ECE 301: Signals and Systems
Homework Assignment #5

Due on November 11, 2015

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Problem 1

Compute the Fourier transform of each of the following signals

(a)

$$x(t) = \begin{cases} 1 + cos(\pi t), & |t| \leq 1 \\ 0, & |t| > 1 \end{cases}$$

(b)

$$\sum_{k=0}^{\infty} a^k \delta(t - kT), |a| < 1$$

(c)

$$[te^{-2t} sin(4t)]u(t)$$

(d)

$$x(t) = \begin{cases} 1 - t^2, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

(e) $x(t)$ as show in Figure 1.

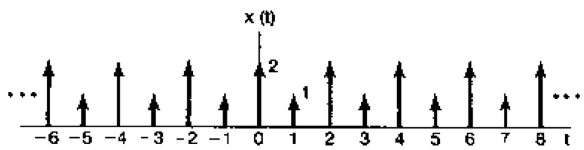

Figure 1: The graph of signal $x(t)$ in (e).

Solution

(a)

$$\begin{aligned} X(jw) &= \int_{-\infty}^{\infty} x(t)e^{-jwt}dt \\ &= \int_{-1}^{1} (1 + cos(\pi t))e^{-jwt}dt \\ &= \frac{2sin(w)}{w} + \frac{sin(w)}{\pi - w} - \frac{sin(w)}{\pi + w} \end{aligned}$$

(b)

$$\begin{aligned} X(jw) &= \int_{-\infty}^{\infty} x(t)e^{-jwt}dt \\ &= \int_{-\infty}^{\infty} (\sum_{k=0}^{\infty} a^k e^{-jwkT} \delta(t - kT))dt \\ &= \sum_{k=0}^{\infty} a^k e^{-jwkT} \\ &= \frac{1}{1 - \alpha e^{-jwT}} \end{aligned}$$

(c) We have

$$x(t) = (1/2j)te^{-2t}e^{j4t}u(t) - (1/2j)e^{-2t}e^{-j4t}u(t).$$

Therefore,

$$X(jw) = \frac{1/2j}{(2 - j4 + jw)^2} - \frac{1/2j}{(2 + j4 - jw)^2}.$$

(d)

$$\begin{aligned} X(jw) &= \int_{-\infty}^{\infty} x(t)e^{-jwt}dt \\ &= \int_{0}^{1} (1 - t^2)e^{-jwt}dt \\ &= \frac{1}{jw} + \frac{2e^{-jw}}{-w^2} - \frac{2e^{-jw} - 2}{jw^2} \end{aligned}$$

(e) If

$$x_1(t) = \sum_{-\infty}^{\infty} \delta(t - 2k),$$

then

$$x(t) = x_1(t) + x_1(t - 1)$$

Therefore,

$$X(jw) = X_1(jw)[2 + e^{-w}] = \pi \sum_{k=-\infty}^{\infty} \delta(w - k\pi)[2 + (-1)^k]$$

Problem 2

Consider the signal

$$x_0(t) = \begin{cases} e^{-t}, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

Determine the Fourier transform of each of the signals shown in Figure 2. You should be able to do this by explicitly evaluating *only* the transform of $x_0(t)$ and then using properties of the Fourier transform.

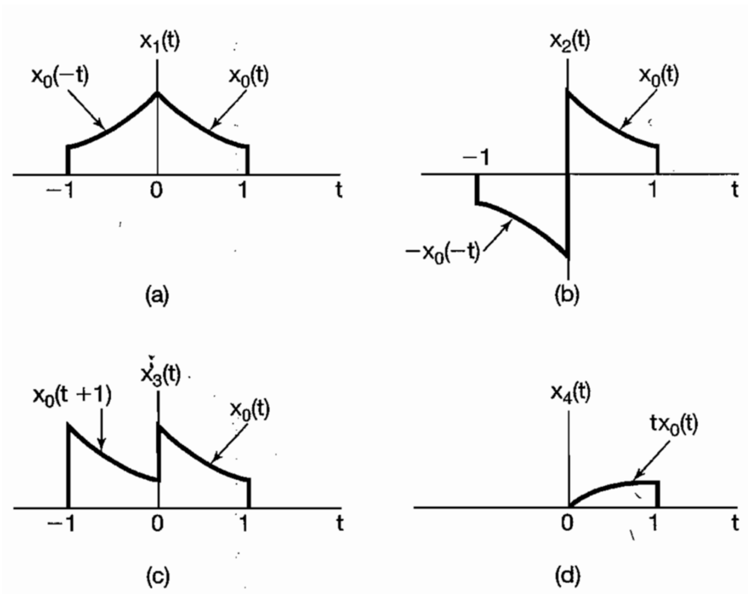

Figure 2: The graph of signals $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$.

