ECE 301: Signals and Systems
Homework Assignment #7

Due on December 11, 2015

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Note: Homework 7 will have only 4 problems since this is the last homework. The first 2 problems are assigned 10 pts, and the last 2 problems are assigned 15 pts. Enjoy the last homework!

Problem 1

A signal $x(t)$ with Fourier transform $X(jw)$ undergoes impulse-train sampling to generate

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

where $T = 10^{-4}$. For each of the following sets of constraints on $x(t)$ and/or $X(jw)$, does the sampling theorem guarantee that $x(t)$ can be recovered exactly from $x_p(t)$?

(a) $X(jw) = 0$ for $|w| > 5000\pi$

(b) $X(jw) = 0$ for $|w| > 15000\pi$

(c) $\text{Re}\{X(jw)\} = 0$ for $|w| > 5000\pi$

(d) $x(t)$ real and $X(jw) = 0$ for $w > 5000\pi$

(e) $x(t)$ real and $X(jw) = 0$ for $w < -15000\pi$

(f) $X(jw) * X(jw) = 0$ for $|w| > 15000\pi$

(g) $|X(jw)| = 0$ for $w > 5000\pi$

Solution

(a) The Nyquist rate for the given signal is $2 \times 5000\pi = 10000\pi$. Therefore, in order to be able to recover $x(t)$ from $x_p(t)$, the sampling period must at most be $T_{max} = \frac{2\pi}{10000\pi} = 2 \times 10^{-4}$ sec. Since the sampling period used is $T = 10^{-4} < T_{max}$, $x(t)$ can be recovered from $x_p(t)$.

(b) The Nyquist rate for the given signal is $2 \times 15000\pi = 30000\pi$. Therefore, in order to be able to recover $x(t)$ from $x_p(t)$, the sampling period must at most be $T_{max} = \frac{2\pi}{30000\pi} = 0.66 \times 10^{-4}$ sec. Since the sampling period used is $T = 10^{-4} > T_{max}$, $x(t)$ cannot be recovered from $x_p(t)$.

(c) Here, $\text{Im}\{X(jw)\}$ is not specified. Therefore, the Nyquist rate for the signal $x(t)$ is indeterminate. This implies that one cannot guarantee that $x(t)$ would be recoverable from $x_p(t)$.

(d) Since $x(t)$ is real, we may conclude that $X(jw) = 0$ for $|w| > 5000\pi$. Therefore, the answer to this part is identical to that of part (a).

(e) Since $x(t)$ is real, we may conclude that $X(jw) = 0$ for $|w| > 15000\pi$. Therefore, the answer to this part is identical to that of part (b).

(f) If $X(jw) = 0$ for $|w| > w_1$, then $X(jw) * X(jw) = 0$ for $|w| > 2w_1$. Therefore, in this part, $X(jw) = 0$ for $|w| > 7500\pi$. The Nyquist rate for this signal is $2 \times 7500\pi = 15000\pi$. Therefore, in order to be able to recover $x(t)$ from $x_p(t)$, the sampling period must at most be $T_{max} = \frac{2\pi}{15000\pi} = 1.33 \times 10^{-4}$ sec. Since the sampling period used is $T = 10^{-4} < T_{max}$, $x(t)$ can be recovered from $x_p(t)$.

(g) If $|X(jw)| = 0$ for $w > 5000\pi$, then $X(jw) = 0$ for $w > 5000\pi$. Therefore, the answer to this part is identical to the answer of part (a).

Problem 2

A signal $x[n]$ has a Fourier transform $X(e^{jw})$ that is zero for $\frac{\pi}{4} \leq |w| \leq \pi$. Another signal

$$g[n] = x[n] \sum_{k=-\infty}^{\infty} \delta[n - 1 - 4k]$$

is generated. Specify the frequency response $H(e^{jw})$ of a lowpass filter that produces $x[n]$ as output when $g[n]$ is the input.

