Note: Homework 7 will have only 4 problems since this is the last homework. The first 2 problems are assigned 10 pts, and the last 2 problems are assigned 15 pts. Enjoy the last homework!

Problem 1

A signal \( x(t) \) with Fourier transform \( X(jw) \) undergoes impulse-train sampling to generate

\[
x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)
\]

where \( T = 10^{-4} \). For each of the following sets of constraints on \( x(t) \) and/or \( X(jw) \), does the sampling theorem guarantee that \( x(t) \) can be recovered exactly from \( x_p(t) \)?

(a) \( X(jw) = 0 \) for \( |w| > 5000\pi \)
(b) \( X(jw) = 0 \) for \( |w| > 15000\pi \)
(c) \( \text{Re}\{X(jw)\} = 0 \) for \( |w| > 5000\pi \)
(d) \( x(t) \) real and \( X(jw) = 0 \) for \( w > 5000\pi \)
(e) \( x(t) \) real and \( X(jw) = 0 \) for \( w < -15000\pi \)
(f) \( X(jw) \ast X(jw) = 0 \) for \( |w| > 15000\pi \)
(g) \( |X(jw)| = 0 \) for \( w > 5000\pi \)

Solution

(a) The Nyquist rate for the given signal is \( 2 \times 5000\pi = 10000\pi \). Therefore, in order to be able to recover \( x(t) \) from \( x_p(t) \), the sampling period must at most be \( T_{\text{max}} = \frac{2\pi}{10000\pi} = 2 \times 10^{-4} \) sec. Since the sampling period used is \( T = 10^{-4} < T_{\text{max}} \), \( x(t) \) can be recovered from \( x_p(t) \).

(b) The Nyquist rate for the given signal is \( 2 \times 15000\pi = 30000\pi \). Therefore, in order to be able to recover \( x(t) \) from \( x_p(t) \), the sampling period must at most be \( T_{\text{max}} = \frac{2\pi}{30000\pi} = 0.66 \times 10^{-4} \) sec. Since the sampling period used is \( T = 10^{-4} > T_{\text{max}} \), \( x(t) \) cannot be recovered from \( x_p(t) \).

(c) Here, \( \text{Im}\{X(jw)\} \) is not specified. Therefore, the Nyquist rate for the signal \( x(t) \) is indeterminate. This implies that one cannot guarantee that \( x(t) \) would be recoverable from \( x_p(t) \).

(d) Since \( x(t) \) is real, we may conclude that \( X(jw) = 0 \) for \( |w| > 5000\pi \). Therefore, the answer to this part is identical to that of part (a).

(e) Since \( x(t) \) is real, we may conclude that \( X(jw) = 0 \) for \( |w| > 15000\pi \). Therefore, the answer to this part is identical to that of part (b).

(f) If \( X(jw) = 0 \) for \( |w| > w_1 \), then \( X(jw) \ast X(jw) = 0 \) for \( |w| > 2w_1 \). Therefore, in this part, \( X(jw) = 0 \) for \( |w| > 7500\pi \). The Nyquist rate for this signal is \( 2 \times 7500\pi = 15000\pi \). Therefore, in order to be able to recover \( x(t) \) from \( x_p(t) \), the sampling period must at most be \( T_{\text{max}} = \frac{2\pi}{15000\pi} = 1.33 \times 10^{-4} \) sec. Since the sampling period used is \( T = 10^{-4} < T_{\text{max}} \), \( x(t) \) can be recovered from \( x_p(t) \).

(g) If \( |X(jw)| = 0 \) for \( w > 5000\pi \), then \( X(jw) = 0 \) for \( w > 5000\pi \). Therefore, the answer to this part is identical to the answer of part (a).
Problem 2

A signal $x[n]$ has a Fourier transform $X(e^{jw})$ that is zero for $\pi/4 \leq |w| \leq \pi$. Another signal

$$g[n] = x[n] \sum_{k=-\infty}^{\infty} \delta[n-1-4k]$$

is generated. Specify the frequency response $H(e^{jw})$ of a lowpass filter that produces $x[n]$ as output when $g[n]$ is the input.

Solution

Let $p[n] = \sum_{k=-\infty}^{\infty} \delta[n-1-4k]$. Then we obtain

$$P(e^{jw}) = e^{-jw \frac{2\pi}{4}} \sum_{k=-\infty}^{\infty} \delta(w - 2\pi k/4) = \frac{\pi}{2} \sum_{k=-\infty}^{\infty} e^{-j \frac{2\pi k}{4}} \delta(w - 2\pi k/4)$$

Therefore,

$$G(e^{jw}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{j\theta}) X(e^{j(w-\theta)} d\theta)$$

$$= \frac{1}{4} \sum_{k=0}^{3} e^{-j 2\pi k/4} X(e^{j(w-2\pi k/4)})$$

Since $X(e^{jw}) = 0$ for $\pi/4 \leq |w| \leq \pi$, $G(e^{jw})$ is as shown in Figure 1.