Solution

For the given signal $x_0(t)$, we use the Fourier transform analysis to evaluate the corresponding Fourier transform

$$X_0(jw) = \frac{1 - e^{-(1+jw)}}{1 + jw}.$$

(a) We know that

$$x_1(t) = x_0(t) + x_0(-t).$$

Using the linearity and time reversal properties of the Fourier transform we have

$$X_1(jw) = X_0(jw) + X_0(-jw) = \frac{2 - 2e^{-1}cos(w) - 2we^{-1}sin(w)}{1 + w^2}$$

(b) We know that

$$x_2(t) = x_0(t) - x_0(-t).$$

Using the linearity and time reversal properties of the Fourier transform we have

$$X_2(jw) = X_0(jw) - X_0(-jw) = j[\frac{-2w + 2e^{-1}sin(w) + 2we^{-1}cos(w)}{1 + w^2}]$$

Problem 2 continued on next page. . .

(c) We know that

$$x_3(t) = x_0(t) + x_0(t+1).$$

Using the linearity and time reversal properties of the Fourier transform we have

$$X_3(jw) = X_0(jw) + e^{jw} X_0(-jw) = \frac{1 + e^{jw} - e^{-1}(1 + e^{-jw})}{1 + jw}$$

(d) We know that

$$x_4(t) = t x_0(t).$$

Using the linearity and time reversal properties of the Fourier transform we have

$$X_4(jw) = j\frac{d}{dw} X_0(jw) = \frac{1 - 2e^{-1}e^{-jw} - jwe^{-1}e^{-jw}}{(1 + jw)^2}$$

Problem 3

Determine which, if any, of the real signals depicted in Figure 3 have Fourier tranforms that satisfy each of the following conditions:

(a) $\mathfrak{Re}\{X(jw)\} = 0$

(b) $\mathfrak{Im}\{X(jw)\} = 0$

(c) There exists a real α such that $e^{j\alpha w}X(jw)$ is real

(d) $\int_{-\infty}^{\infty} X(jw)dw = 0$

(e) $\int_{-\infty}^{\infty} wX(jw)dw = 0$

(f) $X(jw)$ is periodic

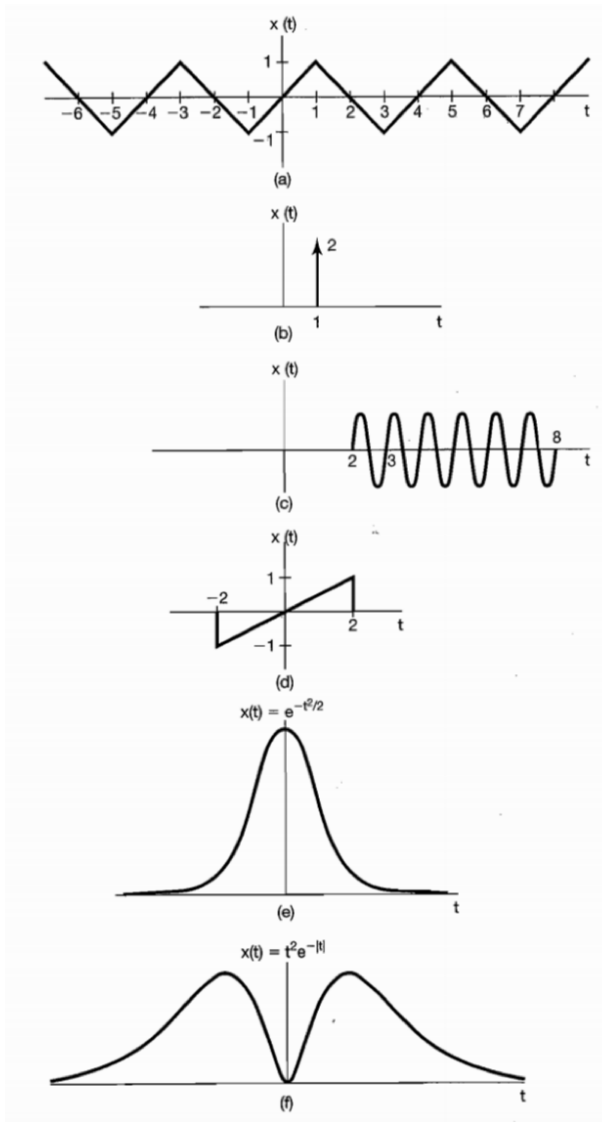

Figure 3: The graph of real signals (a), (b), (c), (d), (e), (f).

Problem 3 continued on next page. . .

Solution

(a) For $\mathfrak{Re}\{X(jw)\} = 0$, the signal $x(t)$ must be real and odd. Therefore, signals in figures (a) and (d) have this property.

(b) For $\mathfrak{Im}\{X(jw)\} = 0$, the signal $x(t)$ must be real and even. Therefore, signals in figures (e) and (f) have this property.

(c) For there to exist a real α such that $e^{j\alpha w}X(jw)$ is real, we require that $x(t+\alpha)$ be a real and even signal. Therefore, signals in figures (a), (b), (e), and (f) have this property.

(d) For this condition to be true, $x(0) = 0$. Therefore, signals in figures (a), (b), (c), (d), and (f) have this property.