Solution

Let $p[n] = \sum_{k=-\infty}^{\infty} \delta[n - 1 - 4k]$. Then we obtain

$$P(e^{jw}) = e^{-jw} \frac{2\pi}{4} \sum_{k=-\infty}^{\infty} \delta(w - 2\pi k/4) = \frac{\pi}{2} \sum_{k=-\infty}^{\infty} e^{-j\frac{2\pi k}{4}} \delta(w - 2\pi k/4)$$

Therefore,

$$\begin{aligned} G(e^{jw}) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{j\theta} X(e^{j(w-\theta)} d\theta)) \\ &= \frac{1}{4} \sum_{k=0}^{3} e^{-j2\pi k/4} X(e^{j(w-2\pi k/4}) \end{aligned}$$

Since $X(e^{jw}) = 0$ for $\pi/4 \leq |w| \leq \pi$, $G(e^{jw})$ is as shown in Figure 1.

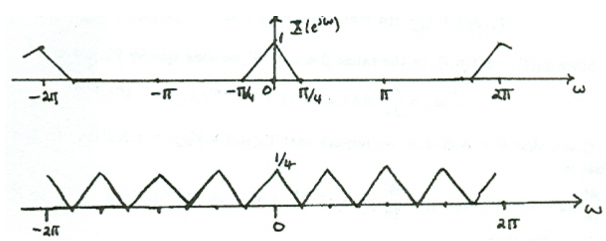

Figure 1: The signal of $X(e^{jw})$ and $G(e^{jw})$.

Clearly, in order to isolate just $X(e^{jw})$ we need to use an ideal lowpass filter with cutoff frequency $\pi/4$ and passband gain of 4. Therefore, in the range $|w| < \pi$,

$$H(e^{jw}) = \begin{cases} 4, & |w| \leq \pi/4 \\ 0, & \pi/4 \leq |w| \leq \pi \end{cases}$$

Problem 3

Let $x(t)$ be a band-limited signal such that $X(jw) = 0$ for $|w| \geq \frac{\pi}{T}$.

(a) If $x(t)$ is sampled using a sampling period T, determine an interpolating function $g(t)$ such that

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} x(nT)g(t - nT).$$

(b) Is the function $g(t)$ unique?

Solution

(a) Let us denote the sampled signal by $x_p(t)$. We have

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT).$$

Since the Nyquist rate for the signal $x(t)$ is $\frac{2\pi}{T}$, we can reconstruct the signal $x_p(t)$. We then know that

$$x(t) = x_p(t) * h(t),$$

where

$$h(t) = \frac{sin(\pi t/T)}{\pi t/T}.$$

Therefore,

$$\frac{dx(t)}{dt} = x_p(t) * \frac{dh(t)}{dt}.$$

Denoting $\frac{dx(t)}{dt}$ by $g(t)$, we have

$$\frac{dx(t)}{dt} = x_p(t) * g(t) = \sum_{n=-\infty}^{\infty} x(nT)g(t - nT).$$

Therefore,

$$g(t) = \frac{dh(t)}{dt} = \frac{cos(\pi t/T)}{t} - \frac{T sin(\pi t/T)}{\pi t^2}.$$

(b) The function $g(t)$ is not unique. We can easily contruct a counterexample which shows that it is different than the $g(t)$ shown above. However, the proof is omitted.

Problem 4

In this problem we develop the dual to the time-domain sampling theorem, whereby a time-limited signal can be reconstructed from *frequency-domain* samples. To develop this result, consider the frequency-domain sampling operation in Figure 2.

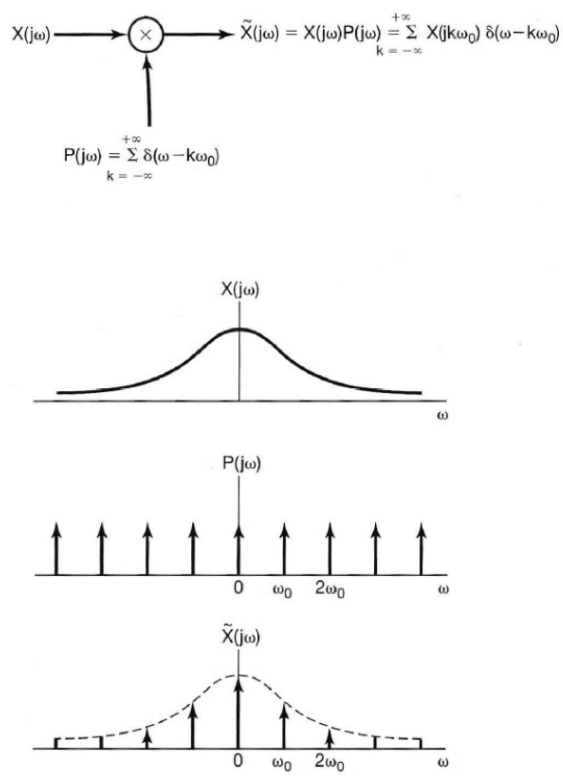

Figure 2: The frequency-domain sampling operation.

