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INTERFERENCE CHANNELS WITH COORDINATED MULTI-POINT  
TRANSMISSION

BY

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DISSERTATION

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# ABSTRACT

Coordinated Multi-Point (CoMP) transmission is an infrastructural enhancement under consideration for next-generation wireless networks. In this dissertation, the capacity gain achieved through CoMP transmission is studied in various models of wireless networks that have practical significance. The capacity gain is analyzed through the degrees of freedom (DoF) criterion. The DoF available for communication provides an analytically tractable way to characterize the capacity of interference channels. The considered channel model has  $K$  transmitter/receiver pairs, and each receiver is interested in one unique message from a set of  $K$  independent messages. Each message can be available at more than one transmitter. The maximum number of transmitters at which each message can be available is defined as the *cooperation order*  $M$ . For fully connected interference channels, it is shown that the asymptotic per user DoF, as  $K$  goes to infinity, remains at  $\frac{1}{2}$  as  $M$  is increased from 1 to 2. Furthermore, the same negative result is shown to hold for all  $M \geq 2$  for any message assignment that satisfies a *local cooperation* constraint. On the other hand, when the assumption of full connectivity is relaxed to *local connectivity*, and each transmitter is connected only to its own receiver as well as  $L$  neighboring receivers, it is shown that local cooperation is optimal. The asymptotic per user DoF is shown to be at least  $\max\{\frac{1}{2}, \frac{2M}{2M+L}\}$  for locally connected channels, and is shown to be  $\frac{2M}{2M+1}$  for the special case of *Wyner's asymmetric model* where  $L = 1$ . An interesting feature of the proposed achievability scheme is that it relies on simple zero-forcing transmit beams and does not require symbol extensions. Also, to achieve the optimal per user DoF for Wyner's model, messages are assigned to transmitters in an asymmetric fashion unlike traditional assignments where message  $i$  has to be available at transmitter  $i$ . It is also worth noting that some receivers have to be inactive, and *fractional reuse* is needed to achieve equal DoF for all users. The obtained results for locally connected channels are then extended to each

of the following scenarios. First, a multiple-antenna transmitters setting is studied to highlight the comparison between dedicating multiple antennas to each message and sharing multiple antennas between messages. Second, an average transmit set size constraint is considered, where instead of imposing a constraint on the number of transmitters carrying each message, the considered constraint is a *backhaul load* constraint that limits the number of messages that can be delivered from a centralized controller to the base station transmitters. Third, an interference channel with block erasures is studied, where long-term fluctuations (shadow fading) in the wireless channel can lead to any link being erased with probability  $p$ . For each value of  $p$ , our goal is to find a fixed assignment of messages to transmitters that maximizes the average per user DoF.

*To my family*

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# CHAPTER 1

## INTRODUCTION

In the past decade, there has been a significant growth in the usage of wireless networks, and in particular, cellular networks, because of the increased data demands (see e.g. [1]-[4]). This has been the driver of recent research for new ways of managing interference in wireless networks.

Due to the superposition and broadcast properties of the wireless medium (see e.g. [5]), interfering signals pose a significant limitation to the rate of communication of users in a wireless network. Hence, it is of interest to understand the fundamental limits of communication in interference channels and to capture the effect of interference on optimal encoding and decoding schemes. The problem of finding the capacity region of even the simple two-user Gaussian interference channel is still an open problem because of several challenges including the identification of optimal choices for both the codebook and decoding scheme. However, approximations exist in the literature, where the capacity region or the sum capacity is known in the special scenario where the interference is strong enough such that the channel can appear to each receiver as a multiple access channel [6], and where the interference is weak enough such that the channel can appear to each receiver as a point to point channel (see [7], [8], and [9]).

### 1.1 Degrees of Freedom of the Fully Connected Interference Channel

Another effective approximation that simplifies the problem of finding the capacity of interference channels is to consider only the sum degrees of freedom (DoF) or the pre-log factor of the sum capacity at high signal-to-noise ratio (SNR). The DoF criterion provides an analytically tractable way to characterize the sum capacity and captures the number of interference-free sessions

that can be supported in a given multi-user channel. One clear advantage of this analysis is that it is not sensitive to the received power of the extracted interference free desired signal at the decoder. Another insight into why the problem is easier for this model, is that it ignores the effect of the Gaussian noise, since at high signal to noise ratio, the effect of interfering signals will be dominant. In [10], an upper bound of  $\frac{K}{2}$  is derived for the DoF of the  $K$ -user fully connected Gaussian interference channel. The key idea makes use of the elimination of the Gaussian noise in the analysis, as the received signal becomes a linear combination of the the transmitted signals with a zero constant term. This leads to the conclusion that for any reliable coding scheme for the case when  $K = 2$ , where each receiver can decode its intended signal, it will also be able to decode the interference. Hence, any value for the DoF achievable for the two-user Gaussian interference channel is also achievable for a two-user Gaussian multiple access channel, and the DoF for the latter is 1. By a simple counting argument, we reach the  $\frac{K}{2}$  upperbound. Interestingly, it is conjectured in [10] that the bound is loose, based on the slackness in the counting argument (the argument simply bounds the DoF for each pair of users by assuming all other signals to be known at these two receivers).

In [11], Cadambe and Jafar showed that the  $\frac{K}{2}$  bound is achievable for a time varying channel, where the channel coefficients are drawn from a continuous joint distribution. The coding scheme employs linear precoding in a manner that exploits the diversity in the channel to *align* all interfering signals in a dimension that does not contain all of the desired signal. By zero forcing the interference, each receiver can recover part of the signal that is linearly independent from a subspace containing all the interference, thus achieving half degree of freedom. It is clear that one cannot design the transmit beams such that interfering signals perfectly align at more than one receiver. However, it is shown in [11] that by using a time extension at each receiver, all vectors corresponding to interfering signals can align in a subspace whose size approaches half the received signal space as the block length goes to infinity.

## 1.2 Asymptotic Interference Alignment

A time division based scheme that *avoids* interference, can only achieve one degree of freedom for the  $K$ -user fully connected Gaussian interference channel, since at each time slot, the channel is equivalent to a single point-to-point channel. To understand why the idea of aligning interference when the received signal space has a dimension that is more than one, can achieve more than a single degree of freedom, consider a three-user channel, where we use three time slots to have a three-dimensional signal space. One of the users (call it user 1) transmits its message in two linearly independent directions, and each of the remaining users uses only one beam. By fixing the direction of one of the beams, and choosing each other beam to align with another transmitted vector at an *unintended* receiver, one can reduce the summation of dimensions of interference subspaces at all receivers by three. We note that the total number of interfering vectors is eight (two at the first receiver, and three at each other receiver). Hence, the total dimension of subspaces containing interference is five, leaving a four-dimensional subspace for the desired signal. Therefore, four degrees of freedom can be achieved over three time slots, or a  $\frac{4}{3}$  DoF per time slot. The details can be found in [11].

Now, we provide a sketch of the asymptotic interference alignment used in [11] to achieve a half degree of freedom per user. Define the sequences,

$$\alpha_n = n^{K(K-1)}, \quad (1.1)$$

$$\gamma_n = \alpha_n + \alpha_{n+1}, \quad (1.2)$$

then the transmitted signal at transmitter  $k$  and time  $t$  is given by,

$$\mathbf{X}_k = \mathbf{B}_n \mathbf{b}_k, \quad (1.3)$$

where the time index is removed for brevity,  $\mathbf{X}_k$ ,  $\mathbf{b}_k$  are the  $\gamma_n \times 1$ ,  $\alpha_n \times 1$ -dimensional vectors, corresponding to the transmitted vector, and the  $\alpha_n$  encoded symbols corresponding to the message of user  $k$ , respectively.  $\mathbf{B}_n$  is the  $\gamma_n \times \alpha_n$ -dimensional matrix whose columns are the elements of the set,

$$\mathcal{B}_n = \left\{ \left( \prod_{j,k \in \{1,2,\dots,K\}, k \neq j} \mathbf{H}_{j,k}^{p_{jk}} \right) \mathbf{w} : \forall p_{jk} \in \{0, \dots, n-1\} \right\}, \quad (1.4)$$

where  $\mathbf{w}$  is a  $\gamma_n \times 1$  vector, whose elements are selected independently from a continuous distribution,  $\mathbf{H}_{j,k}$  is the  $\gamma_n \times \gamma_n$  -dimensional diagonal matrix, representing the channel coefficients from transmitter  $k$  to receiver  $j$  for the  $\gamma_n$  symbol extension.

Now, considering the received message at receiver  $j$ , it can be easily verified that a non-intended message belonging to user  $k$ ,  $k \neq j$ , will arrive at a direction that belongs to the set  $\mathcal{B}_{n+1}$ , i.e.,

$$\mathbf{H}_{j,k}\mathbf{v}_k \in \mathcal{B}_{n+1}. \quad (1.5)$$

Thus, a total degrees of freedom of  $\frac{K}{2} - \epsilon$  can be achieved for any  $\epsilon > 0$ , by choosing  $n$  large enough, since

$$\frac{\alpha_{n+1}}{\alpha_n} \rightarrow 1, \quad (1.6)$$

and it can be shown that the matrix,

$$\mathbf{S} = [\mathbf{H}_{k,k}\mathbf{B}_n \ \mathbf{B}_{n+1}], \quad (1.7)$$

has full rank of  $\gamma_n$  with probability one,  $\forall k \in \{1, 2, \dots, K\}$ , if the channel coefficients are drawn from a continuous joint distribution.

Now, we argue why no more than  $\frac{K}{2}$  DoF is achievable from the perspective of the considered coding scheme. A possible line of thought attempting to increase the achieved DoF by using the above coding scheme, would suggest that not only should beams corresponding to different messages be aligned, but beams originating at the same transmitter should be aligned at an unintended receiver. However, it is still required that these very beams be *distinguishable* at their intended receiver (the desired signal should span a subspace of dimension  $\alpha_n$ ). This is not possible since vectors transmitted by one transmitter undergo the same transformations by the channel, hence if aligned at one receiver, it will be aligned at all other receivers.

Now, we note that the  $\frac{K}{2}$  achievable DoF may not be sufficient to meet the demands of wireless applications in many scenarios of practical interest, and hence, it is of interest to study ways to enhance the infrastructure of wireless networks in order to increase the rate of communication. We also note that it is intuitive to think that asymptotic interference alignment can be extended to achieve more than  $\frac{K}{2}$  DoF if each message can be transmitted

from more than one transmitter. Hence, the transmit beams corresponding to the same message can be transformed differently by the channel.

### 1.3 Coordinated Multi-Point Transmission

Managing wireless interference through infrastructural enhancements is a major consideration for next-generation cellular networks. One example of such an enhancement is in cellular downlink through the assignment of one receiver's message to multiple base station transmitters and managing interference through a Coordinated Multi-Point Transmission (CoMP) scheme [12]. The cost of delivering messages to multiple transmitters over a backhaul link is highlighted in this dissertation.

Motivated by the cellular downlink scenario (see e.g. [13], [14] and [15]), we consider an extension of the information theoretic model of the  $K$ -user Gaussian interference channel, where each message can be available at more than one transmitter. In the extreme case where all messages are known at all transmitters, the channel is equivalent to the Multiple Input Single Output (MISO) broadcast channel with  $K$  antennas at a single transmitter, and  $K$  single antenna receivers. The  $K$ -user MISO broadcast channel has  $K$  degrees of freedom, achievable by a simple linear beam-forming strategy that nulls out each message at exactly  $K - 1$  receivers.

In order to bridge the gap between the two cases of *No Cooperation* and *Full Cooperation*, we study the DoF in a scenario where each message is available at a number of transmitters that is bounded by a maximum transmit set size  $M$ . It is not difficult to see that a gain in the DoF is always achievable for the case where  $M$  is proportional to the number of users. For example, when  $M = K - 1$ , by communicating only  $M$  messages, a linear beam-forming strategy can null out each message at  $M - 1$  receivers, thus achieving  $M$  degrees of freedom, which is considerably larger than  $\frac{K}{2}$  for large fully connected networks. Therefore, it is more interesting to study the possible gain when fixing a value of the maximum transmit set size constraint as the size of the network increases. It is also useful for practical applications to understand the potential gains enabled by CoMP transmission through fixed-sized transmit sets in large networks.

By assuming that each message is available at the transmitter carrying

the same index as the message as well as  $M - 1$  succeeding transmitters, we show in Chapter 3 that an extension of the asymptotic interference alignment scheme of [11] can be used to prove that the DoF of the channel is lower bounded by  $\frac{K+M-1}{2}, \forall K < 10$ , and it is conjectured that this lower bound is valid for all values of  $K$ . We note that this DoF cooperation gain beyond  $K/2$  does not scale linearly with  $K$  as  $K$  goes to infinity. In other words, the asymptotic per user DoF remains  $1/2$ . We then study whether there exists an assignment of messages satisfying the fixed maximum transmit set size constraint that enables the achievability of an asymptotic per user DoF that is strictly greater than  $1/2$ .

### 1.3.1 Assigning Messages to Transmitters

The selection of an assignment of messages to transmitters is instrumental to the potential gain offered by CoMP transmission in interference networks [16]. For example, a message assignment based on the traditional idea of clustering does not lead to DoF gains for the fully connected channel. More precisely, if the network is split into small subnetworks, and the message corresponding to a user in a subnetwork can be available only at all transmitters in the subnetwork, then the DoF of the channel remains  $\frac{K}{2}$ . However, assigning each message to the transmitter carrying the same index as well as  $M - 1$  succeeding transmitters can lead to DoF gains in fully connected networks, as shown in Chapter 3.

The choice of message assignment should be based on the channel connectivity. For example, we show in Chapter 3 that for the fully connected channel, message assignments based on local cooperation cannot lead to a DoF gain that scales with the size of the network. However, we show in Chapter 4 that local cooperation is optimal for locally connected channels, and can be used to achieve gains in the asymptotic per user DoF.

## 1.4 Locally Connected Interference Channels

The assumption of full connectivity is key to the results obtained in [10], [11], [17], and in Chapter 3 of this dissertation. For the fully connected interference channel, interference mitigating schemes are designed to avoid the interfer-

ence caused by all other transmitters in the network. However, in practice, each receiver gets most of the destructive interference from a few dominant interfering transmitters. For example, in cellular networks, the number of dominant interfering transmit signals at each receiver ranges from two to seven. All the interference from the remaining transmitters may contribute to the interference floor, and the improvement obtained by including them in the dominant interferers set may not justify the corresponding overhead. For this reason, we study locally connected channels in Chapter 4, where the channel coefficients between transmitters and receivers that lie at a distance that is greater than some threshold are approximated to equal zero.

For the locally connected channel model, we assume that each transmitter is connected to  $L$  neighboring receivers as well as the receiver carrying its own index,  $\lfloor \frac{L}{2} \rfloor$  preceding receivers and  $\lceil \frac{L}{2} \rceil$  succeeding receivers. The special case of this model where  $L = 1$  is Wyner’s asymmetric model [18]. This special case was considered in [19], and it was assumed that each message is available at the transmitter carrying the same index as well as  $M - 1$  succeeding transmitters. The asymptotic per user DoF was shown under this setting to equal  $\frac{M}{M+1}$ . The achieving scheme relies only on zero-forcing transmit beam-forming. In Chapter 4, we extend this result and characterize the asymptotic per user DoF for Wyner’s asymmetric model as  $\frac{2M}{2M+1}$  under a general cooperation order constraint. The message assignment enabling this result uses only local cooperation, that is, each message is available only at neighboring transmitters. The size of the neighborhood does not scale linearly with the size of the network, and therefore, our assignment scheme enjoys the same advantage as the message assignment considered in [19].

## 1.5 Practical Considerations

### 1.5.1 Channel Knowledge

We assume that global channel state information is available at all transmitters and receivers for all considered models. In practice, the channel coefficients are approximately estimated at the receivers by transmitting known pilot signals, and then they are fed back to the transmitters (see e.g. [20], [21] and [22]). It is a common practice in information theoretic analysis to ignore

the overhead of the estimation and communication of channel coefficients, in order to derive insights relevant to the remaining design parameters of the coding scheme; we follow this layered approach in this dissertation.

### 1.5.2 Symbol Extension and Delay

One major obstacle toward a practical implementation of asymptotic interference alignment is that the achievable DoF is approached only with a very large number of signal dimensions. The feasibility of alignment with a finite symbol extension is studied in [23] and [24]. We also note that CoMP transmission can be used to achieve DoF gains without the need for infinite symbol extensions. For example, assigning each message to two transmitters in a four-user fully connected network can lead to the achievability of  $\frac{9}{4}$  per user DoF, by coding over four time slots, which is greater than the  $\frac{1}{2}$  per user DoF achieved without cooperation using infinite symbol extensions. Also, we show in Chapter 4 how CoMP transmission can lead to scalable DoF gains for locally connected channels without the need for symbol extensions.

### 1.5.3 Backhaul Constraint

The appropriate constraint to consider for the assignment of messages to transmitters should depend on the nature of the backhaul link used in practice (see e.g. [12] and [25]). For example, in the context of heterogeneous networks, the backhaul can be a wireless network. In this case, a study of multi-hop networks can be relevant (see [26]). On the other hand, for the case of wireline or fiber optics backhaul links, the considered model can be useful. In particular, a constraint that bounds the average transmit set size is more relevant than imposing a maximum constraint on each transmit set size. We show in Chapter 5 how the solutions provided to our setting can be used to find solutions for the CoMP transmission problem under an average transmit set size constraint.



#### 1.5.4 Dynamic Interference Management

In Chapter 6, we generalize the solutions obtained for locally connected interference channels to networks with changing topology. The goal is to find a fixed assignment of messages to transmitters that achieves the optimal average per user DoF in an interference network with dynamic connectivity. We note that the topology can change in practice due to either design choices as in heterogeneous network or long-term fluctuations in the channel (deep fading conditions).

#### 1.5.5 Synchronization

It is worth noting that one major practical consideration for the implementation of CoMP transmission coding schemes is the synchronization between different transmitters that carry the same message. The problem of synchronization is outside the scope of this dissertation, and is discussed in [27].

### 1.6 Related Problems

Many existing works studying interference networks with cooperating transmitters use the term *cognitive radios* (e.g. [28], [29], [30], [31], [32]). Cooperation through cumulative message sharing is studied for the fully connected channel in [33], where each message is available at the transmitter carrying the same index and all following transmitters. We use a similar setting of cooperation to that of cumulative message sharing in the coding scheme for locally connected channels in Section 4.3. In another body of work, unlike the considered setting where we assume that transmitters cooperate by sharing complete messages, cooperation through sharing partial message information that is considered as side information is studied (see e.g., [34]). In [35] and [36], the transmitters are allowed to cooperate through noise-free bit pipes or over the air, respectively.

Communication scenarios with cooperating multiple antenna transmitters have been considered in [37] and [38] under the umbrella of the x-channel. However, in the x-channel, mutually exclusive parts of each message are given to different transmitters. This is extended in [39] to allow each part of

each message to be available at more than one transmitter, and in [34] the MIMO x-channel is studied in the setting where transmitters share further side information.

In the considered setting, we implicitly assume the coordinated design of the transmit beams between all transmitters. This kind of coordination is also referred to in the literature as *transmitter cooperation*, even without the sharing of messages (see e.g. [40]). Finally, it is worth noting that CoMP Reception settings have been studied in [17] and [41]. In [17], sharing of analog signals is allowed between receivers, and in [41], neighboring receivers are assumed to share the decoded messages.

## 1.7 Dissertation Outline

In Chapter 2, we provide the system model and notation. The first channel model we consider is the fully connected channel model, which we study in Chapter 3. We show that CoMP transmission can offer DoF gains in Section 3.1, and then study whether the offered DoF gains can scale with the size of the network in Section 3.3. We then relax the assumption of full connectivity and study locally connected channel models in Chapter 4. We show in Section 4.3 that scalable DoF gains are possible for locally connected channels, and that these gains can be enabled through a local cooperation mechanism. We then show in Section 4.4 that local cooperation is optimal for locally connected channels. We extend the obtained results for locally connected channels to a channel with multiple-antenna transmitters in Section 4.7. In Chapter 5, we study the more practically relevant constraint on the average transmit set size, and show that solutions provided for the maximum transmit set size constraint can be used to solve the more general problem. We finally consider the problem of maximizing the average DoF in a network with changing topology in Chapter 6. We provide concluding remarks in Chapter 7

# CHAPTER 2

## SYSTEM MODEL

We use the standard model for the  $K$ -user interference channel [42]. We assume that each transmitter and receiver has a single antenna unless stated otherwise,

$$Y_i(t) = \sum_{j=1}^K H_{i,j}(t)X_j(t) + Z_i(t), \quad (2.1)$$

where  $t$  is the time index,  $X_j(t)$  is the transmitted signal of transmitter  $j$ ,  $Y_i(t)$  is the received signal at receiver  $i$ ,  $Z_i(t)$  is the zero mean unit variance Gaussian noise at receiver  $i$ , and  $H_{i,j}(t)$  is the channel coefficient from transmitter  $j$  to receiver  $i$  over the time slot  $t$ . We remove the time index in the rest of the dissertation for brevity unless it is needed.

### 2.1 Channel Model

In Chapter 3, we consider a fully connected interference channel where all channel coefficients are drawn from a continuous joint distribution. We next consider in Chapter 4 a locally connected channel model where channel coefficients between well separated nodes are approximated to be identically zero. The locally connected channel model is a function of the number of interferers  $L$  as follows:

$$H_{i,j} \text{ is not identically 0 if and only if } i \in \left[ j - \left\lfloor \frac{L}{2} \right\rfloor, j + \left\lceil \frac{L}{2} \right\rceil \right], \quad (2.2)$$

and all channel coefficients that are not identically zero are drawn from a continuous joint distribution. We note that for values of  $L = 1$  and  $L = 2$ , the locally connected channel reduces to the commonly known Wyner's asymmetric and symmetric linear models, respectively [18]. We illustrate

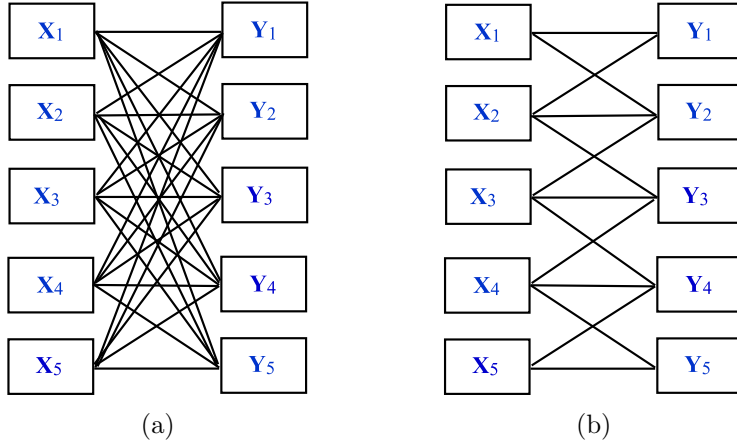


Figure 2.1: Figure showing examples of the considered channel models with a number of users  $K = 5$ . In (a), a fully connected channel model is shown. In (b), a locally connected channel model with connectivity parameter  $L = 2$  is shown.

examples for the described fully and locally connected channel models in Figure 2.1.