(e) For this condition to be true the derivative of $x(t)$ has to be zero at $t = 0$. Therefore, signals in figures (b), (c), (e), and (f) have this property.

(f) The signal in figure (a) and (b) have this property. Figure (a) has this property because it's the integration of shifted square waves. The Fourier transform of the signal that shown in Figure (b) is $X(jw) = 2e^{-jw}$, which is periodic.

Problem 4

The input and the output of a stable and causal LTI system are related by the differential equation

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

(a) Find the impulse response of this system.

(b) What is the response of this system if $x(t) = te^{-2t}u(t)$?

(c) Repeat part (a) for the stable and causal LTI system described by the equation

$$\frac{d^2y(t)}{dt^2} + \sqrt{2}\frac{dy(t)}{dt} + y(t) = 2\frac{d^2x(t)}{dt^2} - 2x(t)$$

Solution

(a) Taking the Fourier transform of both sides of the given differential equation, we obtain

$$-w^2Y(jw) + 6jwY(jw) + 8Y(jw) = 2X(jw)$$

Therefore, we have

$$H(jw) = \frac{Y(jw)}{X(jw)} = \frac{2}{-w^2 + 6jw + 8}$$

Using partial fraction expansion, we obtain

$$H(jw) = \frac{1}{jw+2} - \frac{1}{jw+4}$$

Taking the inverse Fourier transform,

$$h(t) = e^{-2t}u(t) - e^{-4t}u(t)$$

(b) For the given signal $x(t) = te^{-2t}u(t)$, we have

$$X(jw) = \frac{1}{(2+jw)^2}.$$

Therefore,

$$Y(jw) = X(jw)H(jw) = \frac{2}{(-w^2+6jw+8)}\frac{1}{(2+jw)^2}.$$

Using partial fraction expansion, we obtain

$$Y(jw) = \frac{1/4}{jw+2} - \frac{1/2}{(jw+2)^2} + \frac{1}{(jw+2)^3} - \frac{1/4}{jw+4}$$

Taking the inverse Fourier transform,

$$y(t) = \frac{1}{4}e^{-2t}u(t) - \frac{1}{2}te^{-2t}u(t) + t^2e^{-2t}u(t) - \frac{1}{4}e^{-4t}u(t)$$

(c) Taking the Fourier transform of both sides of the given differential equation, we obtain

$$-w^2Y(jw) + \sqrt{2}jwY(jw) + Y(jw) = -2w^2X(jw) - 2X(jw)$$

Problem 4 continued on next page...

Therefore, we have

$$H(jw) = \frac{Y(jw)}{X(jw)} = \frac{2(-w^2 - 1)}{-w^2 + \sqrt{2}jw + 1}$$

Using partial fraction expansion, we obtain

$$H(jw) = 2 + \frac{-\sqrt{2} - 2\sqrt{2}j}{jw - \frac{-\sqrt{2}+j\sqrt{2}}{2}} + \frac{-\sqrt{2} + 2\sqrt{2}j}{jw - \frac{-\sqrt{2}-j\sqrt{2}}{2}}$$

Taking the inverse Fourier transform,

$$h(t) = 2\delta(t) - \sqrt{2}(1+2j)e^{-(1+j)t/\sqrt{2}}u(t) - \sqrt{2}(1-2j)e^{-(1-j)t/\sqrt{2}}u(t)$$

Problem 5

A causal and stable LTI system S has the frequency response

$$H(jw) = \frac{jw + 4}{6 - w^2 + 5jw}$$

(a) Determine a differential equation relating the input $x(t)$ and output $y(t)$ of S.

(b) Determine the impulse response $h(t)$ of S.

(c) What is the output of S when the input is

$$x(t) = e^{-4t}u(t) - te^{-4t}u(t)?$$

Solution

(a) We have

$$\frac{Y(jw)}{X(jw)} = \frac{jw + 4}{6 - w^2 + 5jw}.$$

Cross-multiplying and taking the inverse Fourier transform, we obtain

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t)$$

(b) We have

$$\begin{aligned} H(jw) &= \frac{jw + 4}{6 - w^2 + 5jw} \\ &= \frac{2}{2 + jw} - \frac{1}{3 + jw} \end{aligned}$$

Taking the inverse Fourier transform we obtain,

$$h(t) = 2e^{-2t}u(t) - e^{-3t}u(t)$$

(c) For

$$x(t) = e^{-4t}u(t) - te^{-4t}u(t),$$

we have

$$X(jw) = \frac{1}{4 + jw} - \frac{1}{(4 + jw)^2}.$$

Therefore,

$$Y(jw) = X(jw)H(jw) = \frac{1}{(4 + jw)(2 + jw)} = \frac{1/2}{2 + jw} - \frac{1/2}{4 + jw}$$

Finding the partial fraction expansion of $Y(jw)$ and taking the inverse Fourier transform,

$$y(t) = \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-4t}u(t),$$