![Figure 1: The signal of $X(e^{jw})$ and $G(e^{jw})$.](image)

Clearly, in order to isolate just $X(e^{jw})$ we need to use an ideal lowpass filter with cutoff frequency $\pi/4$ and passband gain of 4. Therefore, in the range $|w| < \pi$,

$$H(e^{jw}) = \begin{cases} 4, & |w| \leq \pi/4 \\ 0, & \pi/4 \leq |w| \leq \pi \end{cases}$$
Problem 3

Let $x(t)$ be a band-limited signal such that $X(jw) = 0$ for $|w| \geq \frac{\pi}{T}$.

(a) If $x(t)$ is sampled using a sampling period $T$, determine an interpolating function $g(t)$ such that

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} x(nT)g(t-nT).$$

(b) Is the function $g(t)$ unique?

Solution

(a) Let us denote the sampled signal by $x_p(t)$. We have

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT).$$

Since the Nyquist rate for the signal $x(t)$ is $\frac{2\pi}{T}$, we can reconstruct the signal $x_p(t)$. We then know that

$$x(t) = x_p(t) * h(t),$$

where

$$h(t) = \frac{\sin(\pi t/T)}{\pi t/T}.$$ 

Therefore,

$$\frac{dx(t)}{dt} = x_p(t) * \frac{dh(t)}{dt}.$$ 

Denoting $\frac{dx(t)}{dt}$ by $g(t)$, we have

$$\frac{dx(t)}{dt} = x_p(t) * g(t) = \sum_{n=-\infty}^{\infty} x(nT)g(t-nT).$$

Therefore,

$$g(t) = \frac{dh(t)}{dt} = \frac{\cos(\pi t/T)}{t} - \frac{T\sin(\pi t/T)}{\pi t^2}.$$ 

(b) The function $g(t)$ is not unique. We can easily construct a counterexample which shows that it is different than the $g(t)$ shown above. However, the proof is omitted.
**Problem 4**

In this problem we develop the dual to the time-domain sampling theorem, whereby a time-limited signal can be reconstructed from frequency-domain samples. To develop this result, consider the frequency-domain sampling operation in Figure 2.

Figure 2: The frequency-domain sampling operation.

(a) Show that

\[ \tilde{x}(t) = x(t) \ast p(t) \]

where \( \tilde{x}(t), x(t), \) and \( p(t) \) are the inverse Fourier transforms of \( \tilde{X}(jw), X(jw), \) and \( P(jw), \) respectively.

(b) Assuming that \( x(t) \) is time-limited so that \( x(t) = 0 \) for \( |t| \geq \frac{\pi}{w_0}, \) show that \( x(t) \) can be obtained from \( \tilde{x}(t) \) through a ”low-time windowing” operation. That is,

\[ x(t) = \tilde{x}(t) w(t) \]

where

\[ w(t) = \begin{cases} w_0, & |t| \leq \frac{\pi}{w_0} \\ 0, & |t| > \frac{\pi}{w_0} \end{cases} \]

(c) Show that \( x(t) \) is not recoverable from \( \tilde{x}(t) \) if \( x(t) \) is not constrained to be zero for \( |t| \geq \frac{\pi}{w_0}. \)
Solution

(a) Since,
\[ \tilde{X}(jw) = X(jw)P(jw), \]
we have
\[ \tilde{x}(t) = x(t) * p(t). \]

(b) Taking the inverse Fourier transform of \( P(jw) \), we have
\[ p(t) = \frac{1}{w_0} \sum_{k=-\infty}^{\infty} \delta(t - \frac{2\pi k}{w_0}). \]

From part (a), we have
\[ \tilde{x}(t) = x(t) * p(t). \]
\[ = \frac{1}{w_0} \sum_{k=-\infty}^{\infty} x(t - \frac{2\pi k}{w_0}). \]

Noting that \( x(t) \) is time-limited so that \( x(t) = 0 \) for \( |t| > \frac{\pi}{w_0} \). we assume that \( x(t) \) is as shown in Figure 3. Then \( \tilde{x}(t) \) is as shown in the figure next to it. Clearly, \( x(t) \) can be recovered from \( \tilde{x}(t) \) by multiplying it with the function
\[ w(t) = \begin{cases} w_0, & |t| \leq \frac{\pi}{w_0} \\ 0, & \text{otherwise} \end{cases} \]

(c) If \( x(t) \) is not constrained to be zero for \( |t| > \frac{\pi}{w_0} \), then \( \tilde{x}(t) \) is as shown in Figure 3. Clearly, there is "time-domain aliasing" between the replicas of \( x(t) \) in \( \tilde{x}(t) \). Therefore, \( x(t) \) cannot be recovered from \( \tilde{x}(t) \).

![Figure 3: The signal of \( x(t) \) (top left), \( \tilde{x}(t) \) obtained in part (b) (top right), and \( \tilde{x}(t) \) obtained in part (c) (bottom).](image)