(a) Show that

$$\tilde{x}(t) = x(t) * p(t)$$

where $\tilde{x}(t)$, $x(t)$, and $p(t)$ are the inverse Fourier transforms of $\tilde{X}(jw)$, $X(jw)$, and $P(jw)$, respectively.

(b) Assuming that $x(t)$ is time-limited so that $x(t) = 0$ for $|t| \geq \frac{\pi}{w_0}$, show that $x(t)$ can be obtained from $\tilde{x}(t)$ through a "low-time windowing" operation. That is,

$$x(t) = \tilde{x}(t)w(t)$$

where

$$w(t) = \begin{cases} w_0, & |t| \leq \frac{\pi}{w_0} \\ 0, & |t| > -\frac{\pi}{w_0} \end{cases}$$

(c) Show that $x(t)$ is not recoverable from $\tilde{x}(t)$ if $x(t)$ is not constrained to be zero for $|t| \geq \frac{\pi}{w_0}$.

Problem 4 continued on next page. . .

Solution

(a) Since,

$$\tilde{X}(jw) = X(jw)P(jw),$$

we have

$$\tilde{x}(t) = x(t) * p(t).$$

(b) Taking the inverse Fourier transform of $P(jw)$, we have

$$p(t) = \frac{1}{w_0} \sum_{k=-\infty}^{\infty} \delta(t - \frac{2\pi k}{w_0}).$$

From part (a), we have

$$\begin{aligned}\tilde{x}(t) &= x(t) * p(t). \\ &= \frac{1}{w_0} \sum_{k=-\infty}^{\infty} x(t - \frac{2\pi k}{w_0}).\end{aligned}$$

Noting that $x(t)$ is time-limited so that $x(t) = 0$ for $|t| > \frac{\pi}{w_0}$. we assume that $x(t)$ is as shown in Figure 3. Then $\tilde{x}(t)$ is as shown in the figure next to it. Clearly, $x(t)$ can be recovered from $\tilde{x}(t)$ by multiplying it with the function

$$w(t) = \begin{cases} w_0, & |t| \leq \frac{\pi}{w_0} \\ 0, & \text{othwise} \end{cases}$$

(c) If $x(t)$ is not contrained to be zero for $|t| > \frac{\pi}{w_0}$, then $\tilde{x}(t)$ is as shown in Figure 3. Clearly, there is "time-domain aliasing" between the replicas of $x(t)$ in $\tilde{x}(t)$. Therefore, $x(t)$ cannot be recovered from $\tilde{x}(t)$.

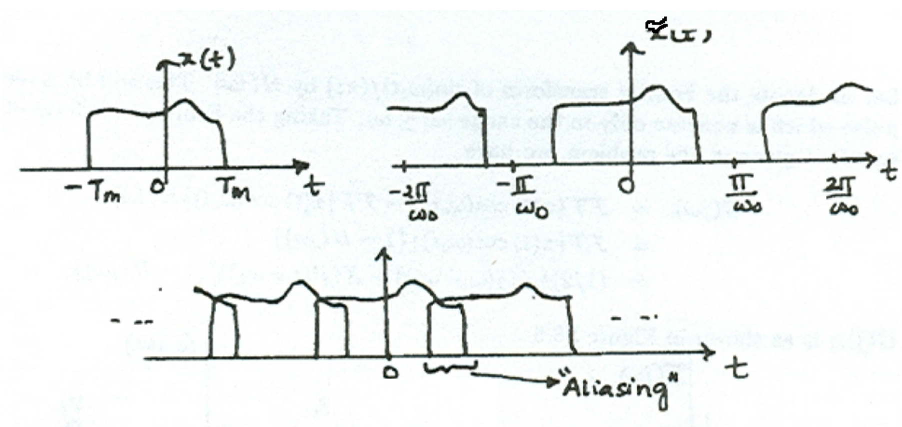

Figure 3: The signal of $x(t)$ (top left), $\tilde{x}(t)$ obtained in part (b) (top right), and $\tilde{x}(t)$ obtained in part (c) (bottom) .