In Chapter 6, we consider a linear interference channel ( $L = 1$ ) with block erasures [43], where in order to consider the effect of long-term fluctuations (shadowing), we assume that communication takes place over blocks of time slots, and let  $p$  be the probability of block erasure. In each block, we assume that for each  $j$ , and each  $i \in \{j, j+1\}$ ,  $H_{i,j} = 0$  with probability  $p$ . Moreover, short-term channel fluctuations allow us to assume that in each time slot, all non-zero channel coefficients are drawn from a continuous joint distribution.

We assume that channel state information is known at all transmitters and receivers for all considered models.

## 2.2 Cooperation Model

For each  $i \in \{1, 2, \dots, K\}$ , let  $W_i$  be the message intended for receiver  $i$ , and  $\mathcal{T}_i \subseteq \{1, 2, \dots, K\}$  be the transmit set of receiver  $i$ , i.e., those transmitters with the knowledge of  $W_i$ . The transmitters in  $\mathcal{T}_i$  cooperatively transmit the message  $W_i$  to the receiver  $i$ . We assume that each transmit set size is upper bounded by a *cooperation order*  $M$ ,

$$|\mathcal{T}_i| \leq M, \forall i \in \{1, 2, \dots, K\}. \quad (2.3)$$

In Chapter 5, instead of the maximum transmit set size constraint of (2.3), we impose an average transmit set size constraint, where the average transmit set size is upper bounded by a *backhaul load*  $B$ ,

$$\frac{\sum_{i=1}^K |\mathcal{T}_i|}{K} \leq B. \quad (2.4)$$

### 2.2.1 Message Assignment Strategy

A message assignment strategy is defined by a sequence of supersets. The  $k^{\text{th}}$  element in the sequence consists of the transmit sets for a  $k$ -user channel. We use message assignment strategies to define a pattern for assigning messages to transmitters in large networks.

A message assignment strategy is defined by a sequence of transmit sets  $(\mathcal{T}_{i,K}), i \in \{1, 2, \dots, K\}, K \in \{1, 2, \dots\}$ . For each positive integer  $K$  and  $\forall i \in \{1, 2, \dots, K\}$ , the transmit sets  $\{\mathcal{T}_{i,K}\}_{i \in \{1, 2, \dots, K\}}$  satisfy the considered cooperation and define a message assignment for a  $K$ -user channel.

### 2.2.2 Local Cooperation

We say that a message assignment strategy satisfies the local cooperation constraint, if and only if there exists a function  $r(K)$  such that  $r(K) = o(K)$ , and for every  $K \in \mathbf{Z}^+$ , the transmit sets defined by the strategy for a  $K$ -user channel satisfies the following,

$$\mathcal{T}_i \subseteq \{i - r(K), i - r(K) + 1, \dots, i + r(K)\}, \forall i \in \{1, 2, \dots, K\}. \quad (2.5)$$

## 2.3 Degrees of Freedom

Let  $P$  be the average transmit power constraint at each transmitter, and let  $\mathcal{W}_i$  denote the alphabet for message  $W_i$ . Then the rates  $R_i(P) = \frac{\log |\mathcal{W}_i|}{n}$  are achievable if the decoding error probabilities of all messages can be simultaneously made arbitrarily small for a large enough coding block length  $n$ , and this holds for almost all channel realizations. The degrees of freedom  $d_i, i \in \{1, 2, \dots, K\}$  are defined as  $d_i = \lim_{P \rightarrow \infty} \frac{R_i(P)}{\log P}$ . The DoF region  $\mathcal{D}$  is the closure of the set of all achievable DoF tuples. The total number of

degrees of freedom ( $\eta$ ) is the maximum value of the sum of the achievable degrees of freedom,  $\eta = \max_{\mathcal{D}} \sum_{i \in \{1, 2, \dots, K\}} d_i$ .

For a sequence of  $K$ -user channels,  $K \in \{1, 2, \dots\}$ , if the limit of the ratio between the degrees of freedom and the number of users exists, then we call the value of this limit the asymptotic per user DoF. We say that a message assignment strategy is *optimal*, if and only if there exists a sequence of coding schemes achieving the asymptotic per user DoF using the transmit sets defined by the message assignment strategy.

## 2.4 Notation

### 2.4.1 Performance Criteria

For the  $K$ -user fully connected channel model considered in Chapter 3, we define  $\eta(K, M)$  as the best achievable DoF  $\eta$  over all choices of transmit sets satisfying the maximum transmit set size constraint in (2.3). We define the asymptotic per user DoF  $\tau(M)$  to measure how  $\eta(K, M)$  scales with  $K$ , while all other parameters are fixed,

$$\tau(M) = \lim_{K \rightarrow \infty} \frac{\eta(K, M)}{K}. \quad (2.6)$$

We use  $\eta^{(\text{loc})}(K, M)$  to denote that maximum achievable DoF under the maximum transmit set size constraint of (2.3) and the additional local cooperation constraint of (2.5) for a  $K$ -user fully connected channel. Similarly, we use  $\tau^{(\text{loc})}(M)$  to denote the local cooperation asymptotic per user DoF for fully connected channels.

For the locally connected channel model of (2.2) and considered in Chapter 4, we use  $\eta_L(K, M)$  to denote the DoF of a  $K$ -user channel with connectivity parameter  $L$ , and  $\tau_L(M)$  to denote the asymptotic per user DoF. We also use  $\eta_L^{(\text{loc})}(K, M)$  and  $\tau_L^{(\text{loc})}(M)$  to denote the corresponding DoF and asymptotic per user DoF values for local cooperation, respectively.

For the locally connected channel model with the average transmit set size constraint of  $B$  (2.4) considered in Chapter 5, we use  $\eta_L^{(\text{avg})}(K, B)$  to denote the DoF of a  $K$ -user channel with connectivity parameter  $L$ , and  $\tau_L^{(\text{avg})}(B)$  to denote the asymptotic per user DoF.

For the linear interference channel with block erasures considered in Chapter 6, we use  $\eta_p(K, M)$  to denote the DoF of a  $K$ -user channel with block erasure probability  $p$  and a maximum transmit set size constraint  $M$ , and  $\tau_p(M)$  to denote the asymptotic per user DoF.

## 2.4.2 Other Notation

We use  $[K]$  to denote the set  $\{1, 2, \dots, K\}$ , and  $\phi$  to denote the empty set. For any set  $\mathcal{A} \subseteq [K]$ , we use the abbreviations  $X_{\mathcal{A}}$ ,  $Y_{\mathcal{A}}$ , and  $Z_{\mathcal{A}}$  to denote the sets  $\{X_i, i \in \mathcal{A}\}$ ,  $\{Y_i, i \in \mathcal{A}\}$ , and  $\{Z_i, i \in \mathcal{A}\}$ , respectively. For  $\mathcal{A}, \mathcal{B} \subseteq [K]$ , we let  $\mathbf{H}_{\mathcal{A}, \mathcal{B}}$  be the  $|\mathcal{A}| \times |\mathcal{B}|$  matrix of channel coefficients between  $X_{\mathcal{B}}$  and  $Y_{\mathcal{A}}$ . We use  $\mathsf{H}(\cdot)$  to denote the binary entropy function.

For any set  $\mathcal{A} \subseteq [K]$ , we define  $C_{\mathcal{A}}$  as the set of messages carried by transmitters with indices in  $\mathcal{A}$ , i.e., the set  $\{i : \mathcal{T}_i \cap \mathcal{A} \neq \phi\}$ , and  $U_{\mathcal{A}}$  as the set of indices of transmitters that exclusively carry the messages for the receivers in  $\mathcal{A}$ , and the complement set  $\bar{U}_{\mathcal{A}}$  is the set of indices of transmitters that carry messages for receivers outside  $\mathcal{A}$ . More precisely,  $\bar{U}_{\mathcal{A}} = \cup_{i \notin \mathcal{A}} \mathcal{T}_i$ .

We call a finite set of real random variables *generic* if and only if each element has a continuous probability distribution conditioned on any subset of the set of all other elements. Note that any subset of a generic set, is generic. For any considered channel model, the set of channel coefficients that are not identically zero is generic.

# CHAPTER 3

## FULLY CONNECTED CHANNEL

Recall that for a  $K$ -user fully connected channel, we use  $\eta(K, M)$  as the best achievable DoF  $\eta$  over all choices of transmit sets satisfying the maximum transmit set size constraint in (2.3), and  $\tau(M)$  to denote the asymptotic per user DoF.

In this chapter, we investigate whether  $\tau(M) > \frac{1}{2}$  for  $M > 1$ , and message assignment strategies that may lead to a positive conclusion.

### 3.1 Achievable Scheme

We know from [10] and [11] that the sum DoF of a fully connected interference channel without cooperation is  $\frac{K}{2}$ , i.e.,  $\eta(K, 1) = \frac{K}{2}$ . We now show that  $\eta(K, M) > \frac{K}{2}$ , for  $M > 1$ , by using the following spiral message assignment for each  $K$ -user channel:

$$\mathcal{T}_i = \begin{cases} \{i, i+1, \dots, i+M-1\}, & \forall i \in [K - (M-1)] \\ \{i, i+1, \dots, K, 1, 2, \dots, M - (K - i + 1)\}, & \forall i \in \{K - (M-2), \dots, K\}. \end{cases}$$

Using this message assignment strategy and an asymptotic interference alignment scheme, we prove the following result.

**Theorem 1.**

$$\eta(K, M) \geq \frac{K + M - 1}{2}, \forall M \leq K < 10 \quad (3.1)$$

*Proof.* The proof is relegated to Section 3.5, and we provide a sketch here. To achieve the stated lower bound, the  $M$  transmitters carrying each message are used to cancel the interference introduced by this message at the first  $M-1$  receivers, thereby allowing each of these receivers to enjoy one degree of freedom. By coding over multiple parallel channels corresponding to different



time slots, we use an interference alignment scheme to align the interfering signals at each other receiver to occupy half the signal space as the number of parallel channels goes to infinity.

The achievable scheme is based on transmit beam-forming. The beam design process is broken into two steps as illustrated in Figure 3.1. First, we transform each parallel CoMP channel into a derived channel. Then, we design an asymptotic interference alignment scheme over the derived channel achieving the required DoF in an asymptotic fashion as the number of parallel channels goes to infinity. Figure 3.2 provides a description of the derived channel for the special case of  $K = 4$  and  $M = 2$ . In order to use asymptotic interference alignment in the achievable scheme, we need to show that at each receiver, polynomial transformations defining a set of derived channel coefficients determined by the receiver index, are algebraically independent as functions of the original channel coefficients. We could verify in MATLAB that this is true for all the values of  $K$  and  $M$  that we checked. Specifically, we checked until  $K \leq 9$ , but we conjecture that the result holds true for any  $K$  and  $M$ .

□

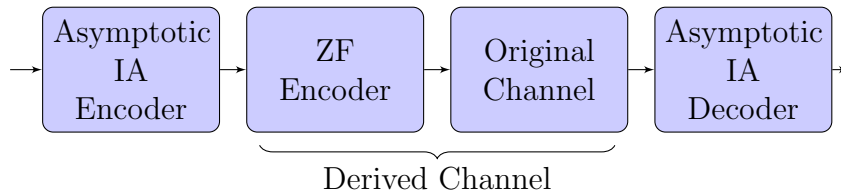


Figure 3.1: Summary of the achievable scheme of Theorem 1.

It is worth noting that using tools from algebraic geometry to study the feasibility of interference alignment was introduced in the context of MIMO interference channels in [44]. Now, We note that the achieved DoF gain due to CoMP transmission shown in Theorem 1 (beyond  $\frac{K}{2}$ ) does not scale with the number of users  $K$ . Hence, the question of whether  $\tau(M) > \frac{1}{2}$  for  $M > 1$  remains open. Here, we note that the spiral message assignment strategy satisfies the local cooperation constraint in (2.5) and in Section 3.3, we show that no gain in the asymptotic per user DoF can be achieved through any message assignment strategy that satisfies the local cooperation constraint.

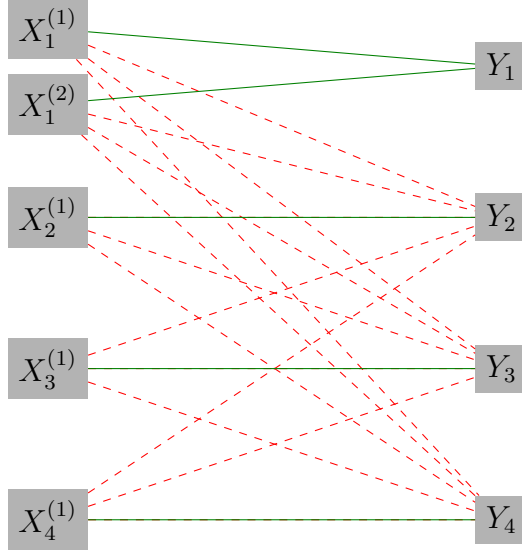


Figure 3.2: The derived channel when  $K = 4$  and  $M = 2$ . The thick green lines indicate the links carrying signal. The dashed red lines indicate the links carrying interference.

## 3.2 DoF Upper Bound

In order to characterize the DoF of the channel  $\tau(M)$ , we need to consider all possible strategies for message assignments satisfying the maximum transmit set size constraint defined in (2.3). Through the following corollary of Lemma 5 in Appendix A, we provide a way to bound the DoF number  $\eta$  of a  $K$ -user fully connected channel with a fixed message assignment, thereby, introducing a criterion for comparing different message assignments satisfying (2.3) using the special cases where this bound holds tightly. Recall that for a set of transmitter indices  $\mathcal{S}$ , the set  $C_{\mathcal{S}}$  is the set of messages carried by transmitters in  $\mathcal{S}$ , and for a set of receiver indices  $\mathcal{A}$ , the set  $U_{\mathcal{A}}$  is the set of indices of transmitters that exclusively carry the messages for the receivers in  $\mathcal{A}$

**Corollary 1.** *For any  $m, \bar{m} : m + \bar{m} \geq K$ , if there exists a set  $\mathcal{S}$  of indices for transmitters carrying no more than  $m$  messages, and  $|\mathcal{S}| = \bar{m}$ , then  $\eta \leq m$ , or more precisely,*

$$\eta \leq \min_{\mathcal{S} \subseteq [K]} \max(|C_{\mathcal{S}}|, K - |\mathcal{S}|). \quad (3.2)$$

*Proof.* For each subset of transmitter indices  $\mathcal{S} \subseteq [K]$ , we apply Lemma 5 with the set  $\mathcal{A}$  defined as follows.

Initially, set  $\mathcal{A}$  as the set of indices for messages carried by transmitters with indices in  $\mathcal{S}$ . That is,  $\mathcal{A} = C_{\mathcal{S}}$ . Now, if  $|\mathcal{A}| < K - |\mathcal{S}|$ , then augment the set  $\mathcal{A}$  with arbitrary message indices such that  $|\mathcal{A}| = K - |\mathcal{S}|$ .

We now note that the above construction guarantees that  $|\mathcal{A}| + |\mathcal{S}| \geq K$  and that  $U_{\mathcal{A}} \subseteq \bar{\mathcal{S}}$ . Hence, using Lemma 5, it suffices to show the existence of functions  $f_1$  and  $f_2$  such that  $f_1(Y_{\mathcal{A}}, X_{\mathcal{S}}) = X_{\bar{\mathcal{S}}} + f_2(Z_{\mathcal{A}})$ , where  $f_2$  is a linear function that does not depend on the transmit power.

Consider the following argument. Given  $Y_{\mathcal{A}}$ ,  $Z_{\mathcal{A}}$ , and  $X_{\mathcal{S}}$ , we can construct the set of signal  $\tilde{Y}_{\mathcal{A}}$  as follows:

$$\begin{aligned} \tilde{Y}_i &= Y_i - \left( \sum_{j \in \mathcal{S}} H_{i,j} X_j + Z_i \right) \\ &= \sum_{j \in \bar{\mathcal{S}}} H_{i,j} X_j, \forall i \in \mathcal{A}. \end{aligned} \tag{3.3}$$

Since the channel is fully connected, by removing the Gaussian noise signals  $Z_{\mathcal{A}}$  and transmit signals in  $X_{\mathcal{S}}$  from received signals in  $Y_{\mathcal{A}}$ , we obtain the set of signals  $\{\tilde{Y}_i : i \in \mathcal{A}\}$ , which has at least  $K - |\mathcal{S}| = |\bar{\mathcal{S}}|$  linear equations in the transmit signals in  $X_{\bar{\mathcal{S}}}$ . Moreover, since the channel coefficients are generic, these equations will be linearly independent with high probability. Now, if we do not remove the noise signals  $Z_{\mathcal{A}}$  from (3.3), then by using  $Y_{\mathcal{A}}$  and  $X_{\mathcal{S}}$ , we can reconstruct  $X_{\bar{\mathcal{S}}} + f_2(Z_{\mathcal{A}})$ , where  $f_2$  depends on the inverse transformation of  $|\bar{\mathcal{S}}|$  linearly independent equations in  $X_{\bar{\mathcal{S}}}$ , and the coefficients of the linear equations depend only on the channel coefficients. □

Refer to Figure 3.3 for an example illustration of Corollary 1.

### 3.3 Asymptotic DoF Cooperation Gain

We now use Corollary 1 to prove upper bounds on the asymptotic per user DoF  $\tau(M)$ .

In an attempt to reduce the complexity of the problem of finding an optimal message assignment strategy, we begin by considering message assignment strategies satisfying the local cooperation constraint defined in Section 2.2.2. We now show that a scalable cooperation DoF gain cannot be achieved using

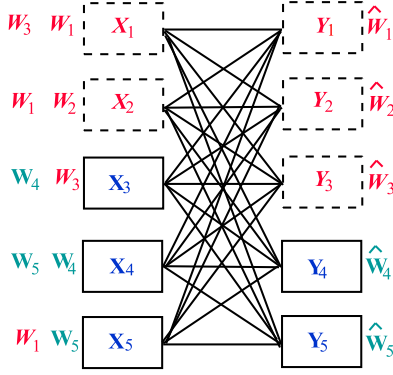


Figure 3.3: Example application of Corollary 1, with  $\mathcal{S} = \{1, 2\}$  and  $C_{\mathcal{S}} = \{1, 2, 3\}$ . Transmit signals with indices in  $\mathcal{S}$ , and messages as well as receive signals with indices in  $C_{\mathcal{S}}$  are shown in tilted red font and dashed boxes. The DoF  $\eta \leq |C_{\mathcal{S}}| = K - |\mathcal{S}| = 3$ .

local cooperation. Recall that  $\tau^{(\text{loc})}(M)$  is the maximum achievable asymptotic per user DoF under the additional local cooperation constraint of (2.5). We obtain the following result.

**Theorem 1.** *Any message assignment strategy satisfying the local cooperation constraint of (2.5) cannot be used to achieve an asymptotic per user DoF greater than that achieved without cooperation. More precisely,*

$$\tau^{(\text{loc})}(M) = \frac{1}{2}, \text{ for all } M. \quad (3.4)$$

*Proof.* Fix  $M \in \mathbf{Z}^+$ . For any value of  $K \in \mathbf{Z}^+$ , we use Corollary 1 with the set  $\mathcal{S} = \{1, 2, \dots, \lceil \frac{K}{2} \rceil\}$ . Note that  $C_{\mathcal{S}} \subseteq \{1, 2, \dots, \lceil \frac{K}{2} \rceil + r(K)\}$ , where  $\lim_{K \rightarrow \infty} \frac{r(K)}{K} = 0$ , and hence, it follows that  $\eta^{(\text{loc})}(K, M) \leq \lceil \frac{K}{2} \rceil + r(K)$ . Finally,  $\tau^{(\text{loc})}(M) = \lim_{K \rightarrow \infty} \frac{\eta^{(\text{loc})}(K, M)}{K} \leq \frac{1}{2}$ . The lower bound follows from [11] without cooperation.  $\square$

We now investigate if it is possible for the cooperation gain to scale linearly with  $K$  for fixed  $M$ . It was shown in Theorem 1 that such a gain is not possible for message assignment strategies that satisfy the local cooperation constraint. Here, we only impose the maximum transmit set size constraint in (2.3) and prove in Theorem 2 an upper bound on  $\tau(M)$  that is tight enough for finding  $\tau(2)$ .

**Theorem 2.** *For any cooperation order constraint  $M \geq 2$ , the following*

upper bound holds for the asymptotic per user DoF,

$$\tau(M) \leq \frac{M-1}{M}. \quad (3.5)$$

*Proof.* For any value of  $M$  and  $K$ , we show that  $\eta(K, M) \leq \frac{K(M-1)}{M} + o(K)$ . For every value of  $K$  such that  $\frac{K-1}{M}$  is an integer, we show that  $\eta(K, M) \leq \frac{K(M-1)+1}{M}$ . When  $\frac{K-1}{M}$  is not an integer, we add  $x = o(K)$  extra users such that  $\frac{K+x-1}{M}$  is an integer, and bound the DoF as follows,

$$\eta(K, M) \leq \eta(K+x, M) \quad (3.6)$$

$$\leq \frac{(K+x)(M-1)+1}{M} \quad (3.7)$$

$$= \frac{K(M-1)}{M} + o(K). \quad (3.8)$$

It then suffices to consider the case where  $\frac{K-1}{M}$  is an integer. The idea is to show that for any assignment of messages satisfying the cooperation order constraint, there exists a set of indices  $\mathcal{S} \subseteq [K]$  for  $\frac{K-1}{M}$  transmitters that do not carry more than  $K - \frac{K-1}{M}$  messages, and then the DoF upper bound follows by applying Corollary 1. More precisely, it suffices to show that the following holds,

$$\forall K : \frac{K-1}{M} \in \mathbf{Z}^+, \exists \mathcal{S} \subseteq [K] : |\mathcal{S}| = \frac{K-1}{M}, |\mathcal{C}_{\mathcal{S}}| = \frac{K(M-1)+1}{M} = K - |\mathcal{S}|. \quad (3.9)$$

We first illustrate simple examples that demonstrate the validity of (3.9). Consider the case where  $K = 3$ ,  $M = 2$ , we need to show in this case that there exists a transmitter that does not carry more than two messages, which follows by the pigeonhole principle since each message can only be available at a maximum of two transmitters. Now, consider the slightly more complex example of  $K = 5$ ,  $M = 2$ , we need to show in this case that there exists a set of two transmitters that do not carry more than three messages. We know that there is a transmitter carrying at most two messages, and we select this transmitter as the first element of the desired set. Without loss of generality, let the two messages available at the selected transmitter be  $W_1$  and  $W_2$ . Now, we need to find another transmitter that carries at most one message among the messages in the set  $\{W_3, W_4, W_5\}$ . Since each of these three messages can be available at a maximum of two transmitters, and we

have four transmitters to choose from, one of these transmitters has to carry at most one of these messages. By adding the transmitter satisfying this condition as the second element of the set, we obtain a set of two transmitters carrying no more than three messages, and (3.9) holds.

