are referred to as Direct Form I realizations of $S$.

## Solution

(a) Realizing that $x_{2}[n]=y_{1}[n]$, we may eliminate these from the two given difference equations. This would give us

$$
2 y_{2}[n]-y_{2}[n-1]+y_{2}[n-3]=x_{1}[n]-5 x_{1}[n-4] .
$$

This is the same as the overall difference equation.
(b) The figures corresponding to the remaining parts of this problem are shown in Figure 1.


Figure 1: The block diagram representations in sub-questions (b), (c), (d), (e), (f).

## Problem 2

Determine the Fourier series representations for the following signals. You shall not only give the Fourier series coefficients, but also give the Fourier series expression of the signals.
(a) Each $x[n]$ or $x(t)$ illustrated in Figure 2.
(b) $x(t)$ periodic with period 2 and

$$
x(t)=e^{-t} \text { for }-1 \leq t \leq 1
$$

(c) $x(t)$ periodic with period 4 and

$$
x(t)= \begin{cases}\sin (\pi t), & 0 \leq t \leq 2 \\ 0, & 2<t \leq 4\end{cases}
$$



Figure 2: The signal of $x[n]$ or $x(t)$ in sub-question (a).

## Solution

(a) (i) It is known that $N=2$. So that

$$
\begin{aligned}
a_{0} & =\frac{1}{N} \sum_{N} x[n] e^{-j k(2 \pi / N) n} \\
& =\frac{1}{2} \sum_{n=0}^{1} x[n] e^{-j \cdot 0 \cdot \pi n} \\
& =\frac{1}{2} \sum_{n=0}^{1} x[n] \\
& =-\frac{1}{2}
\end{aligned}
$$

Meanwhile,

$$
\begin{aligned}
a_{k} & =\frac{1}{N} \sum_{N} x[n] e^{-j k(2 \pi / N) n} \\
& =\frac{1}{2} \sum_{n=0}^{1} x[n] e^{-j k \pi n} \\
& =\frac{1}{2}-(-1)^{k}
\end{aligned}
$$

The Fourier series representation is

$$
x[n]=\sum_{k=-\infty}^{\infty} a_{k} e^{j k(2 \pi / N) n}=\sum_{k=-\infty}^{\infty}\left(\frac{1}{2}-(-1)^{k}\right) e^{j k \pi n}
$$

(ii) It is known that $T=6$. So that

$$
\begin{aligned}
a_{0} & =\frac{1}{N} \int_{T} x(t) e^{-j k(2 \pi / T) t} d t \\
& =\frac{1}{6} \int_{0}^{6} x(t) e^{-j \cdot 0 \cdot(\pi / 3) t} d t \\
& =\frac{1}{6} \int_{0}^{6} x(t) d t \\
& =0
\end{aligned}
$$

Meanwhile,

$$
\begin{aligned}
a_{k} & =\frac{1}{T} \int_{T} x(t) e^{-j k(2 \pi / T) t} d t \\
& =\frac{1}{6} \int_{0}^{6} x(t) e^{-j k(\pi / 3) t} d t \\
& =\frac{1}{6}\left\{-\left(\int_{1}^{2} e^{-j k(\pi / 3) t} d t\right)+\left(\int_{4}^{5} e^{-j k(\pi / 3) t} d t\right)\right\} \\
& =\frac{1}{j k \pi}\left(\cos \left(\frac{2 k \pi}{3}\right)-\cos \left(\frac{k \pi}{3}\right)\right)
\end{aligned}
$$

Note that not only $a_{0}=0$, but also $a_{k \text { even }}=0$.
The Fourier series representation is

$$
x[n]=\sum_{k=-\infty}^{\infty} a_{k} e^{j k(2 \pi / T) t}=\sum_{k=-\infty}^{\infty}\left(\frac{1}{j k \pi}\left(\cos \left(\frac{2 k \pi}{3}\right)-\cos \left(\frac{k \pi}{3}\right)\right)\right) e^{j k(\pi / 3) t}(k \neq 0)
$$