We extend the argument used in the above examples through Lemmas 1 and 2 that are provided in Section 3.6. We know by induction using these lemmas that (3.9) holds, and the theorem statement follows.  $\square$

Together with the achievability result in [11], the statement in Theorem 2 implies the following corollary.

**Corollary 2.** *For any message assignment strategy such that each message is available at a maximum of two transmitters, the asymptotic per user DoF is the same as that achieved without cooperation. More precisely,*

$$\tau(2) = \frac{1}{2}. \quad (3.10)$$

The characterization of  $\tau(M)$  for values of  $M > 2$  remains an open question, as Theorem 2 is only an upper bound. Moreover, the following result shows that the upper bound in Theorem 2 is loose for  $M = 3$ .

**Theorem 3.** *For any message assignment strategy such that each message is available at a maximum of three transmitters, the following bound holds for the asymptotic per user DoF,*

$$\tau(3) \leq \frac{5}{8}. \quad (3.11)$$

*Proof.* In a similar fashion to the proof of Theorem 2, we prove the statement by induction. The idea is to prove the existence of a set  $\mathcal{S}$  with approximately  $\frac{3K}{8}$  transmitter indices, and these transmitters are carrying no more than approximately  $\frac{5K}{8} = K - |\mathcal{S}| + o(K)$  messages, and then use Corollary 1 to derive the DoF outer bound. In the proof of Theorem 2, we used Lemmas 1 and 2 in Section 3.6, to provide the basis and induction step of the proof, respectively. Here, we follow the same path until we show that there exists a set  $\mathcal{S}$  such that  $|\mathcal{S}| = \frac{K+1}{4}$  and  $|C_{\mathcal{S}}| \leq (M-1)|\mathcal{S}| + 1$ , and then we use Lemma 4 in Section 3.6 to provide a stronger induction step that establishes a tighter bound on the size of the set  $C_{\mathcal{S}}$ .

We note that it suffices to show that  $\eta(K, 3) \leq \frac{5K}{8} + o(K)$  for all values of  $K$  such that  $\frac{K+1}{4}$  is an even positive integer, and hence, we make that assumption for  $K$ . Define the following,

$$x_1 = \frac{K+1}{4}, \quad (3.12)$$

$$x_2 = \frac{K-7}{8}, \quad (3.13)$$

$$x_3 = 2x_1 + 1 + x_2. \quad (3.14)$$

Now, we note that

$$x_3 = K - (x_1 + x_2), \quad (3.15)$$

and by induction, it follows from Lemmas 1 and 2 that  $\exists \mathcal{S}_1 \subset [K]$ ,  $|\mathcal{S}_1| = x_1$ ,  $|C_{\mathcal{S}_1}| \leq 2x_1 + 1$ . We now apply induction again with the set  $\mathcal{S}_1$  as a basis, and use Lemma 4 for the induction step to show that  $\exists \mathcal{S}_2 \subset [K]$ ,  $|\mathcal{S}_2| = x_1 + x_2$ ,  $|C_{\mathcal{S}_2}| \leq x_3 = K - |\mathcal{S}_2|$ . Hence, we get the following upper bound using Corollary 1,

$$\begin{aligned} \eta(K, 3) &\leq x_3 \\ &= \frac{5(K+1)}{8}, \end{aligned} \quad (3.16)$$

from which (3.11) holds.  $\square$

### 3.4 Connection between DoF Upper Bound and Bipartite Vertex Expanders

We note that all the DoF upper bounding proofs used so far employ Corollary 1. We now show that under the hypothesis that the upper bound in Corollary 1 is tight for any  $K$ -user fully connected interference channel with a cooperation order constraint  $M$ , then scalable DoF cooperation gains are achievable for any value of  $M \geq 3$ . Hence, a solution to the general problem necessitates the discovery of either new upper bounding techniques or new coding schemes.

In this section, we restrict our attention to upper bounds on  $\tau(M)$  that follow by a direct application of Corollary 1. More precisely, for a  $K$ -user

fully connected channel with an assignment of the transmit sets  $\{\mathcal{T}_i\}_{i \in [K]}$ , define  $B(K, \{\mathcal{T}_i\})$  as the upper bound that follows by Corollary 1 for this channel, i.e.,

$$B(K, \{\mathcal{T}_i\}) = \min_{\mathcal{S} \subseteq [K]} \max(|C_{\mathcal{S}}|, K - |\mathcal{S}|). \quad (3.17)$$

Now, let  $\eta_{\text{out}}(K, M)$  and  $\tau_{\text{out}}(M)$  be the corresponding upper bounds that apply on  $\eta(K, M)$  and  $\tau(M)$ ,

$$\eta_{\text{out}}(K, M) = \max_{\{\mathcal{T}_i\}_{i \in [K]}: \mathcal{T}_i \subseteq [K], |\mathcal{T}_i| \leq M, \forall i \in [K]} B(K, \{\mathcal{T}_i\}), \quad (3.18)$$

$$\tau_{\text{out}}(M) = \lim_{K \rightarrow \infty} \frac{\eta_{\text{out}}(K, M)}{K}. \quad (3.19)$$

All the facts that we stated above about  $\tau(M)$  hold for  $\tau_{\text{out}}(M)$ , as all the upper bounding proofs follow by a direct application of Corollary 1. We now identify a property for message assignment strategies, that lead us to prove that  $\tau_{\text{out}}(M) > \frac{1}{2}, \forall M > 2$ . Note that this does not necessarily imply that  $\tau(M) > \frac{1}{2}, \forall M > 2$ , but it provides some insight into whether this statement might be true [45].

For each possible message assignment, define a bipartite graph with partite sets of size  $K$ . Vertices in one of the partite sets represent transmitters, and vertices in the other set represent messages. There exists an edge between two vertices if and only if the corresponding message is available at the designated transmitter. We note that the maximum transmit set size constraint implies that the maximum degree of nodes in one of the partite sets is bounded by  $M$ . We now observe that for any set  $\mathcal{A}$  of transmitters,  $C_{\mathcal{A}} = \{i : \mathcal{T}_i \cap \mathcal{A} \neq \emptyset\}$  is just the neighboring set  $N_G(\mathcal{A})$  in the corresponding bipartite graph  $G$ . Please refer to Figure 3.4 for an illustration of the bipartite graph representation of message assignments.

Let  $\mathcal{U}_G, \mathcal{V}_G$ , denote the partite sets corresponding to transmitters and messages in graph  $G$ , with respect to order. For all values of  $i \in [K]$ , define the following:

$$e_G(i) = \min_{\mathcal{A} \subseteq \mathcal{U}_G: |\mathcal{A}|=i} |N_G(\mathcal{A})|, \quad (3.20)$$

then we can readily see that

$$\eta_{\text{out}}(K, M) = \max_{G \in \mathcal{G}_M(K)} \min_{i \in [K]} \max(K - i, e_G(i)), \quad (3.21)$$



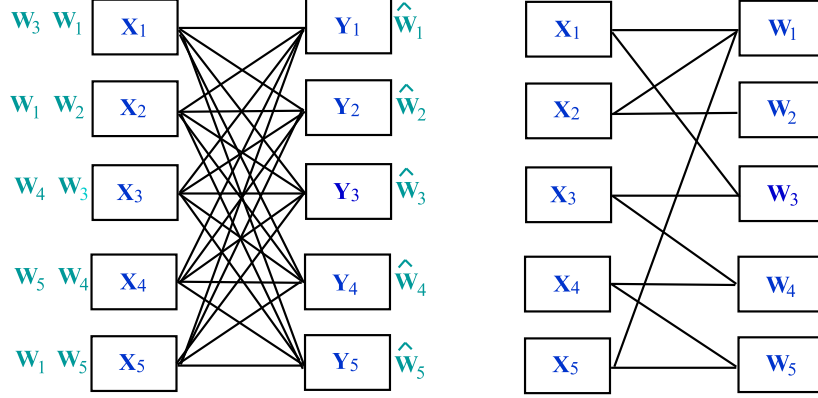


Figure 3.4: The bipartite graph on the right side represents the message assignment for the 5–user channel shown on the left side.

where  $\mathcal{G}_M(K)$  is the set of all bipartite graphs, whose equi-sized partite sets have size  $K$ , and the maximum degree of the nodes in the partite set  $\mathcal{V}_G$  is  $M$ .

For values of  $M > 2$ , Pinsker proved the following result in 1973 [46].

**Theorem 4.** *For any  $M > 2$ ,  $\exists$  a constant  $c > 1$ , and a sequence of  $M$ –regular bipartite graphs  $(G_{M,K})$  whose partite sets have  $K$  vertices, such that the following is true.*

$$\lim_{K \rightarrow \infty} \frac{e_{G_{M,K}}(\alpha K)}{\alpha K} \geq c, \forall 0 < \alpha \leq \frac{1}{2} \quad (3.22)$$

We next show that the above statement implies that  $\tau_{\text{out}}(M) > 2, \forall M > 2$ .

**Corollary 3.**

$$\tau_{\text{out}}(M) > \frac{1}{2}, \forall M > 2. \quad (3.23)$$

*Proof.* For each bipartite graph  $G$  with partite sets of size  $K$ , define  $i_{\min}(G)$  as,

$$i_{\min}(G) = \operatorname{argmin}_i \max(K - i, e_G(i)). \quad (3.24)$$

Now, assume that  $\tau_{\text{out}}(M) \leq \frac{1}{2}$ , then for the sequence  $(G_{M,K})$  chosen as in the statement of Theorem 4,

$$\lim_{K \rightarrow \infty} \frac{\max(K - i_{\min}(G_{M,K}), e_{G_{M,K}}(i_{\min}(G_{M,K})))}{K} \leq \frac{1}{2}. \quad (3.25)$$

It follows that

$$\lim_{K \rightarrow \infty} \frac{K - i_{\min}(G_{M,K})}{K} \leq \frac{1}{2}, \quad (3.26)$$

or,

$$\lim_{K \rightarrow \infty} \frac{i_{\min}(G_{M,K})}{K} \geq \frac{1}{2}. \quad (3.27)$$

But then as  $e_G(i)$  is non-decreasing in  $i$ , (3.22) implies that

$$\lim_{K \rightarrow \infty} \frac{e_{G_{M,K}}(i_{\min}(G_{M,K}))}{K} > \frac{1}{2}. \quad (3.28)$$

Therefore, the result in (3.22) implies that  $\tau_{\text{out}}(M) > \frac{1}{2}, \forall M > 2$ .  $\square$

It is worth noting that a sequence of bipartite graphs satisfying (3.22) is said to define a *vertex expander* as  $K \rightarrow \infty$ . To summarize, we have shown that because of message assignment strategies corresponding to vertex expanders, one cannot apply the bound in (3.17) directly to show that  $\tau(M) = \tau(1) = \frac{1}{2}$  for any  $M > 2$ . Finally, we show that in case the upper bound  $\tau_{\text{out}}(M)$  is tight, then using partial cooperation, the DoF gain can approach that achieved through assigning each message to all transmitters (full cooperation). More precisely, we show the following.

**Theorem 5.**

$$\lim_{M \rightarrow \infty} \tau_{\text{out}}(M) = 1 \quad (3.29)$$

*Proof.* We show that

$$\forall \epsilon > 0, \exists M(\epsilon) : \forall M \geq M(\epsilon), \tau_{\text{out}}(M) > (1 - \epsilon). \quad (3.30)$$

For each positive integer  $K$ , we construct a bipartite graph  $G_M(K)$ , whose partite sets are of order  $K$ , by taking the union of  $M$  *random* perfect matchings between the two partite sets. That is, the matchings are probabilistically independent, and each is drawn uniformly from the set of all possible matchings. One can easily see that the maximum degree of nodes in  $G_M(K)$  is bounded by  $M$ . i.e.,  $\Delta(G_M(K)) \leq M$ , and hence,  $G_M(K) \in \mathcal{G}_M(K)$ . We will prove that for any  $\epsilon > 0$ , there exists an  $M(\epsilon)$  sufficiently large, such that for any  $M \geq M(\epsilon)$ , the probability that each set of  $\epsilon K$  nodes in the partite set  $\mathcal{U}_{G_M(K)}$  have more than  $(1 - \epsilon)K$  neighbors, is bounded away from zero for large enough  $K$ . More precisely, we show that

$$\lim_{K \rightarrow \infty} Pr[\forall \mathcal{A} \subset \mathcal{U}_{G_M(K)} : |\mathcal{A}| = \epsilon K, |N_{G_M(K)}(\mathcal{A})| > (1 - \epsilon)K] > 0, \quad (3.31)$$

and hence, for large enough  $K$ , there exists a graph  $G$  in  $\mathcal{G}_M(K)$  where all subsets of  $\mathcal{U}_G$  of order  $\epsilon K$  have more than  $(1 - \epsilon)K$  neighbors in  $\mathcal{V}_G$ , i.e.,

$$e_G(i) > (1 - \epsilon)K, \forall i \geq \epsilon K, \quad (3.32)$$

and it follows that  $\eta_{\text{out}}(K, M) > (1 - \epsilon)K$ , and (3.30) holds.

We now show that (3.31) holds. Let  $\mathcal{A} \subset \mathcal{U}_{G_M(K)}$ ,  $\mathcal{B} \subset \mathcal{V}_{G_M(K)}$  such that  $|\mathcal{A}| = \epsilon K$ ,  $|\mathcal{B}| = (1 - \epsilon)K$ . For any random perfect matching, the probability that all the neighbors of  $\mathcal{A}$  are in  $\mathcal{B}$  is  $\frac{\binom{(1-\epsilon)K}{\epsilon K}}{\binom{K}{\epsilon K}}$ . By independence of the matchings, we get the following,

$$\begin{aligned} Pr[N_{G_M(K)}(\mathcal{A}) \subseteq \mathcal{B}] &= \left( \frac{\binom{(1-\epsilon)K}{\epsilon K}}{\binom{K}{\epsilon K}} \right)^M \\ &\leq ((1 - \epsilon)^{\epsilon K})^M. \end{aligned} \quad (3.33)$$

A direct application of the union bound results in the following,

$$\begin{aligned} Pr[|N_{G_M(K)}(\mathcal{A})| \leq (1 - \epsilon)K] &\leq \sum_{\mathcal{B} \subset \mathcal{V}_{G_M(K)}: |\mathcal{B}|=(1-\epsilon)K} Pr[N_{G_M(K)}(\mathcal{A}) \subseteq \mathcal{B}] \\ &\leq \binom{K}{(1 - \epsilon)K} (1 - \epsilon)^{\epsilon MK}, \end{aligned} \quad (3.34)$$

and,

$$\begin{aligned} Pr[\exists \mathcal{A} \subset \mathcal{U}_{G_M(K)} : |\mathcal{A}| = \epsilon K, |N_{G_M(K)}(\mathcal{A})| \leq (1 - \epsilon)K] &\leq \sum_{\mathcal{A} \subset \mathcal{U}_{G_M(K)}: |\mathcal{A}|=\epsilon K} Pr[|N_{G_M(K)}(\mathcal{A})| \leq (1 - \epsilon)K] \\ &\leq \binom{K}{\epsilon K} \binom{K}{(1 - \epsilon)K} (1 - \epsilon)^{\epsilon MK} \\ &= \binom{K}{\epsilon K}^2 (1 - \epsilon)^{\epsilon MK} \\ &\stackrel{(a)}{\approx} 2^{2KH(\epsilon)} (1 - \epsilon)^{\epsilon MK} \\ &= 2^{(2H(\epsilon) + \epsilon M \log(1 - \epsilon))K}, \end{aligned} \quad (3.35)$$

where  $H(\cdot)$  is the binary entropy function, and (a) follows as  $\binom{n}{\epsilon n} \approx 2^{nH(\epsilon)}$  for large enough  $n$ . Now, we choose  $M(\epsilon) > \frac{2H(\epsilon)}{-\epsilon \log(1 - \epsilon)}$ , to make the above

exponent negative, and the above probability will be strictly less than unity, i.e., we showed that for any  $M \geq M(\epsilon)$ ,

$$\lim_{K \rightarrow \infty} Pr[\exists \mathcal{A} \subset \mathcal{U}_{G_M(K)} : |\mathcal{A}| = \epsilon K, |N_{G_M(K)}(\mathcal{A})| \leq (1 - \epsilon)K] < 1, \quad (3.36)$$

which implies that (3.31) is true.  $\square$

### 3.5 Proof of Theorem 1

In this section, we show that the DoF of a  $K$ -user fully connected interference channel with a maximum transmit set size constraint of  $M$  is lower-bounded by

$$\eta(K, M) \geq \frac{K + M - 1}{2}.$$

We prove this by assigning each message to the transmitter with the same index as well as  $M - 1$  succeeding transmitters, and arguing that the DoF vector

$$d_i = \begin{cases} 1 & 1 \leq i \leq M - 1 \\ 0.5 & M \leq i \leq K \end{cases}$$

is achievable; i.e., the first  $M - 1$  users benefit from cooperation and achieve 1 degree of freedom, whereas the remaining  $K - M + 1$  users achieve 1/2 degree of freedom just like in the interference channel without cooperation. Conceptually, the achievable scheme in this section is based on converting the CoMP channel into a derived channel and then employing the asymptotic interference alignment scheme on the derived channel, as summarized in Figure 3.1. We now provide a detailed description of the design steps summarized in Figure 3.1.

#### 3.5.1 Derived Channel

Since our objective is to achieve a DoF vector that is asymmetric, the derived channel is also chosen to be asymmetric. The derived channel we consider in this section has two antennas for each of the first  $M - 1$  transmitters, and one antenna for each of the remaining  $K - M + 1$  transmitters. The received

signal at receiver  $i$  is given by,

$$Y_i = \sum_{j=1}^K g_{i,j}^{(1)} X_j^{(1)} + \sum_{j=1}^{M-1} g_{i,j}^{(2)} X_j^{(2)} + Z_i, \quad (3.37)$$

where  $g_{i,j}^{(m)}$  is the derived channel coefficient between the  $m^{\text{th}}$  antenna at transmitter  $j$  and receiver  $i$ , and  $X_j^{(m)}$  is the transmit signal of the  $m^{\text{th}}$  antenna at transmitter  $j$  in the derived channel. We assume that the channel inputs of the CoMP channel are related to the channel inputs of the derived channel through a linear transformation. The contribution of the derived channel input  $X_j^{(m)}$  in the real transmit signals  $X_j, X_{j+1}, \dots, X_{j+M-1}$  is defined by a  $M \times 1$  beam-forming vector, i.e.,

$$\begin{bmatrix} X_j \\ X_{j+1} \\ \vdots \\ X_{j+M-1} \end{bmatrix} = (*) + \mathbf{v}_j^{(m)} X_j^{(m)},$$

where  $(*)$  represents the contribution from other derived channel inputs. It is easy to see that the derived channel coefficients are related to the original channel coefficients as

$$g_{i,j}^{(m)} = \mathbf{H}_{i,\mathcal{T}_j} \mathbf{v}_j^{(m)},$$

for all  $i, j \in [K]$  and appropriate  $m$ . Since we are designing the achievable scheme to achieve 1 degree of freedom for the first  $M-1$  users, it must be that the first  $M-1$  receivers in the derived channel do not see any interference.

### 3.5.2 Zero-Forcing Step

We now explain our choice of the beam-forming vectors that ensures that the first  $M-1$  receivers do not see any interference.

#### ZF Beam Design

We first describe the general idea of constructing a zero-forcing beam. Consider the problem of designing a zero-forcing beam  $\mathbf{v}$  to be transmitted by  $n$  transmit antennas indexed by the set  $\mathcal{T} \subseteq [K]$  such that it does not cause

interference at  $n - 1$  receive antennas indexed by the set  $\mathcal{I} \subseteq [K]$ , i.e.,

$$\mathbf{H}_{\mathcal{I},\mathcal{T}}\mathbf{v} = \mathbf{0}.$$

Since  $\mathbf{H}_{\mathcal{I},\mathcal{T}}$  is an  $(n - 1) \times n$  matrix, the choice for  $\mathbf{v}$  is unique up to a scaling factor. For any arbitrary row vector  $\mathbf{a}$  of length  $n$ , we can use the Laplace expansion to expand the determinant

$$\det \begin{bmatrix} \mathbf{H}_{\mathcal{I},\mathcal{T}} \\ \mathbf{a} \end{bmatrix} = \sum_{j=1}^n a_j c_j,$$

where  $c_j$  is the cofactor of  $a_j$ , that depends only on the channel coefficients in  $\mathbf{H}_{\mathcal{I},\mathcal{T}}$ , and is independent of  $\mathbf{a}$ . By setting the beam-forming vector  $\mathbf{v}$  as  $\mathbf{v} = [c_1 \ c_2 \ \cdots \ c_n]$ , we see that an arbitrary receiver  $i$  sees the signal transmitted along the beam  $\mathbf{v}$  with a strength equal to

$$\mathbf{g} = \mathbf{H}_{\{i\},\mathcal{T}}\mathbf{v} = \det \begin{bmatrix} \mathbf{H}_{\mathcal{I},\mathcal{T}} \\ \mathbf{H}_{\{i\},\mathcal{T}} \end{bmatrix} = \det \mathbf{H}_{\mathcal{I} \cup \{i\},\mathcal{T}}.$$

Clearly, this satisfies the zero-forcing condition  $\mathbf{H}_{\{i\},\mathcal{T}}\mathbf{v} = 0$  for all  $i \in \mathcal{I}$ .

Design of Transmit Beam  $\mathbf{v}_j^{(1)}$  for  $j \geq M$

The signal  $X_j^{(1)}$  is transmitted by the  $M$  transmitters from the transmit set  $\mathcal{T}_j = \{j, j + 1, \dots, j + M - 1\}$ . The corresponding beam  $\mathbf{v}_j^{(1)}$  is designed to avoid the interference at the first  $M - 1$  receivers  $\mathcal{I} = [M - 1]$ . Therefore, we see that the contribution of  $X_j^{(1)}$  at receiver  $i$  is given by

$$g_{i,j}^{(1)} = \det \mathbf{H}_{\mathcal{A},\mathcal{B}}, \tag{3.38}$$

where

$$\begin{aligned} \mathcal{A} &= \{1, 2, \dots, M - 1, i\} \\ \mathcal{B} &= \{j, j + 1, \dots, j + M - 1\}. \end{aligned}$$

Design of Transmit Beams  $\mathbf{v}_j^{(1)}$  and  $\mathbf{v}_j^{(2)}$  for  $j < M$

The signals  $X_j^{(1)}$  and  $X_j^{(2)}$  are transmitted by the  $M$  transmitters from the transmit set  $\mathcal{T}_j = [M]$ . They must avoid interference at the  $M - 2$  receivers

$$\mathcal{I} = \{1, 2, \dots, j - 1, j + 1, \dots, M - 1\}.$$

Since we only need to avoid interference at  $M - 2$  receivers, it is sufficient to transmit each signal from  $M - 1$  transmitters. We use the first  $M - 1$  antennas of the transmit set  $\mathcal{T}_j$  to transmit  $X_j^{(1)}$ , and the last  $M - 1$  antennas of the transmit set  $\mathcal{T}_j$  to transmit  $X_j^{(2)}$ . Thus, we obtain

$$\begin{aligned} g_{i,j}^{(1)} &= \det \mathbf{H}_{\mathcal{A}, \mathcal{B}_1} \\ g_{i,j}^{(2)} &= \det \mathbf{H}_{\mathcal{A}, \mathcal{B}_2}, \end{aligned} \tag{3.39}$$

where

$$\begin{aligned} \mathcal{A} &= \{1, 2, \dots, j - 1, j + 1, M - 1, i\} \\ \mathcal{B}_1 &= \{j, j + 1, \dots, j + M - 2\} \\ \mathcal{B}_2 &= \{j + 1, j + 1, \dots, j + M - 1\}. \end{aligned}$$

Thus, the derived channel (3.37) can be simplified as

$$\begin{aligned} Y_i &= g_{i,i}^{(1)} X_j^{(1)} + g_{i,i}^{(2)} X_j^{(2)} + Z_i, \quad 1 \leq i < M \\ Y_i &= \sum_{j=1}^K g_{i,j}^{(1)} X_j^{(1)} + \sum_{j=1}^{M-1} g_{i,j}^{(2)} X_j^{(2)} + Z_i, \quad M \leq i \leq K, \end{aligned} \tag{3.40}$$

where the derived channel coefficients are as described in (3.38) and (3.39).

### 3.5.3 Asymptotic Interference Alignment

In this section, we consider  $L$  parallel derived channels, and propose a scheme achieving a DoF arbitrary close to  $(K + M - 1)/2$  in the limit  $L \rightarrow \infty$ . We

can combine  $L$  parallel derived channels (3.40) and express them together as

$$\begin{aligned} \underline{Y}_i &= \mathbf{G}_{i,i}^{(1)} \underline{X}_j^{(1)} + \mathbf{G}_{i,i}^{(2)} \underline{X}_j^{(2)} + \underline{Z}_i, \quad 1 \leq i < M \\ \underline{Y}_i &= \sum_{j=1}^K \mathbf{G}_{i,j}^{(1)} \underline{X}_j^{(1)} + \sum_{j=1}^{M_i-1} \mathbf{G}_{i,j}^{(2)} \underline{X}_j^{(2)} + \underline{Z}_i, \quad M \leq i \leq K, \end{aligned}$$

where  $\underline{X}_j^{(m)}$ ,  $\underline{Y}_i$  and  $\underline{Z}_i$  are  $L \times 1$  column vectors and  $\mathbf{G}_{i,j}^{(m)}$  is  $L \times L$  diagonal channel transfer matrix given by

$$\mathbf{G}_{i,j}^{(m)} = \begin{bmatrix} g_{i,j}^{(m)}(1) & & & \\ & g_{i,j}^{(m)}(2) & & \\ & & \ddots & \\ & & & g_{i,j}^{(m)}(L) \end{bmatrix}.$$

The achievable scheme that we propose is based on the asymptotic alignment scheme introduced by Cadambe and Jafar in [11].

**Definition 1** (Cadambe-Jafar (CJ) subspace). *The order- $n$  CJ subspace generated by the diagonal matrices*

$$\mathbf{G}_1, \mathbf{G}_2, \dots, \mathbf{G}_N$$

is defined as the linear subspace spanned by the vectors

$$\{\mathbf{G}_1^{a_1} \mathbf{G}_2^{a_2} \dots \mathbf{G}_N^{a_N} \mathbf{1} : \mathbf{a} \in \mathbb{Z}_+^N \text{ and } \sum_i a_i \leq n\},$$

where  $\mathbf{1}$  is the  $L \times 1$  column vector of all ones. The matrix containing these  $\binom{N+n}{n}$  vectors as columns is said to be the order- $n$  CJ matrix.

Let  $\mathbf{V}$  denote the order- $n$  CJ subspace (and the corresponding matrix) generated by the nontrivial channel matrices carrying interference:

$$\{\mathbf{G}_{i,j}^{(1)}, \mathbf{G}_{i,j}^{(2)} : i \geq M, j < M\} \cup \{\mathbf{G}_{i,j}^{(1)} : i \neq j \geq M\}. \quad (3.41)$$

We use  $\mathbf{V}$ , defined as the transmit beam-forming matrix at every transmitter of the derived channel. The first  $M - 1$  receivers do not see any interference. Therefore, for each  $k < M$ , the receiver  $k$  can decode all the



desired streams free of interference if the matrix

$$\mathbf{M}_k = \begin{bmatrix} \mathbf{G}_{kk}^{(1)}\mathbf{V} & \mathbf{G}_{kk}^{(2)}\mathbf{V} \end{bmatrix}$$

has full column rank. Assuming that the number of rows in  $\mathbf{M}_k$ , equal to the number of parallel channels  $L$ , is greater than or equal to the number of columns, i.e.,  $L \geq 2|\mathbf{V}|$ , the matrix  $\mathbf{M}_k$  has full column rank for generic (original) channel coefficients  $\{H_{i,j}\}$  if the following claim is true. See Corollary 1 in Appendix B for an explanation.

**Claim 1.** *For each  $k < M$ , the polynomials denoted by the variables*

$$\begin{aligned} \{g_{k,k}^{(1)}, g_{k,k}^{(2)}\} &\cup \{g_{i,j}^{(1)}, g_{i,j}^{(2)} : i \geq M, j < M\} \\ &\cup \{g_{i,j}^{(1)} : i \neq j \geq M\} \end{aligned} \quad (3.42)$$

*are algebraically independent.*

For each  $k \geq M$ , the interference seen at receiver  $k$  is limited to the order- $(n+1)$  CJ subspace, denoted by  $\mathbf{INT}$ . Therefore, the receiver  $k$  can decode all the desired streams free of interference if the matrix

$$\mathbf{M}_k = \begin{bmatrix} \mathbf{G}_{kk}^{(1)}\mathbf{V} & \mathbf{INT} \end{bmatrix}$$

has full column rank. Assuming that the number of rows is greater than or equal to the number of columns, i.e.,  $L \geq |\mathbf{V}| + |\mathbf{INT}|$ , the matrix  $\mathbf{M}_k$  has full column rank for generic (original) channel coefficients  $\{H_{i,j}\}$  if the following claim is true.

**Claim 2.** *For each  $k \geq M$ , the polynomials denoted by the variables*

$$\begin{aligned} \{g_{k,k}^{(1)}\} &\cup \{g_{i,j}^{(1)}, g_{i,j}^{(2)} : i \geq M, j < M\} \\ &\cup \{g_{i,j}^{(1)} : i \geq M, j \geq M, i \neq j\} \end{aligned} \quad (3.43)$$

*are algebraically independent.*

To satisfy the requirements on  $L$ , we choose  $L$  as

$$L = \max(2|\mathbf{V}|, |\mathbf{V}| + |\mathbf{INT}|) = |\mathbf{V}| + |\mathbf{INT}|.$$

Observe that

$$|\mathbf{V}| = \binom{N+n}{n} \text{ and } |\mathbf{INT}| = \binom{N+n+1}{n+1},$$

where  $N$  is the number of matrices (3.41) used to generate the CJ subspace, and is given by

$$\begin{aligned} N &= 2(K-M+1)(M-1) + (K-M+1)(K-M) \\ &= (K-M+1)(K+M-2). \end{aligned} \tag{3.44}$$

Therefore, if we let  $\eta(K, M, L)$  be the maximum achievable DoF by coding over at most  $L$  parallel channels, then we obtain the following,

$$\begin{aligned} \eta(K, M, L) &\geq \frac{2(M-1)|\mathbf{V}| + (K-M+1)|\mathbf{V}|}{L} \\ &= \frac{(K+M-1)|\mathbf{V}|}{|\mathbf{V}| + |\mathbf{INT}|} \\ &= \frac{K+M-1}{2 + \frac{N}{n+1}}. \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} \eta(K, M) &= \limsup_{L \rightarrow \infty} \eta(K, M, L) \\ &\geq \lim_{n \rightarrow \infty} \frac{K+M-1}{2 + \frac{N}{n+1}} \\ &= \frac{K+M-1}{2}. \end{aligned}$$

### 3.5.4 Proof of Algebraic Independence

We use the Jacobian criterion of Lemma 6 in Appendix B to prove Claims 1 and 2. Recall that each derived channel coefficient is a polynomial in  $K^2$  variables  $\{H_{i,j} : 1 \leq i, j, \leq K\}$ . Let  $\mathbf{g}$  denote the vector consisting of the polynomials specified by the derived channel coefficients in the respective claims. The exact description of the polynomials can be obtained from (3.38) and (3.39) in Section 3.5.2. The number of polynomials in Claims 1 and 2 is equal to  $N+2$  and  $N+1$ , respectively, where  $N$  is given by (3.44). From Lemma 6 in Appendix B, we see that a collection of polynomials is

algebraically independent if and only if the corresponding Jacobian matrix has full row rank. It can be easily verified that  $N + 2 \leq K^2$ , and hence  $N + 1 \leq K^2$ , for any  $K$  and  $M$ , which is a necessary condition for the corresponding Jacobian matrices to have full row rank. It is easy to verify that the Jacobian matrices corresponding to the polynomials in Claims 1 and 2 have full row rank using symbolic toolbox of MATLAB for any fixed  $K$  and  $M$ . In particular, we verified that the Jacobian matrices have full row rank for all values of  $M < K \leq 9$ .

### 3.6 Auxiliary Lemmas for Large Networks Upper Bounds

**Lemma 1.** *For a  $K$ -user channel where each message is available at a maximum of  $M$  transmitters, there exists a transmitter carrying at most  $M$  messages,*

$$\text{There exists } i \in [K] \text{ such that } |C_{\{i\}}| \leq M.$$

*Proof.* The statement follows by the pigeonhole principle, since the following holds,

$$\sum_{i=1}^K |C_{\{i\}}| = \sum_{i=1}^K |\mathcal{T}_i| \leq MK. \quad (3.45)$$

□

**Lemma 2.** *For a  $K$ -user channel, if each message is available at a maximum of  $M$  transmitters, and  $M \geq 2$ , if there exists a set  $\mathcal{A}$  of  $n$  transmitters carrying at most  $(M - 1)n + 1$  messages, then there exists a set  $\mathcal{B}$  of  $n + 1$  transmitters carrying at most  $(M - 1)(n + 1) + 1$  messages. More precisely, if  $\exists \mathcal{A} \subset [K]$  such that  $|\mathcal{A}| = n < K$ , and  $|C_{\mathcal{A}}| \leq (M - 1)n + 1$ , then  $\exists \mathcal{B} \subseteq [K]$  such that  $|\mathcal{B}| = n + 1$ , and  $|C_{\mathcal{B}}| \leq (M - 1)(n + 1) + 1$ .*

*Proof.* We only consider the case where  $K > (M - 1)(n + 1) + 1$ , as otherwise, the statement trivially holds. In this case, we can show that

$$M(K - |C_{\mathcal{A}}|) < (K - n)((M - 1)(n + 1) + 2 - |C_{\mathcal{A}}|). \quad (3.46)$$

The proof of (3.46) is available in Lemma 3 below. Note that the left-hand side in (3.46) is the maximum number of message instances for messages

outside the set  $C_{\mathcal{A}}$ , i.e.,

$$\begin{aligned} \sum_{i \in [K], i \notin \mathcal{A}} |C_{\{i\}} \setminus C_{\mathcal{A}}| &\leq M(K - |C_{\mathcal{A}}|) \\ &< (K - n)((M - 1)(n + 1) + 2 - |C_{\mathcal{A}}|). \end{aligned} \quad (3.47)$$

Since the number of transmitters outside the set  $\mathcal{A}$  is  $K - n$ , it follows by the pigeonhole principle that there exists a transmitter whose index is outside  $\mathcal{A}$  and carries at most  $(M - 1)(n + 1) + 1 - |C_{\mathcal{A}}|$  messages whose indices are outside  $C_{\mathcal{A}}$ . More precisely,

$$\exists i \in [K] \setminus \mathcal{A} : |C_{\{i\}} \setminus C_{\mathcal{A}}| \leq (M - 1)(n + 1) + 1 - |C_{\mathcal{A}}|. \quad (3.48)$$

It follows that there exists a transmitter whose index is outside the set  $\mathcal{A}$  and can be added to the set  $\mathcal{A}$  to form the set  $\mathcal{B}$  that satisfies the statement.  $\square$

**Lemma 3.** *For a  $K$ -user channel, if each message is available at a maximum of  $M$  transmitters and  $K \geq (M - 1)(n + 1) + 1$ ,  $M \geq 2$ , and there exists a set  $\mathcal{S} \subseteq [K]$  such that  $|\mathcal{S}| \leq (M - 1)n + 1$ , then the following holds,*

$$M(K - |\mathcal{S}|) < (K - n)((M - 1)(n + 1) + 2 - |\mathcal{S}|). \quad (3.49)$$

*Proof.* We first prove the statement for the case where  $|\mathcal{S}| = (M - 1)n + 1$ . This directly follows as,

$$\begin{aligned} M(K - |\mathcal{S}|) &= M(K - ((M - 1)n + 1)) \\ &\leq M(K - (n + 1)) \\ &< M(K - n) \\ &= (K - n)((M - 1)(n + 1) + 2 - |\mathcal{S}|). \end{aligned} \quad (3.50)$$

In order to complete the proof, we note that each decrement of  $|\mathcal{S}|$  leads to an increase in the left hand side by  $M$ , and in the right-hand side by  $K - n$ ,

and,

$$\begin{aligned}
K - n &\geq (M - 1)(n + 1) + 1 - n \\
&= (M - 2)n + M \\
&\geq M.
\end{aligned} \tag{3.51}$$

□

**Lemma 4.** *For a  $K$ -user channel, if each message is available at a maximum of  $M = 3$  transmitters, and there exists a set  $\mathcal{A}$  of  $n$  transmitters carrying at most  $n + \frac{K+1}{4} + 1$  messages, and  $\frac{K+1}{4} \leq n < K$ , then there exists a set  $\mathcal{B}$  of  $n + 1$  transmitters carrying at most  $n + \frac{K+1}{4} + 2$  messages. More precisely, if  $\exists \mathcal{A} \subset [K]$  such that  $|\mathcal{A}| = n$ ,  $\frac{K+1}{4} \leq n < K$ , and  $|C_{\mathcal{A}}| \leq n + \frac{K+1}{4} + 1$ , then  $\exists \mathcal{B} \subset [K]$  such that  $|\mathcal{B}| = n + 1$ ,  $|C_{\mathcal{B}}| \leq n + \frac{K+1}{4} + 2$ .*

*Proof.* The proof follows in a similar fashion to that of Lemma 2. Let  $x = n + \frac{K+1}{4} + 1$ . We only consider the case where  $K > x + 1$ , as otherwise, the proof is trivial. We first assume the following,

$$3(K - |C_{\mathcal{A}}|) < (K - n) \left( n + \frac{K+1}{4} + 3 - |C_{\mathcal{A}}| \right). \tag{3.52}$$

Now, it follows that

$$\begin{aligned}
\sum_{i \in [K], i \notin \mathcal{A}} |C_{\{i\}} \setminus C_{\mathcal{A}}| &\leq M(K - |C_{\mathcal{A}}|) \\
&< (K - n) \left( n + \frac{K+1}{4} + 3 - |C_{\mathcal{A}}| \right),
\end{aligned} \tag{3.53}$$

and hence,

$$\exists i \in [K] \setminus \mathcal{A} : |C_{\{i\}} \setminus C_{\mathcal{A}}| \leq n + \frac{K+1}{4} + 2 - |C_{\mathcal{A}}|, \tag{3.54}$$

and then the set  $\mathcal{B} = \mathcal{A} \cup \{i\}$  satisfies the statement of the lemma. Finally,

we need to show that (3.52) is true. For the case where  $|C_{\mathcal{A}}| = x$ ,

$$\begin{aligned}
3x &= \frac{3K}{4} + \frac{15}{4} + 3n \\
&= (2n + K) + \left(n - \frac{K}{4} + \frac{15}{4}\right) \\
&> 2n + K,
\end{aligned} \tag{3.55}$$

and hence,  $3(K - x) < 2(K - n)$ , which implies (3.52) for the case where  $|C_{\mathcal{A}}| = x$ . Moreover, we note that each decrement of  $|C_{\mathcal{A}}|$  increases the left-hand side of (3.52) by 3 and the right-hand side by  $(K - n)$ , and we know that,

$$\begin{aligned}
K &> x + 1 \\
&= n + \frac{K + 1}{4} + 2 \\
&\geq n + 2,
\end{aligned} \tag{3.56}$$

and hence,  $K - n \geq 3$ , so there is no loss of generality in assuming that  $|C_{\mathcal{A}}| = x$  in the proof of (3.52), and the proof is complete.  $\square$

# CHAPTER 4

## LOCALLY CONNECTED CHANNELS

In Section 2.1, we defined the locally connected channel model as a function of the number of dominant interferers per receiver  $L$ , by connecting each transmitter to  $\lfloor \frac{L}{2} \rfloor$  preceding receivers and  $\lceil \frac{L}{2} \rceil$  succeeding receivers. Recall that for a  $K$ -user locally connected channel with connectivity parameter  $L$ , we use  $\eta_L(K, M)$  as the best achievable DoF  $\eta$  over all choices of transmit sets satisfying the maximum transmit set size constraint in (2.3), and  $\tau_L(M)$  to denote the asymptotic per user DoF.

For the locally connected channel model where  $L > 1$ , let  $x = \lfloor \frac{L}{2} \rfloor$ . We silence the first  $x$  transmitters, deactivate the last  $x$  receivers, and relabel the transmit signals to obtain a  $(K - x)$ -user channel, where transmitter  $j$  is connected to receivers in the set  $\{Y_i : i \in \{j, j + 1, \dots, j + L\}\}$ . We note that the new channel model gives the same value of  $\tau_L(M)$  as the original one, since  $x = o(K)$ . Unless explicitly stated otherwise, we will be using this equivalent model in the rest of this chapter. More precisely, we consider the following channel model,

$$H_{i,j} \text{ is not identically 0 if and only if } i \in [j, j + 1, \dots, j + L], \quad (4.1)$$

and all non-zero channel coefficients are generic. We show an example construction of the equivalent channel model in Figure 4.1.

### 4.1 Prior Work

In [19], the special case of Wyner's asymmetric model ( $L = 1$ ) was considered, and the spiral message assignment strategy mentioned in Section 3.1 was fixed, i.e., each message is assigned to its own transmitter as well as  $M - 1$  following transmitters. The asymptotic per user DoF was then characterized

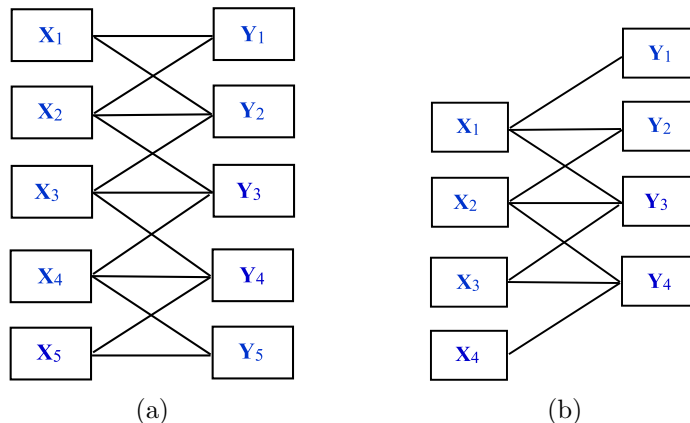


Figure 4.1: Construction of the equivalent locally connected channel model with a number of users  $K = 5$  and connectivity parameter  $L = 2$ . In (a), the original model of (2.2) is shown. In (b), the new model is shown.

as  $\frac{M}{M+1}$ . This shows for our problem that

$$\tau_1(M) \geq \frac{M}{M+1}. \quad (4.2)$$

In [47, Remark 2], a message assignment strategy was described to enable the achievability of an asymptotic per user DoF as high as  $\frac{2M-1}{2M}$ , it can be easily verified that this is indeed true, and hence, we know that

$$\tau_1(M) \geq \frac{2M-1}{2M}. \quad (4.3)$$

The main difference in the strategy described in [47, Remark 2] from the spiral message assignment strategy considered in [19], is that unlike the spiral strategy, messages are assigned to transmitters in an asymmetric fashion, where we say that a message assignment is symmetric if and only if for all  $j, i \in [K], j > i$ , the transmit set  $\mathcal{T}_j$  is obtained by shifting forward the indices of the elements of the transmit set  $\mathcal{T}_i$  by  $(j - i)$ .

We show that both the message assignment strategy analyzed in [19] and the one suggested in [47] are suboptimal for  $L = 1$ , and the value of  $\tau_1(M)$  is in fact strictly larger than the bounds in (4.2) and (4.3). The key idea enabling our result is that each message need not be available at the transmitter carrying its own index. We start by illustrating a simple example for the case of no cooperation ( $M = 1$ ) that highlights the idea behind our scheme.