(b) It is known that $T=2$. So that

$$
\begin{aligned}
a_{k} & =\frac{1}{T} \int_{T} x(t) e^{-j k(2 \pi / T) t} d t \\
& =\frac{1}{2} \int_{-1}^{1} e^{-t} e^{-j k \pi t} d t \\
& =\frac{1}{2} \int_{-1}^{1} e^{(-j k \pi-1) t} d t \\
& =\frac{1}{2(1+j k \pi)}\left[e^{1+j \pi k}-e^{-1-j \pi k}\right]
\end{aligned}
$$

for all $k$.
The Fourier series representation is

$$
x[n]=\sum_{k=-\infty}^{\infty} a_{k} e^{j k(2 \pi / T) t}=\sum_{k=-\infty}^{\infty}\left(\frac{1}{2(1+j k \pi)}\left[e^{1+j \pi k}-e^{-1-j \pi k}\right]\right) e^{j k \pi t}
$$

(c) It is known that $T=4$. So that

$$
\begin{aligned}
a_{0} & =\frac{1}{T} \int_{T} x(t) e^{-j k 2 \pi / T t} d t \\
& =\frac{1}{4} \int_{0}^{2} \sin (\pi t) e^{-j \cdot 0 \cdot(2 \pi / 4) t} d t \\
& =\frac{1}{4} \int_{0}^{2} \sin (\pi t) d t \\
& =0 \\
a_{k} & =\frac{1}{T} \int_{T} x(t) e^{-j k 2 \pi / T t} d t \\
& =\frac{1}{4} \int_{0}^{2} \sin (\pi t) e^{-j k(\pi / 2) t} d t \\
& =\frac{1}{8 j} \int_{0}^{2}\left(e^{j \pi t}-e^{-j \pi t}\right) e^{-j k(\pi / 2) t} d t \\
& =\frac{1}{8 j}\left[\frac{e^{j \pi(2-k)-1}}{j \pi(1-k / 2)}+\frac{e^{j \pi(2+k)-1}}{j \pi(1+k / 2)}\right] \\
& =\frac{e^{-j \pi k}}{\pi\left(k^{2}-4\right)}=\frac{(-1)^{k}}{\pi\left(k^{2}-4\right)}
\end{aligned}
$$

The Fourier series representation is

$$
x[n]=\sum_{k=-\infty}^{\infty} a_{k} e^{j k(2 \pi / T) t}=\sum_{k=-\infty}^{\infty}\left(\frac{(-1)^{k}}{\pi\left(k^{2}-4\right)}\right) e^{j k(\pi / 2) t}(k \neq 0)
$$

## Problem 3

Consider the following three continuous-time signals with a fundamental period of $T=1 / 2$ :

$$
\begin{aligned}
& x(t)=\cos (4 \pi t), \\
& y(t)=\sin (4 \pi t), \\
& z(t)=x(t) y(t) .
\end{aligned}
$$

(a) Determine the Fourier series coefficients of $x(t)$.
(b) Determine the Fourier series coefficients of $y(t)$.
(c) Use the results of parts (a) and (b), along with the multiplication property of the continuous-time Fourier series, to determine the Fourier series coefficients of $z(t)=x(t) y(t)$.
(d) Determine the Fourier series coefficients of $z(t)$ through direct expansion of $z(t)$ in trigonometric form, and compare your result with that of part(c).

Note that you shall only give the nonzero Fourier series coefficients as answers.

## Solution

(a)

$$
x(t)=\cos (4 \pi t)=\frac{1}{2} e^{4 \pi t}+\frac{1}{2} e^{-4 \pi t}
$$

So that the nonzero FS coefficients of $x(t)$ are $a_{1}=a_{-1}=1 / 2$.
(b)

$$
y(t)=\sin (4 \pi t)=\frac{1}{2 j} e^{4 \pi t}-\frac{1}{2 j} e^{-4 \pi t}
$$

So that the nonzero FS coefficients of $x(t)$ are $b_{1}=1 / 2 j, b_{-1}=-1 / 2 j$.
(c) Using the multiplication property, we know that

$$
z(t)=x(t) y(t) \stackrel{F S}{\leftrightarrow} c_{k}=\sum_{t=-\infty}^{\infty} a_{l} b_{k-l}
$$

Therefore,

$$
c_{k}=a_{k} * b_{k}=\sum_{t=-\infty}^{\infty} a_{l} b_{k-l}=\frac{1}{4 j} \delta[k-2]-\frac{1}{4 j} \delta[k+2]
$$

This implies that the nonzero Fourier series coefficients of $z(t)$ are $c_{2}=\frac{1}{4 j}, c_{-2}=-\frac{1}{4 j}$.
(d) We have

$$
z(t)=\sin (4 t) \cos (4 t)=\frac{1}{2} \sin (8 t)=\frac{1}{4 j} e^{8 \pi t}-\frac{1}{4 j} e^{-8 \pi t}
$$

Therefore, the nonzero Fourier series coefficients of $z(t)$ are $c_{2}=\frac{1}{4 j}, c_{-2}=-\frac{1}{4 j}$, the same with part (c).