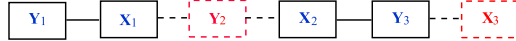


Figure 4.2: Achieving  $2/3$  per user DoF for  $M = L = 1$ . Each transmitter is carrying a message for the receiver connected to it by a solid line. The figure shows only signals corresponding to the first three users in a general  $K$ -user network. Signals in dashed boxes are deactivated. Note that the deactivation of  $X_3$  splits this part of the network from the rest.

## 4.2 Example: $M = L = 1$

Let  $W_1$  be available at the first transmitter,  $W_3$  be available at the second transmitter, and deactivate both the second receiver and the third transmitter. Then it is easily seen that messages  $W_1$  and  $W_3$  can be received without interfering signals at their corresponding receivers. Moreover, the deactivation of  $X_3$  splits this part of the network from the rest, i.e., the same scheme can be repeated by assigning  $W_4, W_6$ , to the transmitters with transmit signals  $X_4, X_5$ , respectively, and so on. Thus, two degrees of freedom can be achieved for each set of three users, thereby, achieving an asymptotic per user DoF of  $\frac{2}{3}$ . The described message assignment is depicted in Figure 4.2. It is evident now that a constraint that is only a function of the load on the *backhaul* link may lead to a discovery of better message assignments than the one considered in [19]. In Section 4.3, we show that the optimal message assignment strategy under the cooperation order constraint (2.3) is different from the spiral strategy. The above described message assignment strategy and coding scheme for the special case of  $M = L = 1$  are shown to be optimal. It is worth noting that the optimality of a TDMA scheme in this case follows as a special case from a general result in [48], where necessary and sufficient conditions on channel connectivity and message assignment are derived for TDMA schemes to be optimal.

## 4.3 Achieving Scalable DoF Cooperation Gains

In this section, we consider a simple linear precoding coding scheme, where each message is assigned to a set of transmitters with successive indices, and a zero-forcing transmit beam-forming strategy is employed [49]. The transmit

signal at the  $j^{\text{th}}$  transmitter is given by,

$$X_j = \sum_{i:j \in \mathcal{T}_i} X_{j,i}, \quad (4.4)$$

where  $X_{j,i}$  depends only on message  $W_i$ .

Using simple zero-forcing transmit beams with a fractional reuse scheme that activates only a subset of transmitters and receivers in each channel use, we extend the example in Section 4.2 to achieve scalable DoF cooperation gains for any value of  $M > \frac{L}{2}$ .

**Theorem 6.** *The following lower bound holds for the asymptotic per user DoF of a locally connected channel with connectivity parameter  $L$ ,*

$$\tau_L(M) \geq \max \left\{ \frac{1}{2}, \frac{2M}{2M+L} \right\}, \forall M \in \mathbf{Z}^+. \quad (4.5)$$

*Proof.* Showing that  $\tau_L(M) \geq \frac{1}{2}, \forall M \geq 1$  follows by a straightforward extension of the asymptotic interference alignment scheme of [11], and hence, it suffices to show that  $\tau_L(M) \geq \frac{2M}{2M+L}$ .

We treat the network as a set of clusters, each consisting of consecutive  $2M + L$  transceivers. The last  $L$  transmitters of each cluster are deactivated to eliminate *inter-cluster* interference. It then suffices to show that  $2M$  DoF can be achieved in each cluster. Without loss of generality, consider the cluster with users of indices in the set  $[2M + L]$ . We define the following subsets of  $[2M + L]$ ,

$$\begin{aligned} \mathcal{S}_1 &= [M], \\ \mathcal{S}_2 &= \{L + M + 1, L + M + 2, \dots, L + 2M\}. \end{aligned}$$

We next show that each user in  $\mathcal{S}_1 \cup \mathcal{S}_2$  achieves one degree of freedom, while messages  $\{W_{M+1}, W_{M+2}, \dots, W_{L+M}\}$  are not transmitted. In the proposed scheme, users in the set  $\mathcal{S}_1$  are served by transmitters in the set  $\{X_1, X_2, \dots, X_M\}$  and users in the set  $\mathcal{S}_2$  are served by transmitters in the set  $\{X_{M+1}, X_{M+2}, \dots, X_{2M}\}$ . Let the message assignments be as follows,

$$\mathcal{T}_i = \begin{cases} \{i, i + 1, \dots, M\}, & \forall i \in \mathcal{S}_1, \\ \{i - L, i - L - 1, \dots, M + 1\}, & \forall i \in \mathcal{S}_2. \end{cases}$$

Now, we note that messages with indices in  $\mathcal{S}_1$  are not available outside transmitters with indices in  $[M]$ , and hence, do not cause interference at receivers with indices in  $\mathcal{S}_2$ . Also, messages with indices in  $\mathcal{S}_2$  are not available at transmitters with indices in  $[M]$ , and hence, do not cause interference at receivers with indices in  $\mathcal{S}_1$ .

In order to complete the proof by showing that each user in  $\mathcal{S}_1 \cup \mathcal{S}_2$  achieves one degree of freedom, we next show that transmissions corresponding to messages with indices in  $\mathcal{S}_1(\mathcal{S}_2)$  do not cause interference at receivers with indices in the same set. To avoid redundancy, we only describe in detail the design of transmit beams for message  $W_1$  to cancel its interference at all receivers in  $\mathcal{S}_1$  except its own receiver. First, the encoding of  $W_1$  into  $X_{1,1}$  at the first transmitter is done in a way that is oblivious to the existence of other receivers in the network except the first receiver, and a capacity achieving code for the point-to-point link  $H_{1,1}$  is used [50]. We then design  $X_{2,1}$  at the second transmitter to cancel the interference caused by  $W_1$  at the second receiver, i.e.,

$$X_{2,1} = -\frac{H_{2,1}}{H_{2,2}}X_{1,1}. \quad (4.6)$$

Similarly, the transmit beam  $X_{3,1}$  is then designed to cancel the interference caused by  $W_1$  at the third receiver. The transmit beams  $X_{i,1}, i \in \{2, 3, \dots, M\}$  are successively designed with respect to order of the index  $i$  such that the received signal due to  $X_{i,1}$  at the  $i^{\text{th}}$  receiver cancels the interference caused by  $W_1$ .

In general, the availability of channel state information at the transmitters allows a design for the transmit beams for message  $W_i$  that delivers it to the  $i^{\text{th}}$  receiver with a capacity achieving point-to-point code and simultaneously cancels its effect at receivers with indices in the set  $\mathcal{C}_i$ , where,

$$\mathcal{C}_i = \begin{cases} \{i+1, i+2, \dots, M\}, & \forall i \in \mathcal{S}_1, \\ \{i-1, i-2, \dots, L+M+1\}, & \forall i \in \mathcal{S}_2. \end{cases}$$

Note that both  $\mathcal{C}_M$  and  $\mathcal{C}_{L+M+1}$  equal the empty set, because both  $W_M$  and  $W_{L+M+1}$  do not contribute to interfering signals at receivers with indices in the set  $\mathcal{S}_1 \cup \mathcal{S}_2$ . We conclude that each receiver with index in the set  $\mathcal{S}_1 \cup \mathcal{S}_2$  suffers only from Gaussian noise, thereby enjoying one degree of freedom.  $\square$

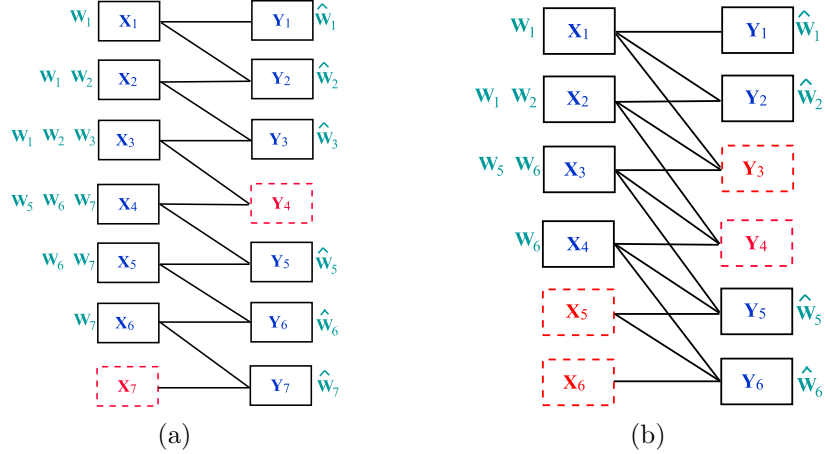


Figure 4.3: The figure shows the assignment of messages in the proof of Theorem 6 for the case where  $M = 3, L = 1$  in (a), and the case where  $M = L = 2$  in (b). Only signals corresponding to the first cluster are shown. Signals in dashed red boxes are deactivated. Note that the last  $L$  transmit signals are deactivated to eliminate inter-cluster interference. Also, messages  $\{W_{M+1}, \dots, W_{M+L}\}$  are not transmitted, while each other message with indices in  $\{1, 2, \dots, 2M + L\}$  has one degree of freedom.

Refer to Figure 4.3 for an illustration of the above described coding scheme. We note that in the above coding scheme, some messages are not being transmitted in order to allow for interference-free communication for the remaining messages. It is worth noting that this can be done while maintaining fairness in the allocation of the available DoF over all users through *fractional reuse* in a system where multiple sessions of communication take place, and different sets of receivers are deactivated in different sessions, e.g., in different time slots or different sub-carriers (in an OFDM system).

#### 4.4 Irreducible Message Assignments and Optimality of Local Cooperation

In order to find an upper bound on the per user DoF  $\tau_L(M)$ , we have to consider all possible message assignment strategies satisfying the cooperation order constraint (2.3). In this section, we characterize necessary conditions for the optimal message assignment. The constraints we provide for transmit sets are governed by the connectivity pattern of the channel. For example, for the case where  $M = 1$ , any assignment of message  $W_i$  to a transmitter

that is not connected to  $Y_i$  is *reducible*, i.e., the rate of transmitting message  $W_i$  has to be zero for these assignments, and hence, removing  $W_i$  from its carrying transmitter does not reduce the sum rate in these cases.

We now introduce a graph theoretic representation that simplifies the presentation of the necessary conditions on irreducible message assignments. For message  $W_i$ , and a fixed transmit set  $\mathcal{T}_i$ , we construct the following graph  $G_{W_i, \mathcal{T}_i}$  that has  $[K]$  as its set of vertices, and an edge exists between any given pair of vertices  $x, y \in [K]$  if and only if:

- $x, y \in \mathcal{T}_i$ .
- $|x - y| \leq L$ .

Vertices corresponding to transmitters connected to  $Y_i$  are given a special mark, i.e., vertices with labels in the set  $\{i, i - 1, \dots, i - L\}$  are marked for the considered channel model. Refer to Figure 4.4 for an example illustration of  $G_{W_i, \mathcal{T}_i}$ .

We now have the following statement.

**Lemma 1.** *For any  $k \in \mathcal{T}_i$  such that the vertex  $k$  in  $G_{W_i, \mathcal{T}_i}$  is not connected to a marked vertex, removing  $k$  from  $\mathcal{T}_i$  does not decrease the sum rate.*

*Proof.* Let  $\mathcal{S}$  denote the set of indices of vertices in a component with no marked vertices. We need to show that removing any transmitter index in  $\mathcal{S}$  from  $\mathcal{T}_i$  does not decrease the sum rate. Let  $\mathcal{S}'$  be the set of indices of received signals that are connected to at least one transmitter with an index in  $\mathcal{S}$ . To prove the lemma, we consider two scenarios, where we add a *tilde* over symbols denoting signals belonging to the second scenario. For the first scenario,  $W_i$  is made available at all transmitters with indices in  $\mathcal{S}$ . Let  $Q$  be a random variable that is independent of all messages and has the same distribution as  $W_i$ , then for the second scenario,  $W_i$  is not available at any transmitter with an index in  $\mathcal{S}$ , and a realization  $q$  of  $Q$  is generated and given to all transmitters with indices in  $\mathcal{S}$  before communication starts. Moreover, the given realization  $Q = q$  contributes to the encoding of  $\tilde{X}_{\mathcal{S}}$  in the same fashion as a message  $W_i = q$  contributes to  $X_{\mathcal{S}}$ . Assuming a reliable communication scheme for the first scenario that uses a large block length  $n$ , the following argument shows that the achievable sum rate is also achievable after removing  $W_i$  from the designated transmitters. And therefore, proving

that removing any transmitter in  $\mathcal{S}$  from  $\mathcal{T}_i$  does not decrease the sum rate.

$$\begin{aligned}
n \sum_j R_j &= \sum_j \mathsf{H}(W_j) \\
&\stackrel{(a)}{\leq} \sum_j I(W_j; Y_j^n) + o(n) \\
&= \sum_{j \in \mathcal{S}'^c} I(W_j, Y_j^n) + \sum_{j \in \mathcal{S}'} I(W_j; Y_j^n) + o(n) \\
&\stackrel{(b)}{=} \sum_{j \in \mathcal{S}'^c} I(W_j, \tilde{Y}_j^n) + \sum_{j \in \mathcal{S}'} I(W_j; Y_j^n) + o(n) \\
&\stackrel{(c)}{=} \sum_{j \in \mathcal{S}'^c} I(W_j, \tilde{Y}_j^n) + \sum_{j \in \mathcal{S}'} I(W_j; \tilde{Y}_j^n) + o(n) \\
&= \sum_j \mathsf{H}(W_j) - \mathsf{H}(W_j | \tilde{Y}_j^n) + o(n),
\end{aligned}$$

where  $\mathsf{H}(\cdot)$  is the entropy function for discrete random variables, (a) follows from Fano's inequality, (b) follows as the difference between the two scenarios lies in the encoding of  $X_{\mathcal{S}}$  which affects only  $Y_{\mathcal{S}'}$ , and (c) holds because any two transmitters carrying  $W_i$  and connected to a receiver whose index is in  $\mathcal{S}'$  must belong to the same component, and hence,  $W_i$  contributes to  $Y_{\mathcal{S}'}$  only through  $X_{\mathcal{S}}$ , it follows that  $(W_j, Y_j^n)$  has the same joint distribution as  $(W_j, \tilde{Y}_j^n)$  for every  $j \in \mathcal{S}'$ . Now, it follows that

$$\sum_j \mathsf{H}(W_j | \tilde{Y}_j^n) = o(n), \tag{4.7}$$

and hence, the rates  $R_j, j \in [K]$  are achievable in the second scenario.  $\square$

We call a message assignment *irreducible* if no element in it can be removed without decreasing the sum rate. The following corollary to Lemma 1 characterizes a necessary condition for any message assignment satisfying the cooperation order constraint in (2.3) to be irreducible. Recall that two vertices in a graph  $G$  are at a distance  $d$  if and only if the shortest path in  $G$  between the two vertices has  $d$  edges.

**Corollary 4.** *Let  $\mathcal{T}_i$  be an irreducible message assignment and  $|\mathcal{T}_i| \leq M$ , then  $\forall k \in [K], k \in \mathcal{T}_i$  only if the vertex  $k$  in  $G_{W_i, \mathcal{T}_i}$  lies at a distance that is less than or equal  $M - 1$  from a marked vertex.*

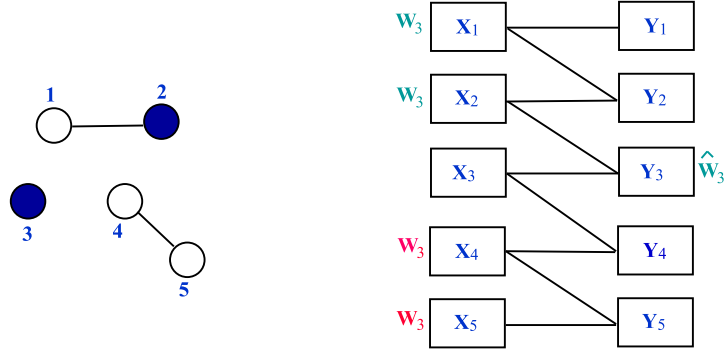


Figure 4.4: Figure showing the construction of  $G_{W_3, \mathcal{T}_3}$  in a 5–user channel with  $L = 1$ . Marked vertices are represented with filled circles.  $W_3$  can be removed at both  $X_4$  and  $X_5$  without decreasing the sum rate, as the corresponding vertices lie in a component that does not contain a marked vertex.

Note that in the considered channel model, the above result implies that  $\mathcal{T}_i \subseteq \{i - ML, i - ML + 1, \dots, i + (M - 1)L\}$ , from which we obtain the following result.

**Theorem 7.** *Local cooperation is optimal for locally connected channels,*

$$\tau_L^{(loc)}(M) = \tau_L(M), \forall M, L \in \mathbf{Z}^+. \quad (4.8)$$

And so we note that even though local cooperation does not achieve a scalable DoF gain for the fully connected channel, not only does it achieve a scalable gain when the connectivity assumption is relaxed to local connectivity, but the confinement to local cooperation no longer results in a loss in the available DoF.

## 4.5 DoF Upper Bounds

In this section, we prove upper bounds on  $\tau_1(M)$  and  $\tau_L(1)$  that establishes the tightness of the lower bound in Theorem 6 for the special cases where either  $L = 1$  or  $M = 1$ . First, in order to assess the optimality of the coding scheme introduced in Section 4.3 for arbitrary values of the system parameters, we prove a general upper bound for a class of coding schemes that only employs a zero-forcing transmit beam-forming strategy.

### 4.5.1 ZF Transmit Beam-Forming

Consider only coding schemes with transmit signals of the form (4.4) and each message is either not transmitted or allocated one degree of freedom. More precisely, let  $\tilde{Y}_j = Y_j - Z_j, \forall j \in [K]$ , then in addition to the constraint in (4.4), it is either the case that the mutual information  $I(\tilde{Y}_j; W_j) = 0$  or it is the case that  $W_j$  completely determines  $\tilde{Y}_j$ . Note that  $\tilde{Y}_j$  can be determined from  $W_j$  for the case where user  $j$  enjoys interference-free communication and  $I(W_j; \tilde{Y}_j) = 0$  for the other case where  $W_j$  is not transmitted. We say that the  $j^{\text{th}}$  receiver is *active* if and only if  $I(\tilde{Y}_j; W_j) > 0$ . Note that using zero-forcing transmit beam-forming, if the  $j^{\text{th}}$  receiver is active, then  $I(W_i; Y_j) = 0, \forall i \neq j$ .

Let  $\tau_L^{(\text{zf})}(M)$  denote the asymptotic characterization of the per user DoF under the restriction to the above described class of coding schemes. In Theorem 8 below, we show that the coding scheme in the proof of Theorem 6 achieves the optimal value of  $\tau_L^{(\text{zf})}(M)$ . We first prove Lemma 2 that bounds the number of receivers at which the interference of a given message can be cancelled.

For a set  $\mathcal{S} \subseteq [K]$ , let  $\mathcal{V}_{\mathcal{S}}$  be the set of indices for active receivers connected to transmitters with indices in  $\mathcal{S}$ . More precisely,  $\mathcal{V}_{\mathcal{S}} = \{j : I(\tilde{Y}_j; W_j) > 0, \mathcal{S} \cap \{j, j-1, \dots, j-L\} \neq \phi\}$ , where  $\phi$  is the empty set. To obtain the following results, we assume that for each transmitter in  $\mathcal{T}_i$ , message  $W_i$  contributes to the transmit signal of this transmitter. i.e.,  $\forall j \in \mathcal{T}_i, I(W_i, X_j) > 0$ . Note that this assumption does not introduce a loss in generality, because otherwise the transmitter can be removed from  $\mathcal{T}_i$ . We need Lemma 2 for the proof of the upper bound on  $\tau_L^{(\text{zf})}(M)$  in Theorem 8.

**Lemma 2.** *For any message  $W_i$ , the number of active receivers connected to at least one transmitter carrying the message is no greater than the number of transmitters carrying the message.*

$$|\mathcal{V}_{\mathcal{T}_i}| \leq |\mathcal{T}_i|. \quad (4.9)$$

*Proof.* We only consider the non-trivial case where  $\mathcal{T}_i \neq \phi$ . For each receiver  $j \in \mathcal{V}_{\mathcal{T}_i}$ , there exists a transmit signal  $X_{k,i}, k \in [K]$  such that conditioned on all other transmit signals, the received signal  $Y_j$  is correlated with the message  $W_i$ . More precisely,  $I(W_i; Y_j | \{X_{v,i}, v \in [K], v \neq k\}) > 0$ . Now,



since we impose the constraint  $I(W_i; Y_j) = 0, \forall j \in \mathcal{V}_{\mathcal{T}_i}$ , the interference seen at all receivers in  $\mathcal{V}_{\mathcal{T}_i}$  has to be cancelled. Finally, since the probability of a zero Lebesgue measure set of channel realizations is zero, the  $|\mathcal{T}_i|$  transmit signals carrying  $W_i$  cannot be designed to cancel  $W_i$  at more than  $|\mathcal{T}_i| - 1$  receivers for almost all channel realizations.  $\square$

**Theorem 8.** *Under the restriction to ZF Transmit Beam-Forming coding schemes (interference avoidance), the asymptotic per user DoF of a locally connected channel with connectivity parameter  $L$  is given by*

$$\tau_L^{(zf)}(M) = \frac{2M}{2M + L}. \quad (4.10)$$

*Proof.* The proof of the lower bound is the same as the proof of Theorem 6 for the case where  $\frac{2M}{2M+L} > \frac{1}{2}$ . It then suffices to show that  $\tau_L^{(zf)}(M) \leq \frac{2M}{2M+L}$ .

In order to prove the upper bound, we show that the sum degree of freedom in each set  $\mathcal{S} \subseteq [K]$  of consecutive  $2M + L$  users is bounded by  $2M$ . We now focus on proving this statement by fixing a set  $\mathcal{S}$  of consecutive  $2M + L$  users, and make the following definitions. For a user  $i \in [\mathcal{S}]$ , let  $\mathcal{U}_i$  be the set of active users in  $\mathcal{S}$  with an index  $j > i$ , i.e.,

$$\mathcal{U}_i = \{j : j > i, j \in \mathcal{S}, I(\tilde{Y}_j; W_j) > 0\}.$$

Similarly, let  $\mathcal{D}_i$  be the set of active users in  $\mathcal{S}$  with an index  $j < i$ ,

$$\mathcal{D}_i = \{j : j < i, j \in \mathcal{S}, I(\tilde{Y}_j; W_j) > 0\}.$$

Assume that  $\mathcal{S}$  has at least  $2M + 1$  active users, then there is an active user in  $\mathcal{S}$  that lies in the middle of a subset of  $2M + 1$  active users in  $\mathcal{S}$ . More precisely,  $\exists i \in \mathcal{S} : |\mathcal{T}_i| > 0, |\mathcal{U}_i| \geq M, |\mathcal{D}_i| \geq M$ , we let this middle user have the  $i^{\text{th}}$  index for the rest of the proof.

Let  $s_{min}$  and  $s_{max}$  be the users in  $\mathcal{S}$  with minimum and maximum indices, respectively, i.e.,  $s_{min} = \min_s \{s : s \in \mathcal{S}\}$  and  $s_{max} = \max_s \{s : s \in \mathcal{S}\}$ , we then consider the following cases to complete the proof,

**Case 1:**  $W_i$  is being transmitted from a transmitter that is connected to the receiver with index  $s_{min}$ , i.e.,  $\exists s \in \mathcal{T}_i : s \in \{s_{min}, s_{min} - 1, \dots, s_{min} - L\}$ . It follows from Lemma 1 that  $\mathcal{V}_{\mathcal{T}_i} \supseteq \mathcal{D}_i \cup \{i\}$ , and hence,  $|\mathcal{V}_{\mathcal{T}_i}| \geq M + 1$ , which contradicts (4.9), as  $|\mathcal{T}_i| \leq M$ .

**Case 2:**  $W_i$  is being transmitted from a transmitter that is connected to the receiver with index  $s_{max}$ , i.e.,  $\exists s \in \mathcal{T}_i : s \in \{s_{max}, s_{max} - 1, \dots, s_{max} - L\}$ . It follows from Lemma 1 that  $\mathcal{V}_{\mathcal{T}_i} \supseteq \mathcal{U}_i \cup \{i\}$ , and hence,  $|\mathcal{V}_{\mathcal{T}_i}| \geq M + 1$ , which again contradicts (4.9).

**Case 3:** For the remaining case, there is no transmitter in  $\mathcal{T}_i$  that is connected to any of the receivers with indices  $s_{min}$  and  $s_{max}$ . In this case, it follows from Lemma 1 that  $\mathcal{T}_i$  does not contain a transmitter that is connected to a receiver with an index less than  $s_{min}$  or greater than  $s_{max}$ , and hence, all the receivers connected to transmitters carrying  $W_i$  belong to  $\mathcal{S}$ . It follows that at least  $L + |\mathcal{T}_i|$  receivers in  $\mathcal{S}$  are connected to one or more transmitter in  $\mathcal{T}_i$ , and since  $\mathcal{S}$  has at least  $2M + 1$  active receivers, then any subset of  $L + |\mathcal{T}_i|$  receivers in  $\mathcal{S}$  has to have at least  $2M + 1 - ((2M + L) - (L + |\mathcal{T}_i|)) = |\mathcal{T}_i| + 1$  active receivers, and the statement is proved by reaching a contradiction to (4.9) in the last case.  $\square$

## 4.5.2 Wyner's Asymmetric Model

Now, we consider the special case of  $L = 1$ , and prove that the lower bound stated in Theorem 6 is tight in this case [51]. We use Lemma 5 in Appendix A to prove the DoF upper bound for Wyner's model. Recall that for any set of receiver indices  $\mathcal{A} \subseteq [K]$ , we use  $U_{\mathcal{A}}$  as the set of indices of transmitters that exclusively carry the messages for the receivers in  $\mathcal{A}$ , and the complement set  $\bar{U}_{\mathcal{A}}$  is the set of indices of transmitters that carry messages for receivers outside  $\mathcal{A}$ . More precisely,  $\bar{U}_{\mathcal{A}} = \cup_{i \notin \mathcal{A}} \mathcal{T}_i$ .

**Theorem 9.** *The asymptotic per user DoF for Wyner's asymmetric model with CoMP transmission is given by,*

$$\tau_1(M) = \frac{2M}{2M + 1}, \forall M \in \mathbf{Z}^+. \quad (4.11)$$

*Proof.* The lower bound follows from Theorem 6. In order to prove the converse, we use Lemma 5 with a set  $\mathcal{A}$  of size  $K \frac{2M}{2M+1} + o(K)$ . We also prove the upper bound for the channel after removing the first  $M$  transmitters ( $X_{[M]}$ ), while noting that this will be a valid bound on  $\tau_1(M)$  since the number of removed transmitters is  $o(K)$ .

Inspired by the coding scheme in the proof of Theorem 6, we define the

set  $\mathcal{A}$  as the set of receivers that are *active* in the coding scheme. That is, the complement set  $\bar{\mathcal{A}} = \{i : i \in [K], i = (2M + 1)(j - 1) + M + 1, j \in \mathbf{Z}^+\}$ . We know from Corollary 4 that messages belonging to the set  $W_{\bar{\mathcal{A}}}$  do not contribute to transmit signals with indices that are multiples of  $2M + 1$ , i.e.,  $i \in U_{\mathcal{A}}$  for all  $i \in [K]$  that is a multiple of  $2M + 1$ . More precisely, let the set  $\mathcal{S}$  be defined as follows:

$$\mathcal{S} = \{i : i \in [K], i \text{ is a multiple of } 2M + 1\},$$

then  $\mathcal{S} \subseteq U_{\mathcal{A}}$ . In particular,  $X_{\mathcal{S}} \subseteq X_{U_{\mathcal{A}}}$ , and hence it suffices using Lemma 5 to show the existence of linear functions  $f_1$  and  $f_2$  such that  $f_1(Y_{\mathcal{A}}, X_{\mathcal{S}}) = X_{\bar{\mathcal{S}}} \setminus X_{[M]} + f_2(Z_{\mathcal{A}})$ , where the coefficients of the function  $f_2$  depend only on the channel coefficients.

In what follows we show how to reconstruct a noisy version of the signals in the set  $\{X_{M+1}, X_{M+2}, \dots, X_{2M}\} \cup \{X_{2M+2}, X_{2M+3}, \dots, X_{3M+1}\}$ , where the reconstruction noise depends only on  $Z_{\mathcal{A}}$  in a linear fashion. Then it will be clear by symmetry how to reconstruct the rest of transmit signals in the set  $X_{\bar{\mathcal{S}}} \setminus X_{[M]}$ . Since  $X_{2M+1} \in X_{\mathcal{S}}$  and  $Y_{2M+1}$  is also given,  $X_{2M} + Z_{2M+1}$  can be reconstructed. Now, with the knowledge of  $X_{2M} + Z_{2M+1}$  and  $Y_{2M}$ , we can reconstruct  $X_{2M-1} + Z_{2M} - Z_{2M+1}$ , and so by iterative processing, a noisy version of all transmit signals in the set  $\{X_{M+1}, X_{M+2}, \dots, X_{2M}\}$  can be reconstructed, where the noise is a linear function of the signals  $\{Z_{M+2}, Z_{M+3}, \dots, Z_{2M+1}\}$ . In a similar fashion, given  $X_{2M+1}$  and  $Y_{2M+2}$ , the signal  $X_{2M+2} + Z_{2M+2}$  can be reconstructed. Then with the knowledge of  $Y_{2M+3}$ , we can reconstruct  $X_{2M+3} + Z_{2M+3} - Z_{2M+2}$ , and we can proceed along this path to reconstruct a noisy version of all transmit signals in the set  $\{X_{2M+2}, X_{2M+3}, \dots, X_{3M+1}\}$ , where the noise is a linear function of the signals  $\{Z_{2M+2}, Z_{2M+3}, \dots, Z_{3M+1}\}$ . This proves the existence of linear functions  $f_1$  and  $f_2$  such that  $f_1(Y_1, X_{\mathcal{S}}) = X_{\bar{\mathcal{S}}} \setminus X_{[M]} + f_2(Z_{\mathcal{A}})$ , and the coefficients for  $f_2$  do not depend on the transmit power constraint  $P$ , and so by Lemma 5 we obtain the converse of Theorem 9.  $\square$

In Figure 4.5 (b), we illustrate how the proof works for the case where  $M = 3$ . Note that the missing received signals  $\{Y_4, Y_{11}, \dots\}$  in the upper bound proof correspond to the inactive receivers in the coding scheme.

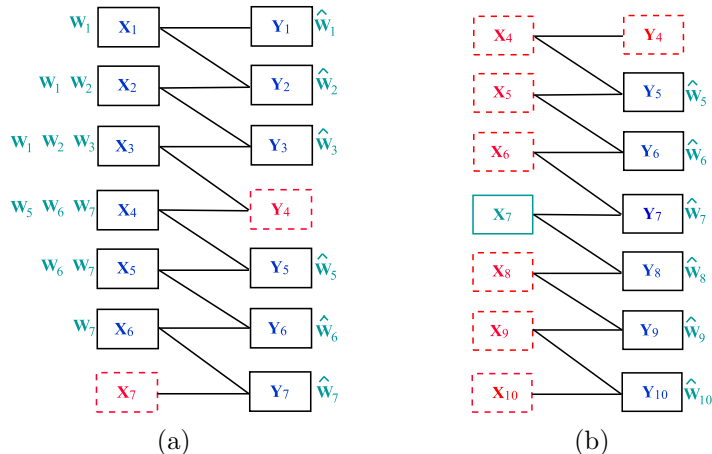


Figure 4.5: Figure illustrating the proof of Theorem 9 for  $M = 3$ ,  $\tau(3) = \frac{6}{7}$ . In (a), the message assignments in the first cluster for the proposed coding scheme are illustrated. Note that both  $X_7$  and  $Y_4$  are deactivated. In (b), an illustration of the upper bound is shown. The messages  $W_4$  and  $W_{11}$  cannot be available at  $X_7$ , hence it can be reconstructed from  $W_A$ . A noisy version of all transmit signals shown in figure can be reconstructed from  $X_7$  and the signals  $\{Y_5, \dots, Y_{10}\}$ , where the reconstruction noise is a linear function of  $\{Z_5, \dots, Z_{10}\}$ .

### No Cooperation

We note that even for the case of *no cooperation*, an asymptotic per user DoF of more than  $\frac{1}{2}$  per user DoF is achievable, i.e.,  $\tau_1(1) = \frac{2}{3}$ . Also, it is straightforward to see that the interference alignment scheme can be generalized to show that  $\tau_L(1) \geq \frac{1}{2}$  for any locally connected channel with parameter  $L$ . The next theorem generalizes the upper bound in [10] for locally connected channels, where each message can be available at one transmitter that is not necessarily the transmitter carrying its own index. In particular, we show that  $\tau_L(1) > \frac{1}{2}$  only if  $L = 1$ .

Lemma 3 serves as a building block for the upper bound proof in Theorem 10. We define  $\mathcal{R}_i$  as the set of indices of received signals that are connected to transmitter  $X_i$ , i.e.,  $\mathcal{R}_i = \{i, i + 1, \dots, i + L\}$ . Note that as we are considering the case of *no cooperation*, hence,  $\mathcal{T}_i$  contains only one element. Recall that  $d_i$  denotes the available DoF for the communication of message  $W_i$ .

**Lemma 3.** *If  $\mathcal{T}_i = \{X_j\}$ , then  $d_i + d_s \leq 1, \forall s \in \mathcal{R}_j, s \neq i$ .*

*Proof.* We assume that all messages other than  $W_i$  and  $W_s$  are deterministic,

and then apply Lemma 5 in Appendix A with the set  $\mathcal{A} = \{s\}$ , and functions  $f_1$  and  $f_2$  defined such that the following holds,

$$\begin{aligned} f_1(Y_s, X_{[K]\setminus\{j\}}) &= H_{s,j}^{-1}(Y_s - \mathbf{H}_{\{s\},[K]\setminus\{j\}}X_{[K]\setminus\{j\}}) \\ &= X_j + H_{s,j}^{-1}Z_s \\ &= X_j + f_2(Z_s), \end{aligned}$$

and then the bound follows.  $\square$

**Theorem 10.** *Without cooperation ( $M = 1$ ), the asymptotic per user DoF of locally connected channels is given by,*

$$\tau_L(1) = \begin{cases} \frac{2}{3}, & \text{if } L = 1, \\ \frac{1}{2}, & \text{if } L \geq 2. \end{cases}$$

*Proof.* The case where  $L = 1$  is a special case of the result in Theorem 9. The lower bound for the case where  $L \geq 2$  follows by assigning each message to the transmitter with the same index, and a simple extension of the asymptotic interference alignment scheme of [11], and hence, it suffices to show that

$$\tau_L(1) \leq \frac{1}{2}, \forall L \geq 2. \quad (4.12)$$

In this proof, we use the original locally connected channel model defined in (2.2). Each transmitter is connected to  $\lfloor \frac{L}{2} \rfloor$  preceding receivers and  $\lceil \frac{L}{2} \rceil$  succeeding receivers. In order to prove the theorem statement, we establish the stronger statement,

$$\eta_L(K, 1) \leq \frac{K+1}{2}, \forall K, \forall L \geq 2. \quad (4.13)$$

We prove (4.13) by induction. The basis to the induction step is given by the following,

$$\text{For } M = 1, \forall L, d_1 + d_2 \leq 1. \quad (4.14)$$

The proof of (4.14) follows from Lemma 3 and the fact that all transmitters connected to  $Y_1$  are also connected to  $Y_2$ . In order to state the induction step, we first define  $B_k$  as a Boolean variable that is true if and only if the following is true:

- $\frac{\sum_{i=1}^k d_i}{k} \leq \frac{1}{2}$ .
- $d_{k-1} + d_k \leq 1$ .

The induction step is given by the following:

$$\text{For } L \geq 2, k \geq 2, \text{ if } B_k \text{ is true, then either } B_{k+1} \text{ or } B_{k+2} \text{ is true.} \quad (4.15)$$

In order to prove (4.15), consider the assignment of message  $W_{k+1}$ , and note that  $W_{k+1}$  is available at a transmitter connected to  $Y_{k+1}$ . Now, note that  $\forall L \geq 2$ , the channel model of (6.1) implies that any transmitter connected to  $Y_{k+1}$  is either connected to  $Y_{k+2}$ , or to both  $Y_k$  and  $Y_{k-1}$ . The proof follows by considering these two cases separately.

**Case 1:** If  $W_{k+1}$  is available at a transmitter that is connected to  $Y_{k+2}$ , then it follows from Lemma 3 that  $d_{k+1} + d_{k+2} \leq 1$ . Since  $B_k$  is true, it follows that  $\frac{\sum_{i=1}^k d_i}{k} \leq \frac{1}{2}$ , and hence,  $\frac{\sum_{i=1}^{k+2} d_i}{k+2} \leq \frac{1}{2}$ . In this case, (4.15) holds since  $B_{k+2}$  is true.

**Case 2:** If  $W_{k+1}$  is available at a transmitter that is connected to both  $Y_k$  and  $Y_{k-1}$ , then it follows from Lemma 3 that  $d_{k+1} + d_k \leq 1$ , and  $d_{k+1} + d_{k-1} \leq 1$ . Now, since  $B_k$  is true, it follows that  $d_k + d_{k-1} \leq 1$ , and hence,

$$\frac{d_{k+1} + d_k + d_{k-1}}{3} \leq \frac{1}{2}. \quad (4.16)$$

Also, since  $B_k$  is true, we know that  $\frac{\sum_{i=1}^{k-2} d_i}{k-2} \leq \frac{1}{2}$ , and hence, we get from (4.16) that  $\frac{\sum_{i=1}^{k+1} d_i}{k+1} \leq \frac{1}{2}$ . In this case, (4.15) holds since  $B_{k+1}$  is true.

It follows by induction from (4.14) and (4.15) that it is either the case that  $B_{K-1}$  is true, or  $B_K$  is true. If  $B_{K-1}$  is true, then  $\sum_{i=1}^{K-1} d_i \leq \frac{K-1}{2}$ , and the DoF number  $\eta \leq \frac{K+1}{2}$ . If  $B_K$  is true, then it follows that the DoF number  $\eta \leq \frac{K}{2}$ .

□

## 4.6 Discussion: SISO Interference Channels with Maximum Transmit Set Size Constraint

There are two design parameters in the considered problem, the message assignment strategy satisfying the maximum transmit set size constraint,

and the design of transmit beams. We characterized the asymptotic per user DoF when one of the design parameters is restricted to a special choice, i.e., restricting message assignment strategies by a local cooperation constraint or restricting the design of transmit beams to zero-forcing transmit beams. The restriction of one of the design parameters can significantly simplify the problem because of the inter-dependence of the two design parameters. On one hand, the achievable scheme is enabled by the choice of the message assignment strategy, and on the other hand, the assignment of messages to transmitters is governed by the technique followed in the design of transmit beams, e.g. zero-forcing transmit beam-forming or interference alignment. In the following, we discuss each of the design parameters.

#### 4.6.1 Message Assignment Strategy

The assignment of each message to more than one transmitter (CoMP transmission) creates a virtual Multiple Input Single Output (MISO) network. A real MISO network, where multiple dedicated antennas are assigned to the transmission of each message (see e.g. [52]), differs from the created virtual one in two aspects. First, in a CoMP transmission setting, the same transmit antenna can carry more than one message. Second, for locally connected channels, the number of receivers at which a message causes undesired interference depends on the number of transmit antennas carrying the message. We study MISO networks in Section 4.7.

For fully connected channels, the number of receivers at which a message causes undesired interference is the same regardless of the size of the transmit set as long as it is non-empty. The only aspect that governs the assignment of messages to transmitters is the pattern of overlap between transmit sets corresponding to different messages. It is expected that the larger the sizes of the intersections between sets of messages carried by different transmit antennas, the more dependent the coefficients of the virtual MISO channel are, and hence, the lower the available DoF. For the spiral assignments of messages considered in Section 3.1,  $|\mathcal{T}_i \cap \mathcal{T}_{i+1}| = M - 1$ , and the same value holds for the size of the intersection between sets of messages carried by successive transmitters. In general, local cooperation implies large intersections between sets of messages carried by different transmitters, and hence, the

negative conclusion we reached for  $\tau^{(\text{loc})}(M)$ .

For the case where we are restricted to zero-forcing transmit beam-forming as in Section 4.3, the number of receivers at which each message causes undesired interference governs the choice of transmit sets, and hence, we saw that for locally connected channels, the message assignment strategy illustrated in Theorem 6 selects transmit sets that consist of successive transmitters, to minimize the number of receivers at which each message should be cancelled. This strategy is optimal under the restriction to zero-forcing transmit beam-forming schemes.

#### 4.6.2 Design of Transmit Beams

While it was shown in Section 3.1 that CoMP transmission accompanied by both zero-forcing transmit beams and asymptotic interference alignment can achieve a DoF cooperation gain beyond what can be achieved using only transmit zero-forcing, this is not obvious for locally connected channels. Unlike in the fully connected channel, the addition of a transmitter to a transmit set in a locally connected channel may result in an increase in the number of receivers at which the message causes undesired interference.

We note that unlike the asymptotic interference alignment scheme, the zero-forcing transmit beam-forming scheme illustrated in Section 4.3 does not need symbol extensions, since it achieves the stated DoF of Theorem 6 in one channel realization. However, it is not clear whether asymptotic interference alignment can be used to show an asymptotic per user DoF cooperation gain beyond that achieved through simple zero-forcing transmit beam-forming; we believe that the answer to this question is closely related to both problems that remain open after this work, i.e., characterizing  $\tau(M)$  and  $\tau_L(M)$ .

### 4.7 Multiple Antenna Transmitters

In order to compare between the cases of having dedicated versus shared antennas for the transmission of each message, we consider in this section the scenario where each transmitter is equipped with  $N$  antennas. We use the standard model for the  $K$ -user interference channel with  $N$ -antenna trans-



mitters and single antenna receivers,

$$Y_i = \sum_{j=1}^K \sum_{n=1}^N H_{i,j}^{(n)} X_j^{(n)} + Z_i, \quad (4.17)$$

where  $X_i^{(n)}(t)$  is the transmitted signal of the  $n^{\text{th}}$  antenna at transmitter  $i$ , and  $H_{i,j}^{(n)}$  is the channel coefficient from the  $n^{\text{th}}$  antenna at transmitter  $j$  to receiver  $i$ . The condition for the equivalent channel model of (4.1) is here extended to the following,

$$H_{i,j}^{(n)} \text{ is not identically 0 if and only if } i \in [j, j+1, \dots, j+L], n \in [N], \quad (4.18)$$

and all channel coefficients that are not identically zero are generic.

In [53], we extended the characterization of the asymptotic per user DoF under the restriction to zero-forcing transmit beam-forming coding schemes to the considered locally connected channel with multiple antenna transmitters. More precisely, let  $\tau_L^{(\text{zf})}(M, N)$  be the asymptotic per user DoF for locally connected channels with connectivity parameter  $L$ ,  $N$  antennas at each transmitter, a maximum transmit set size constraint  $M$ , and under the restriction to the class of zero-forcing transmit beam-forming coding schemes that are defined as follows.

Recall from Section 4.5.1 that  $\tilde{Y}_j = Y_j - Z_j, \forall j \in [K]$ , then we impose the following constraints on allowed coding schemes:

- The transmit signal at the  $n^{\text{th}}$  antenna of the  $j^{\text{th}}$  transmitter is given by

$$X_j^{(n)} = \sum_{i:j \in \mathcal{T}_i} X_{j,i}^{(n)}, \quad (4.19)$$

where  $X_{j,i}^{(n)}$  depends only on message  $W_i$ .

- It is either the case that the mutual information  $I(\tilde{Y}_j; W_j) = 0$  or it is the case that  $W_j$  completely determines  $\tilde{Y}_j$ . Note that  $\tilde{Y}_j$  can be determined from  $W_j$  for the case where user  $j$  enjoys interference-free communication and  $I(W_j; \tilde{Y}_j) = 0$  for the other case where  $W_j$  is not transmitted.

We say that the  $j^{\text{th}}$  receiver is *active* if and only if  $I(\tilde{Y}_j; W_j) > 0$ . Note that using zero-forcing transmit beam-forming, if the  $j^{\text{th}}$  receiver is active, then

$$I(W_i; Y_j) = 0, \forall i \neq j.$$

The results of Theorem 6 and Theorem 8 can be extended to obtain the following characterization.