## Problem 4

Let $x(t)$ be a periodic signal whose Fourier series coefficients are

$$
a_{k}= \begin{cases}2, & k=0 \\ j\left(\frac{1}{2}\right)^{|k|}, & \text { otherwise }\end{cases}
$$

Known that the period of signal $x(t)$ is $T$, use the Fourier series properties to answer the following questions:
(a) Is $x(t)$ real?
(b) Is $x(t)$ even?
(c) Is $\frac{d x(t)}{d t}$ even?

## Solution

(a) If $x(t)$ is real, then $x(t)=x^{*}(t)$. This implies that for $x(t), a_{k}=a_{-k}^{*}$. Since this is not true in this case, for example,

$$
a_{1}=j\left(\frac{1}{2}\right) \neq-j\left(\frac{1}{2}\right)=a_{-1}^{*}
$$

Therefore, $x(t)$ is not real.
(b) If $x(t)$ is even, then $x(t)=x(-t)$. This implies that for $x(t), a_{k}=a_{-k}$. Since this is true in this case, because,

$$
a_{k}=j\left(\frac{1}{2}\right)^{|k|}=j\left(\frac{1}{2}\right)^{|k|}=a_{-k}
$$

Therefore, $x(t)$ is even.
(c) We have

$$
g(t)=\frac{d x(t)}{d t} \stackrel{F S}{\leftrightarrow} b_{k}=j k \frac{2 \pi}{T} a_{k},
$$

where $T$ refer to the period of signal $x(t)$. Therefore,

$$
b_{k}= \begin{cases}0, & k=0 \\ -k(1 / 2)^{|k|}(2 \pi / T), & \text { otherwise }\end{cases}
$$

Since $b_{k}$ is not even, $g(t)$ is not even.

## Problem 5

Let

$$
x[n]= \begin{cases}1, & 0 \leq n \leq 7 \\ 0, & 8 \leq n \leq 9\end{cases}
$$

be a periodic signal with fundamental period $N=10$ and Fourier series coefficients $a_{k}$. Also, let

$$
g[n]=x[n]-x[n-1]
$$

(a) Draw the graph of signal $g[n]$, and determine the fundamental period of $g[n]$.
(b) Determine the Fourier series coefficients of $g[n]$.
(c) Using the Fourier series coefficients of $g[n]$ and the First-Difference property, determine $a_{k}$ for $k \neq 0$.

## Solution

(a) For $0 \leq n \leq 9$, We have

$$
g[n]=x[n]-x[n-1]= \begin{cases}1, & n=0 \\ 0, & 1 \leq n \leq 7 \\ -1, & n=8 \\ 0, & n=9\end{cases}
$$

This period begin to show again in the following 10 points, it's clearly to draw the conclusion that $g[n]$ is periodic with period of 10 . The graph of signal $g[n]$ is shown in Figure 3.


Figure 3: The graph of signal $g[n]$.
(b) It is known that $T=10$. So that the FS coefficients of $g[n]$ is $b_{k}$, which is

$$
\begin{aligned}
b_{k} & =\frac{1}{N} \sum_{N} g[n] e^{-j k(2 \pi / N) n} \\
& =\frac{1}{10} \sum_{n=0}^{9} g[n] e^{-j k(\pi / 5) n} \\
& =\frac{1}{10}\left(1-e^{-j 8 k(\pi / 5)}\right)
\end{aligned}
$$

(c) Since $g[n]=x[n]-x[n-1]$, the FS coefficients $a_{k}$ and $b_{k}$ must be related as

$$
b_{k}=a_{k}-e^{-j k(\pi / 5)} a_{k}
$$

Therefore,

$$
a_{k}=\frac{b_{k}}{1-e^{-j k(\pi / 5)}}=\frac{1}{10} \frac{1-e^{-j 8 k(\pi / 5)}}{1-e^{-j k(\pi / 5)}}
$$