**Theorem 11.** *Under the restriction to ZF Transmit Beam-Forming coding schemes (interference avoidance), the asymptotic per user DoF of a locally connected channel with connectivity parameter  $L$  and  $N$ -antenna transmitters is given as follows.*

*If  $MN \geq M + L$ , then  $\tau_L^{(zf)}(M, N) = 1$ , otherwise,*

$$\tau_L^{(zf)}(M, N) = \frac{2MN}{M(N+1) + L}. \quad (4.20)$$

*Proof.* The proofs of the lower and upper bounds are available in Section 4.7.1 and Section 4.7.2, respectively.  $\square$

#### 4.7.1 Coding Scheme

We first consider the case where  $MN < L + M$  by treating the network as clusters, each consisting of consecutive  $M(N+1) + L$  transceivers. The last  $L$  transmitters in each cluster are deactivated to eliminate inter-cluster interference, and hence, it suffices to show that  $2MN$  DoF can be achieved in each cluster. Without loss of generality, consider the cluster with users of indices in the set  $[M(N+1) + L]$ . Define the following subsets of  $[M(N+1) + L]$ ,

$$\mathcal{S}_1 = [MN] \quad (4.21)$$

$$\mathcal{S}_2 = \{L + M + 1, L + M + 2, \dots, L + M(N + 1)\}, \quad (4.22)$$

where in the proposed scheme, messages with indices in the set  $[M(N+1) + L] \setminus (\mathcal{S}_1 \cup \mathcal{S}_2)$  are not transmitted and the corresponding receivers are deactivated. The remaining messages are assigned as follows:

$$\mathcal{T}_i = \begin{cases} \{1, 2, \dots, M\}, & \forall i \in \mathcal{S}_1 \\ \{MN + 1, MN + 2, \dots, M(N + 1)\}, & \forall i \in \mathcal{S}_2, \end{cases}$$

and all other transmitters in the cluster are deactivated. In other words, the first  $MN$  messages in the cluster are assigned to transmitters in the set  $[M]$ , and the last  $MN$  messages in the cluster are assigned to the  $M$  transmitters

with indices in the set  $\{MN + 1, MN + 2, \dots, M(N + 1)\}$ .

Now, we note that messages with indices in  $\mathcal{S}_1$  are not available outside transmitters with indices in  $[M]$ , and hence, do not cause interference at receivers with indices in  $\mathcal{S}_2$ . Also, messages with indices in  $\mathcal{S}_2$  are not available at transmitters with indices in  $[MN]$ , and hence, do not cause interference at receivers with indices in  $\mathcal{S}_1$ .

In order to complete the proof by showing that each user in  $\mathcal{S}_1 \cup \mathcal{S}_2$  achieves one degree of freedom, we next show that transmissions corresponding to messages with indices in  $\mathcal{S}_1(\mathcal{S}_2)$  do not cause interference at receivers with indices in the same set. Let  $\mathcal{S}_{1,j}$  denote the set  $\{(j - 1)N + 1, (j - 1)N + 2, \dots, jN\}$  where  $j \in [M]$ , and consider the design of transmit beams for messages  $W_i, i \in \mathcal{S}_{1,j}$ . Our aim is to create an interference-free communication between the  $(i - (j - 1)N)^{th}$  antenna at the  $j^{th}$  transmitter to the  $i^{th}$  receiver. We prove this by showing the existence of an assignment for the transmit signals  $\{X_{k,i}^{(n)} : n \in [N], k \in [M], (k, n) \neq (j, i - (j - 1)N)\}$  to cancel the interference caused by  $W_i$  at the  $MN - 1$  receivers in  $\mathcal{S}_1 \setminus \{i\}$ . Consider the design of the transmit beam at the  $n^{th}$  antenna of the  $k^{th}$  transmitter  $X_{k,i}^{(n)}$ , where  $n \in [N], k \in [M], (k, n) \neq (j, i - (j - 1)N)$ , and note that given all other transmit signals carrying  $W_i$ ,  $X_{k,i}^{(n)}$  can be designed such that the interference caused by  $W_i$  at the  $((k - 1)N + n)^{th}$  receiver is canceled. Therefore, the interference cancellation constraints pose a system of  $MN - 1$  equations in  $MN - 1$  variables, where each equation is assigned a distinct variable, that can be set to satisfy it, given any assignment of the other  $MN - 2$  variables. We now show a simple algorithm that finds an assignment for the variables to satisfy the equations. Fix an order on the above mentioned equations and label them from 1 to  $MN - 1$ , and recall that each equation is assigned a distinct variable that can be set to satisfy it given all other variables. For the  $x^{th}$  equation,  $x \in [MN + 1]$ , let that above mentioned distinct variable have the label  $x$ . In the first step of the algorithm, the first variable is removed by setting it as a function of all other variables to satisfy the first equation, and we have a reduced problem of  $MN - 2$  equations in  $MN - 2$  variables. Similarly, in the  $x^{th}$  step of the algorithm, the  $x^{th}$  variable is set as a function of all variables in the set  $\{x + 1, \dots, MN + 1\}$  to satisfy the  $x^{th}$  equation. Now, once we reach the  $(MN + 1)^{st}$  and final step, the  $(MN + 1)^{st}$  variable will be set to satisfy the  $(MN + 1)^{st}$  equation, and recursively, all variables will be set to satisfy all the equations. A solution is found using the above

described algorithm for almost all channel realizations, as the assumption of a generic set of channel coefficients leads to linearly independent equations, almost surely.

Note that the validity of the above argument relies on the fact that  $\forall j \in [M]$ , the  $j^{\text{th}}$  transmitter is connected to all receivers in the set  $\{(j-1)N+1, (j-1)N+2, \dots, jN\}$ . This follows as we consider the case where  $MN < L+M$  which implies that  $jN < L+j, \forall j \in [M]$ , and the  $j^{\text{th}}$  transmitter is connected to receivers with indices in the set  $\{j, j+1, \dots, L+j\}$ .

We finally note that the channel between transmitters with indices in the sequence  $(M(N+1), M(N+1)-1, \dots, MN+1)$  and receivers with indices in the sequence  $(L+M(N+1), L+M(N+1)-1, \dots, L+M+1)$  has the same *connectivity pattern* as the channel between transmitters with indices in the sequence  $(1, 2, \dots, M)$  and receivers with indices in the sequence  $(1, 2, \dots, MN)$ , and hence the argument in the previous paragraph can be used to construct transmit beams for messages  $W_i, i \in \mathcal{S}_2$  such that each user in  $\mathcal{S}_2$  gets access to an interference-free transmission (one degree of freedom).

The proof is simpler for the case where  $NM \geq L+M$ . Let  $x_{\min} = \min_x \{x \in \mathbf{N} : Nx \geq L+x\}$ , then  $x_{\min} \leq M$ , and the messages are assigned as  $\mathcal{T}_i = \{i, i+1, \dots, i+x_{\min}-1\}$ . Consider the design of transmit beams for message  $W_i$ . Our aim is to allow for interference-free communication between the first antenna at the  $i^{\text{th}}$  transmitter and the  $i^{\text{th}}$  receiver, and eliminate the interference caused by  $W_i$  at all receivers in the set  $\{i+1, \dots, i+L+x_{\min}-1\}$ . In a similar fashion to the above described proof, each receiver in  $\{i+1, \dots, i+L+x_{\min}-1\}$  is assigned a distinct transmit signal from the set  $\{X_{j,i}^{(n)}, n \in [N], j \in \mathcal{T}_i, (j,n) \neq (i,1)\}$ , where given all other transmit signals, that transmit signal can be set to cancel the interference caused by  $W_i$  at that receiver, and hence, there exists a setting for all transmit signals carrying  $W_i$  that cancels its interference at all receivers connected to transmitters in  $\mathcal{T}_i$  other than its own receiver.

Please refer to Figure 4.6 for an illustration of the above described coding scheme.

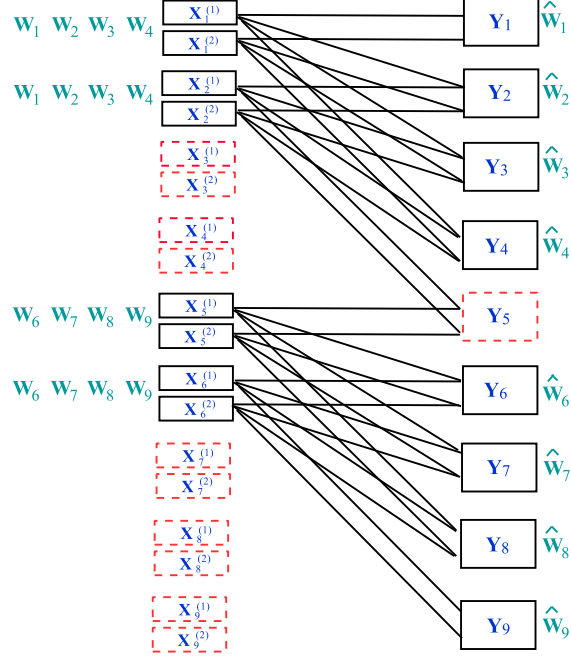


Figure 4.6: Figure showing the assignment of messages in the proof of Theorem 11 for the case where  $N = M = 2$  and  $L = 3$ . Only signals corresponding to the first cluster are shown. Signals in dashed red boxes are deactivated. Note that the last  $L$  signals are deactivated to eliminate inter-cluster interference. Also,  $W_5$  is not transmitted, while each other message with indices in  $\{1, \dots, 9\}$  has one degree of freedom.

#### 4.7.2 ZF Transmit Beam-Forming Upper Bound

We first extend Lemma 2 of Section 4.5.1 to draw an upper bound on the number of active receivers connected to a transmit set in the considered multiple antenna transmitters setting.

**Lemma 4.** *For any message  $W_i$ , the number of active receivers connected to at least one transmitter carrying the message is no greater than the number of transmit antennas carrying the message,*

$$|\mathcal{V}_{\mathcal{T}_i}| \leq N|\mathcal{T}_i|. \quad (4.23)$$

*Proof.* We only consider the non-trivial case where  $\mathcal{T}_i \neq \phi$ . For each receiver  $j \in \mathcal{V}_{\mathcal{T}_i}$ , there exists a transmit signal  $X_{k,i}^{(n)}$ ,  $k \in [K], n \in [N]$  such that conditioned on all other transmit signals, the received signal  $Y_j$  is correlated

with the message  $W_i$ . More precisely,

$$I\left(W_i; Y_j | \{X_{v,i}^{(m)}, (v, m) \in [K] \times [N], (v, m) \neq (k, n)\}\right) > 0.$$

Now, since we impose the constraint  $I(W_i; Y_j) = 0, \forall j \in \mathcal{V}_{\mathcal{T}_i}$ , the interference seen at all receivers in  $\mathcal{V}_{\mathcal{T}_i}$  has to be cancelled. Finally, since the probability of a zero Lebesgue measure set of channel realizations is zero, the  $N|\mathcal{T}_i|$  transmit signals carrying  $W_i$  cannot be designed to cancel  $W_i$  at more than  $N|\mathcal{T}_i| - 1$  receivers for almost all channel realizations.  $\square$

The proof of the upper bound on  $\tau_L^{(\text{zf})}(M, N)$  is an extension of the proof of Theorem 8 for the multiple antenna transmitters setting. We show that the sum degree of freedom in each set  $\mathcal{S} \subseteq [K]$  of consecutive  $M(N+1) + L$  users is bounded by  $2MN$ . We now focus on proving this statement by fixing a set  $\mathcal{S}$  of consecutive  $M(N+1) + L$  users, and make the following definitions. For a user  $i \in [\mathcal{S}]$ , let  $\mathcal{U}_i$  be the set of active users in  $\mathcal{S}$  with an index  $j > i$ , i.e.,

$$\mathcal{U}_i = \{j : j > i, j \in \mathcal{S}, I(\tilde{Y}_j; W_j) > 0\}.$$

Similarly, let  $\mathcal{D}_i$  be the set of active users in  $\mathcal{S}$  with an index  $j < i$ ,

$$\mathcal{D}_i = \{j : j < i, j \in \mathcal{S}, I(\tilde{Y}_j; W_j) > 0\}.$$

Assume that  $\mathcal{S}$  has at least  $2MN + 1$  active users, then there is an active user in  $\mathcal{S}$  that lies in the middle of a subset of  $2MN + 1$  active users in  $\mathcal{S}$ . More precisely,  $\exists i \in \mathcal{S} : |\mathcal{T}_i| > 0, |\mathcal{U}_i| \geq MN, |\mathcal{D}_i| \geq MN$ , we let this middle user have the  $i^{\text{th}}$  index for the rest of the proof.

Let  $s_{\min}$  and  $s_{\max}$  be the users in  $\mathcal{S}$  with minimum and maximum indices, respectively, i.e.,  $s_{\min} = \min_s \{s : s \in \mathcal{S}\}$  and  $s_{\max} = \max_s \{s : s \in \mathcal{S}\}$ , we then consider the following cases to complete the proof,

**Case 1:**  $W_i$  is being transmitted from a transmitter that is connected to the receiver with index  $s_{\min}$ , i.e.,  $\exists s \in \mathcal{T}_i : s \in \{s_{\min}, s_{\min} - 1, \dots, s_{\min} - L\}$ . It follows from Lemma 1 that  $\mathcal{V}_{\mathcal{T}_i} \supseteq \mathcal{D}_i \cup \{i\}$ , and hence,  $|\mathcal{V}_{\mathcal{T}_i}| \geq MN + 1$ , which contradicts (4.9), as  $|\mathcal{T}_i| \leq M$ .

**Case 2:**  $W_i$  is being transmitted from a transmitter that is connected to the receiver with index  $s_{\max}$ , i.e.,  $\exists s \in \mathcal{T}_i : s \in \{s_{\max}, s_{\max} - 1, \dots, s_{\max} - L\}$ . It follows from Lemma 1 that  $\mathcal{V}_{\mathcal{T}_i} \supseteq \mathcal{U}_i \cup \{i\}$ , and hence,  $|\mathcal{V}_{\mathcal{T}_i}| \geq MN + 1$ , which again contradicts (4.9).

**Case 3:** For the remaining case, there is no transmitter in  $\mathcal{T}_i$  that is connected to any of the receivers with indices  $s_{min}$  and  $s_{max}$ . In this case, it follows from Lemma 1 that  $\mathcal{T}_i$  does not contain a transmitter that is connected to a receiver with an index less than  $s_{min}$  or greater than  $s_{max}$ , and hence, all the receivers connected to transmitters carrying  $W_i$  belong to  $\mathcal{S}$ . It follows that at least  $L + |\mathcal{T}_i|$  receivers in  $\mathcal{S}$  are connected to one or more transmitter in  $\mathcal{T}_i$ , and since  $\mathcal{S}$  has at least  $2MN + 1$  active receivers, then any subset of  $L + |\mathcal{T}_i|$  receivers in  $\mathcal{S}$  has to have at least  $2MN + 1 - ((M(N + 1) + L) - (L + |\mathcal{T}_i|)) = MN + |\mathcal{T}_i| - (M - 1)$ , and hence,

$$\begin{aligned} |\mathcal{V}_{\mathcal{T}_i}| &\geq MN + |\mathcal{T}_i| - (M - 1) \\ &= N|\mathcal{T}_i| + (M - |\mathcal{T}_i|)(N - 1) + 1 \\ &\geq N|\mathcal{T}_i| + 1, \end{aligned}$$

and the statement is proved by reaching a contradiction to (4.9) in the last case.

### 4.7.3 Successive Transmit Sets Upper Bound

We note that in the coding scheme used to prove Theorem 11, we use a message assignment that satisfies the irreducible message assignments condition in Corollary 4. Furthermore, each transmit set consists of a successive set of transmitter indices. More precisely,

$$\begin{aligned} \mathcal{T}_i &= \{s, s + 1, \dots, s + x - 1\}, \\ s &\in \{i - L - (x - 1), i - L - (x - 1) + 1, \dots, i\}, \\ x &\in \{1, 2, \dots, M\}, \end{aligned} \tag{4.24}$$

and hence, assigning each message to a successive set of transmitters is a property of the optimal message assignments with the restriction to zero-forcing transmit beam-forming coding schemes. While we observe that Lemma 1 does not imply that transmit sets have to consist of successive transmitter indices, it might be intuitive to think that such a condition is necessary as it minimizes the number of receivers at which each message causes undesired interference.

Let  $\tilde{\tau}_L(M, N)$  be the maximum achievable per user DoF for the considered channel model with parameters  $M$ ,  $N$ , and  $L$ , where only message assignments satisfying (4.24) are considered. We now provide a DoF upper bound for general values of the system parameters.

**Theorem 12.** *Under the restriction to successive transmit sets defined in (4.24), the asymptotic per user DoF of a locally connected channel with connectivity parameter  $L$  and  $N$ -antenna transmitters is given as follows.*

*If  $MN \geq M + L$ , then  $\tilde{\tau}_L(M, N) = 1$ , otherwise,*

$$\frac{2MN}{M(N+1)+L} \leq \tilde{\tau}_L(M, N) \leq \frac{M(N+1)+L-1}{M(N+1)+L}. \quad (4.25)$$

*Proof.* Since the coding scheme used to prove Theorem 11 is based on a message assignment that satisfies (4.24), the lower bound follows from the same coding scheme. We only need to show the upper bound for the case where  $NM < L + M$ . We apply Lemma 5 in Appendix A with the set  $\mathcal{A}$  defined as follows. We view the network as clusters, each consisting of successive  $M(N+1)+L$  users, and we exclude from  $\mathcal{A}$  the  $(L+M)^{th}$  receiver from each cluster. It then suffices to show that the condition in Lemma 5 holds for this choice of the set  $\mathcal{A}$ . More precisely, let the set  $\mathcal{A}$  be defined as follows,

$$\mathcal{A} = \{i, i \in [K], i \neq (M(N+1)+L)(j-1)+L+M, \forall j \in \mathbf{Z}^+\}. \quad (4.26)$$

We then need to show that there exist functions  $f_1$  and  $f_2$ , such that  $f_1(Y_{\mathcal{A}}, X_{U_{\mathcal{A}}}) = X_{\bar{U}_{\mathcal{A}}} + f_2(Z_{\mathcal{A}})$ , where the definition of  $f_2$  does not depend on the transmit power. The function  $f_2$  that we construct is a linear function whose coefficients depend only on the channel coefficients.

We first show the existence of functions  $f_1$  and  $f_2$  for the case where each transmitter has a single antenna, i.e.,  $N = 1$ . Note that using the condition in (4.24), we know that for any message with an index that lies at the intersection between the set  $\bar{\mathcal{A}}$  and a given cluster, all members of its transmit set have indices that belong to the same cluster. We now show how to reconstruct transmit signals in  $X_{\bar{U}_{\mathcal{A}}}$  that lie in the first cluster. That is, transmit signals in the set  $\{X_i : i \in \mathcal{T}_{L+M}\}$ , and the rest of the proof for the remaining clusters will follow similarly. Note that because of (4.24), we know that  $\mathcal{T}_{L+M} \subseteq [L+2M-1]$ , and also  $L+2M \notin \mathcal{T}_i, \forall i \notin \mathcal{A}$ . Now, given



$Y_1$  and  $\frac{Z_1}{H_{1,1}^{(1)}}$ , one can obtain  $X_1^{(1)}$  as  $X_1^{(1)} = \frac{Y_1 - Z_1}{H_{1,1}^{(1)}}$ . Also, given  $X_1^{(1)}$ ,  $Y_2$ , and a linear function of  $Z_2$  whose coefficients depend only on the channel coefficients, one can obtain  $X_2^{(1)}$ . Similarly, transmit signals  $X_3^{(1)}, \dots, X_{L+M-1}^{(1)}$  can be reconstructed from  $Y_{[L+M-1]}$  and a linear function of the noise signals  $Z_{[L+M-1]}$ . It remains to show how to obtain transmit signals in the set  $\{X_{L+M}^{(1)}, X_{L+M+1}^{(1)}, \dots, X_{L+2M-1}^{(1)}\}$ . We note that the relation between those transmit signals and the signals  $\{Y_i : i \in \{L+M+1, \dots, L+2M\}\}$  and  $\{Z_i : i \in \{L+M+1, \dots, L+2M\}\}$  is given as follows,

$$\begin{bmatrix} \tilde{Y}_{L+M+1} - Z_{L+M+1} \\ \tilde{Y}_{L+M+2} - Z_{L+M+2} \\ \vdots \\ \tilde{Y}_{L+2M} - Z_{L+2M} \end{bmatrix} = M_1 \begin{bmatrix} X_{L+M}^{(1)} \\ X_{L+M+1}^{(1)} \\ \vdots \\ X_{L+2M-1}^{(1)} \end{bmatrix}, \quad (4.27)$$

where  $\forall i \in \{L+M+1, \dots, L+2M\}$ ,  $\tilde{Y}_i = Y_i - \sum_{j=1}^{L+M-1} H_{i,j}^{(1)} X_j^{(1)} - H_{i,L+2M}^{(1)} X_{L+2M}^{(1)}$ , and  $M_1$  is the  $M \times M$  matrix defined as the matrix in (4.28),

$$\begin{bmatrix} H_{L+M+1,L+M}^{(1)} & H_{L+M+1,L+M+1}^{(1)} & 0 & 0 & \dots & 0 \\ H_{L+M+2,L+M}^{(1)} & H_{L+M+2,L+M+1}^{(1)} & H_{L+M+2,L+M+2}^{(1)} & 0 & \dots & 0 \\ \vdots & & & & & \\ H_{L+2M,L+M}^{(1)} & H_{L+2M,L+M+1}^{(1)} & \dots & \dots & H_{L+2M,L+2M-2}^{(1)} & H_{L+2M,L+2M-1}^{(1)} \end{bmatrix}, \quad (4.28)$$

Now, if  $M_1$  is invertible, then all the transmit signals in  $X_{[L+2M-1]}$  can be reconstructed, and it follows that all the transmit signals encoding the message  $W_{L+M}$  can be obtained. We show in Section 4.8.1 that the matrix  $M_1$  is full rank for almost all channel realizations. By constructing a similar proof for the remaining clusters, the upper bound proof for the case where  $N = 1$  is complete.

We next prove the statement for the case where  $N > 1$ . As in the above proof, we show how to obtain the transmit signals carrying  $W_{L+M}$  in the first cluster, then the proof follows similarly for the remaining clusters. Let  $i$  be the smallest index in  $\mathcal{T}_{L+M}$ , then we know from (4.24) that  $i \in [L+M]$  and that  $\mathcal{T}_{L+M} \subseteq \{i, i+1, \dots, i+M-1\}$ . Hence, it suffices to show how to obtain the transmit signals in the set  $X_{\mathcal{S}}$  where  $\mathcal{S} = \{i, \dots, i+M-1\}$  from  $Y_{\mathcal{A}}$ ,  $X_{U_{\mathcal{A}}} \setminus X_{\mathcal{S}}$ , and a linear function of  $Z_{\mathcal{A}}$  whose coefficients depend only on the channel coefficients. Let  $\tilde{Y}_k = Y_k - \sum_{j \in U_{\mathcal{A}} \setminus \mathcal{S}, n \in [N]} H_{k,j}^{(n)} X_j^{(n)}$ ,  $\forall k \in [K]$ , then

each of the signals  $\tilde{Y}_k - Z_k, k \in [M(N + 1) + L]$  is a (possibly zero) linear combination of the transmit signals in  $X_S$ . As  $|\mathcal{S}| = M$ , and each transmitter has  $N$  antennas, then we need at least  $MN$  such linear combinations to be able to reconstruct  $X_S$ . In order to do so, we pick  $MN + 1$  received signals, among which at most one is in the set  $Y_{\bar{\mathcal{A}}}$ . We also will show that the linear equations corresponding to any  $MN$  signals of the picked ones are linearly independent, and hence suffice to reconstruct  $X_S$ . By observing that we are considering the case where  $NM < L + M$ , or in particular that  $L+1 \geq M(N-1)+2$ , we pick the  $MN+1$  received signals as the  $M-1$  signals  $\{Y_i, \dots, Y_{i+M-2}\}$  together with the last  $M(N-1)+2$  signals connected to the transmitter with index  $i+M-1$ . i.e., the set  $\{i+M-1+x, i+M+x, \dots, i+M-1+L\}$ , where  $x = L+1 - (M(N-1)+2)$ . The relation between those signals and  $X_S$  can be described as,

$$\begin{bmatrix} \tilde{Y}_i - Z_i \\ \vdots \\ \tilde{Y}_{i+M-2} - Z_{i+M-2} \\ \tilde{Y}_{i+M-1+x} - Z_{i+M-1+x} \\ \vdots \\ \tilde{Y}_{i+M-1+L} - Z_{i+M-1+L} \end{bmatrix} = M_N \begin{bmatrix} X_i^{(1)} \\ \vdots \\ X_i^{(N)} \\ X_{i+1}^{(1)} \\ \vdots \\ \vdots \\ X_{i+M-1}^{(N)} \end{bmatrix}, \quad (4.29)$$

where  $M_N$  is the  $MN + 1 \times MN$  matrix given in (4.30),

$$\begin{bmatrix} H_{i,i}^{(1)} & \dots & H_{i,i}^{(N)} & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ H_{i+1,i}^{(1)} & \dots & \dots & \dots & H_{i+1,i+1}^{(N)} & 0 & \dots & \dots & \dots & 0 \\ \vdots & & & & & & & & & \\ H_{i+M-2,i}^{(1)} & \dots & \dots & \dots & \dots & H_{i+M-2,i+M-2}^{(N)} & 0 & \dots & \dots & 0 \\ H_{i+M-1+x,i}^{(1)} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & H_{i+M-1+x,i+M-1}^{(N)} & \dots \\ \vdots & & & & & & & & & \\ H_{i+L,i}^{(1)} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & H_{i+L,i+M-1}^{(N)} \\ 0 & \dots & 0 & H_{i+L+1,i+1}^{(1)} & \dots & \dots & \dots & \dots & \dots & H_{i+L+1,i+M-1}^{(N)} \\ \vdots & & & & & & & & & \\ 0 & \dots & \dots & \dots & \dots & 0 & H_{i+M-1+L,i+M-1}^{(1)} & \dots & H_{i+M-1+L,i+M-1}^{(N)} & \dots \end{bmatrix}. \quad (4.30)$$

We note here that the missing received signal  $Y_{L+M}$  can be one of the considered  $MN + 1$  received signals. However, we show in Section 4.8.2 that any  $MN \times MN$  sub-matrix of  $M_N$  is full rank for all values of  $N > 1$ , hence proving that the transmit signals in  $X_S$  can be obtained from the remain-

ing  $MN$  received signals and the corresponding linear combinations of the Gaussian noise signals, where the linear coefficients depend only on channel coefficients. This completes the proof for the case where  $N > 1$ .  $\square$

By carefully inspecting the lower and upper bounds, we note that they coincide for the case where  $NM = L + M - 1$ . However, we recall that even in this special case, the tight characterization of the asymptotic per user DoF is available only under the restriction to successive transmit sets as defined in (4.24).

We also note that the lower bound in Theorem 12 is not optimal for general values of the parameters, as for fixed values of  $N$  and  $M$ ,  $\frac{2MN}{M(N+1)+L} \rightarrow 0$  as  $L \rightarrow \infty$ , while we know that the interference alignment scheme introduced in [11] can be applied in the considered channel model to achieve a per user DoF number of  $\frac{1}{2}$  without using multiple antennas or assigning any message to more than one transmitter. Furthermore, it is not intuitive to think that the  $\frac{L+M(N+1)-1}{L+M(N+1)}$  upper bound is tight in general, as for fixed  $L$ , it has a lower value for the case where two antennas are dedicated for the transmission of each message ( $N = 2, M = 1$ ) than the case where each message is allowed to be available at two antennas that may be carrying other messages as well ( $N = 1, M = 2$ ).

#### 4.7.4 Discussion: Dedicated versus Shared Antennas

Consider a comparison between two different scenarios. In the first, each message can be transmitted from a single transmitter that has  $x$  antennas, i.e.,  $N = x, M = 1$ , while in the second scenario, each message can be transmitted from  $x$  single antenna transmitters, i.e.,  $N = 1, M = x$ . We note that the number of receivers at which a given message causes undesired interference is  $L$  in the first scenario, and is at least  $L + x - 1$  in the second. This leads to the result that  $\tau_L^{(\text{zf})}(M = 1, N = x) > \tau_L^{(\text{zf})}(M = x, N = 1), \forall x > 1$ . It is worth noting that the number of receivers at which each message causes undesired interference is not the only difference between the considered scenarios. In particular, other differences between the two scenarios affect the available DoF when considering general coding schemes beyond the simple zero-forcing transmit beam-forming scheme. In the fully connected model, the number of receivers at which a given message causes undesired interfer-

ence is the same for both considered scenarios. However, the per user DoF number for the first scenario where  $x$  antennas are dedicated to each message is  $\frac{x}{x+1}$ , while for the case where each transmitter has a single antenna, and  $M = x = 2$ , the per user DoF number  $\tau(M = 2)$  is shown to be  $\frac{1}{2}$  in Chapter 3.

## 4.8 Proof of Multiple Antenna Transmitters Upper Bound

### 4.8.1 Proof of Non-Singularity of the Matrix $M_1$

The matrix  $M_1$  has the following form,

$$\begin{bmatrix} a_{1,1} & a_{1,2} & 0 & 0 & 0 & 0 & \dots & \dots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & 0 & 0 & \dots & \dots & 0 \\ \vdots & & & & & & & & \\ 0 & \dots & 0 & \dots & a_{M-2,M-L-1} & \dots & \dots & a_{M-2,M-1} & 0 \\ 0 & \dots & \dots & 0 & 0 & a_{M-1,M-L} & \dots & \dots & a_{M-1,M} \\ 0 & \dots & \dots & 0 & 0 & 0 & a_{M,M-L+1} & \dots & a_{M,M} \end{bmatrix}, \quad (4.31)$$

where  $a_{i,j} = 0$  if  $(i - j) \geq L$  or  $(i - j) < -1$ , and the set of all other entries is generic. We show that any matrix of this form is full rank with high probability for any positive integer value of the connectivity parameter  $L$ . The statement holds trivially for the case where  $M = 1$  as  $a_{1,1} \neq 0$  with high probability, hence, in the rest of the proof, we only consider the case where  $M > 1$ .

For a matrix of the form in (4.31), assume there exists a linear combination of the rows that equals zero and has coefficients  $\alpha_1, \alpha_2, \dots, \alpha_M$  where  $\alpha_i$  is the coefficient of the  $i^{\text{th}}$  row and not all the coefficients equal zero. It follows that  $\alpha_{M-1}a_{M-1,M} + \alpha_M a_{M,M} = 0$ . Now, for the case where  $\alpha_{M-1} = 0$ , since  $a_{M,M} \neq 0$  with high probability, it almost surely follows that  $\alpha_M = 0$ . Also, if  $a_{M-2,M-1} \neq 0$  then  $\alpha_{M-2} = 0$ . We conclude that if  $\alpha_{M-1} = 0$ , then for almost all realizations of the elements it must be the case that  $\alpha_i = 0, \forall i \in [M]$ , hence, we only consider the case where  $\alpha_{M-1} \neq 0$ . Consider the following new matrix obtained by replacing the last two rows by one row that is a

linear combination of them in the direction that nulls the last entry. More precisely, the new matrix has the following form,

$$\begin{bmatrix} a_{1,1}^{(1)} & a_{1,2}^{(1)} & 0 & 0 & 0 & \dots & \dots & 0 \\ a_{2,1}^{(1)} & a_{2,2}^{(1)} & a_{2,3}^{(1)} & 0 & 0 & \dots & \dots & 0 \\ \vdots & & & & & & & \\ 0 & \dots & 0 & a_{M-2,M-L-1}^{(1)} & \dots & \dots & a_{M-2,M-1}^{(1)} & 0 \\ 0 & \dots & 0 & 0 & a_{M-1,M-L}^{(1)} & \dots & a_{M-1,M-1}^{(1)} & 0 \end{bmatrix}, \quad (4.32)$$

where  $\forall j \in [M]$ ,  $a_{i,j}^{(1)} = a_{i,j}$ ,  $\forall i \in [M-2]$  and  $a_{M-1,j}^{(1)} = a_{M-1,j} + \frac{\alpha_M}{\alpha_{M-1}} a_{M,j} = a_{M-1,j} - \frac{a_{M-1,M}}{a_{M,M}} a_{M,j}$ . Note that  $\alpha_1, \alpha_2, \dots, \alpha_{M-1}$  are the coefficients for a linear combination of the rows of the new matrix that equals zero. In particular, it follows that the following  $M-1 \times M-1$  matrix is rank deficient,

$$\begin{bmatrix} a_{1,1}^{(1)} & a_{1,2}^{(1)} & 0 & 0 & 0 & \dots & \dots \\ a_{2,1}^{(1)} & a_{2,2}^{(1)} & a_{2,3}^{(1)} & 0 & 0 & \dots & \dots \\ \vdots & & & & & & \\ 0 & \dots & 0 & a_{M-2,M-L-1}^{(1)} & \dots & \dots & a_{M-2,M-1}^{(1)} \\ 0 & \dots & 0 & 0 & a_{M-1,M-L}^{(1)} & \dots & a_{M-1,M-1}^{(1)} \end{bmatrix}. \quad (4.33)$$

Note that  $a_{i,j}^{(1)} = 0$  if  $(i-j) \geq L$  or  $(i-j) < -1$ , and the set of all other entries is generic, and hence, the form in (4.33) is the same as the form in (4.31) with  $M$  replaced by  $M-1$ . Now, by repeating application of the above argument, we find that a matrix of the below form is rank deficient,

$$\begin{bmatrix} a_{1,1}^{(M-2)} & a_{1,2}^{(M-2)} \\ a_{2,1}^{(M-2)} & a_{2,2}^{(M-2)} \end{bmatrix}, \quad (4.34)$$

where  $\forall k \in \{2, 3, \dots, M-2\}$ ,  $\forall j \in [M]$ ,  $a_{i,j}^{(k)} = a_{i,j}^{(k-1)}$ ,  $\forall i \in [M-k-1]$ , and  $a_{M-k,j}^{(k)} = a_{M-k,j}^{(k-1)} - \frac{a_{M-k,M-k+1}^{(k-1)}}{a_{M-k+1,M-k+1}^{(k-1)}} a_{M-k+1,j}^{(k-1)}$ . We now note that in each step of the argument, the set of non-zero entries remains generic, hence the set of elements in the matrix of the form in (4.34) is generic. It follows that a matrix of the form in (4.34) is full rank with high probability, thereby reaching a contradiction to the assumption of rank deficiency of the matrix of the form in (4.31) for almost all realizations of its elements.

## 4.8.2 Proof of Non-Singularity of any $MN \times MN$ Sub-Matrix of $M_N$

The matrix  $M_N$  has the following form,

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,N} & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ a_{2,1} & \dots & \dots & \dots & a_{2,2N} & 0 & \dots & \dots & \dots & 0 \\ \vdots & & & & & & & & & \\ a_{M-1,1} & \dots & \dots & \dots & \dots & \dots & a_{M-1,(M-1)N} & 0 & \dots & 0 \\ a_{M,1} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & a_{M,MN} \\ \vdots & & & & & & & & & \\ a_{M(N-1)+2,1} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & a_{M(N-1)+2,MN} \\ 0 & \dots & 0 & a_{M(N-1)+3,N+1} & \dots & \dots & \dots & \dots & \dots & a_{M(N-1)+3,MN} \\ \vdots & & & & & & & & & \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & a_{MN+1,(M-1)N+1} & \dots & a_{MN+1,MN} \end{bmatrix}, \quad (4.35)$$

where  $N > 1$ , and the set of all the elements that are not identically zeros in (4.35) is *generic*. The proof follows similar steps to those followed in the above proof of non-singularity of  $M_1$ . For the case where  $M = 1$ , the set of all elements of any  $N \times N$  submatrix of any matrix of the form in (4.35) is generic, hence the statement holds for this case. We only consider the case where  $M > 1$  in the rest of the proof.

For any matrix that has the form in (4.35), consider a submatrix that is obtained by removing one of the rows, and assume that there exists a linear combination of the remaining rows that equals zero, hence, there is a linear combination of the  $MN + 1$  rows of the original matrix that equals zero and has coefficients  $\alpha_1, \alpha_2, \dots, \alpha_{MN+1}$  where  $\alpha_i$  is the coefficient of the  $i^{th}$  row, and there exists  $i^* \in [MN + 1]$  such that  $\alpha_{i^*} = 0$ .

Let  $\beta_N$  be the number of rows where the last  $N$  entries are not identically zeros and not including the row indexed  $i^*$  whose corresponding coefficient  $\alpha_{i^*} = 0$ . If  $\beta_N \leq N$  then all the corresponding coefficients are zeros almost surely, i.e.,  $\alpha_i = 0, \forall i \in \{M, M + 1, \dots, MN + 1\}$ . It follows that,

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{M-1} \end{bmatrix} \begin{bmatrix} a_{1,1} & \dots & a_{1,N} & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ a_{2,1} & \dots & \dots & \dots & a_{2,2N} & 0 & \dots & \dots & \dots & 0 \\ \vdots & & & & & & & & & \\ a_{M-1,1} & \dots & \dots & \dots & \dots & \dots & a_{M-1,(M-1)N} & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}, \quad (4.36)$$

and it follows from (4.36) that  $\alpha_i = 0, \forall i \in [M - 1]$  almost surely, thereby, contradicting the assumption that not all the coefficients are zeros, and hence,

we only consider the case where  $\beta_N > N$ . Now, consider the following matrix obtained by replacing the rows in (4.35) with a generic set of elements in the last  $N$  entries, by  $N$  fewer rows that form a basis for a subspace whose vector elements have zeros in the last  $N$  entries,

$$\begin{bmatrix} a_{1,1}^{(1)} & \cdots & a_{1,MN}^{(1)} \\ \vdots & & \\ a_{x,1}^{(1)} & \cdots & a_{x,MN}^{(1)} \end{bmatrix}, \quad (4.37)$$

where,

$$x = \begin{cases} (M-1)N & \text{if } i^* \in \{(M-1)N+1, \dots, MN+1\} \\ (M-1)N+1 & \text{otherwise} \end{cases}, \quad (4.38)$$

and  $a_{i,j}^{(1)} = a_{i,j}$ ,  $\forall i \in [M-1], j \in [MN]$ . In order to describe the remaining elements  $a_{i,j}^{(1)}$ ,  $i \in \{M, M+1, \dots, x\}, j \in [MN]$ , we first define the matrices  $A$ ,  $B$ ,  $C$ , and  $D$ , as follows,

$$A = \begin{bmatrix} a_{M,(M-1)N+1} & \cdots & a_{M,MN} \\ \vdots & & \\ a_{x,(M-1)N+1} & \cdots & a_{x,MN} \end{bmatrix}. \quad (4.39)$$

The  $N \times N$  matrix  $B$  is defined as follows for the case where  $i^* \in \{(M-1)N+1, \dots, MN+1\}$ ,

$$B = \begin{bmatrix} a_{(M-1)N+1,(M-1)N+1} & \cdots & a_{(M-1)N+1,MN} \\ \vdots & & \\ a_{i^*-1,(M-1)N+1} & \cdots & a_{i^*-1,MN} \\ a_{i^*+1,(M-1)N+1} & \cdots & a_{i^*+1,MN} \\ \vdots & & \\ a_{MN+1,(M-1)N+1} & \cdots & a_{MN+1,MN} \end{bmatrix}, \quad (4.40)$$

and for the case where  $i^* \in [(M-1)N]$ ,

$$B = \begin{bmatrix} a_{(M-1)N+2,(M-1)N+1} & \cdots & a_{(M-1)N+2,MN} \\ \vdots & & \\ a_{MN+1,(M-1)N+1} & \cdots & a_{MN+1,MN} \end{bmatrix}. \quad (4.41)$$

Note that as we consider the case where the number of rows with a generic set of elements in the last  $N$  entries is greater than  $N$ , it follows that  $B$  is full rank almost surely, hence, the following definition of the matrix  $C$  is valid,

$$C = -AB^{-1}. \quad (4.42)$$

For the case where  $i^* \in \{(M-1)N+1, \dots, MN+1\}$ , the  $N \times MN$  matrix  $D$  is defined as follows,

$$D = \begin{bmatrix} a_{(M-1)N+1,1} & \cdots & a_{(M-1)N+1,MN} \\ \vdots & & \\ a_{i^*-1,1} & \cdots & a_{i^*-1,MN} \\ a_{i^*+1,1} & \cdots & a_{i^*+1,MN} \\ \vdots & & \\ a_{MN+1,1} & \cdots & a_{MN+1,MN} \end{bmatrix}, \quad (4.43)$$

and for the case where  $i^* \in [(M-1)N]$ ,

$$D = \begin{bmatrix} a_{(M-1)N+2,1} & \cdots & a_{(M-1)N+2,MN} \\ \vdots & & \\ a_{MN+1,1} & \cdots & a_{MN+1,MN} \end{bmatrix}. \quad (4.44)$$

Now, the elements  $a_{i,j}^{(1)}, i \in \{M, M+1, \dots, x\}, j \in [MN]$  are obtained as follows,

$$\begin{bmatrix} a_{M,1}^{(1)} & \cdots & a_{M,MN}^{(1)} \\ \vdots & & \\ a_x^{(1)} & \cdots & a_{x,MN}^{(1)} \end{bmatrix} = \begin{bmatrix} a_{M,1} & \cdots & a_{M,MN} \\ \vdots & & \\ a_x & \cdots & a_{x,MN} \end{bmatrix} + CD. \quad (4.45)$$

We next show that the new matrix in (4.37) has the following form,

$$\begin{bmatrix} a_{1,1}^{(1)} & \cdots & a_{1,N}^{(1)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ a_{2,1}^{(1)} & \cdots & \cdots & \cdots & a_{2,2N}^{(1)} & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & & & & \\ a_{M-1,1}^{(1)} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{M-1,(M-1)N}^{(1)} & 0 & \cdots & 0 \\ a_{M,1}^{(1)} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{M,(M-1)N}^{(1)} & 0 & \cdots & 0 \\ \vdots & & & & & & & & & \\ a_{x,1}^{(1)} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{x,(M-1)N}^{(1)} & 0 & \cdots & 0 \end{bmatrix}, \quad (4.46)$$



where the set of all the elements that are not marked with zeros is generic. Moreover,

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_x \end{bmatrix} \begin{bmatrix} a_{1,1}^{(1)} & \dots & a_{1,MN}^{(1)} \\ \vdots & & \\ a_{x,1}^{(1)} & \dots & a_{x,MN}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}. \quad (4.47)$$

Since the first  $M - 1$  rows in (4.35) have zero entries in the last  $N$  positions, we know that,

$$\begin{bmatrix} \alpha_M & \alpha_{M+1} & \dots & \alpha_{MN+1} \end{bmatrix} \begin{bmatrix} a_{M,(M-1)N+1} & \dots & a_{M,MN} \\ \vdots & & \\ a_{MN+1,(M-1)N+1} & \dots & a_{MN+1,MN} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}. \quad (4.48)$$

Let  $E$  be the  $1 \times N$  vector defined as follows,

$$E = \begin{cases} \begin{bmatrix} \alpha_{(M-1)N+1} & \dots & \alpha_{i^*-1} & \alpha_{i^*+1} & \alpha_{MN+1} \end{bmatrix} & \text{if } i^* \in \{(M-1)N+1, \dots, MN+1\}, \\ \begin{bmatrix} \alpha_{(M-1)N+2} & \dots & \alpha_{MN+1} \end{bmatrix} & \text{otherwise.} \end{cases} \quad (4.49)$$

Now, the equality in (4.48) implies that,

$$\begin{bmatrix} \alpha_M & \dots & \alpha_x \end{bmatrix} A = -EB, \quad (4.50)$$

and consequently,

$$\begin{bmatrix} \alpha_M & \dots & \alpha_x \end{bmatrix} C = E. \quad (4.51)$$

It follows that,

$$\begin{aligned}
& \begin{bmatrix} \alpha_M & \dots & \alpha_x \end{bmatrix} \begin{bmatrix} a_{M,1}^{(1)} & \dots & a_{M,MN}^{(1)} \\ \vdots & & \\ a_x^{(1)} & \dots & a_{x,MN}^{(1)} \end{bmatrix} \\
&= \begin{bmatrix} \alpha_M & \dots & \alpha_x \end{bmatrix} \begin{bmatrix} a_{M,1} & \dots & a_{M,MN} \\ \vdots & & \\ a_x & \dots & a_{x,MN} \end{bmatrix} + \begin{bmatrix} \alpha_M & \dots & \alpha_x \end{bmatrix} CD \\
&= \begin{bmatrix} \alpha_M & \dots & \alpha_x \end{bmatrix} \begin{bmatrix} a_{M,1} & \dots & a_{M,MN} \\ \vdots & & \\ a_x & \dots & a_{x,MN} \end{bmatrix} + ED \\
&= \begin{bmatrix} \alpha_M & \dots & \alpha_{MN+1} \end{bmatrix} \begin{bmatrix} a_{M,1} & \dots & a_{M,MN} \\ \vdots & & \\ a_{MN+1,1} & \dots & a_{MN+1,MN} \end{bmatrix}. \tag{4.52}
\end{aligned}$$

We can now see that (4.47) follows from (4.52) and the fact that  $a_{i,j}^{(1)} = a_{i,j}, \forall i \in [M-1], j \in [MN]$ .

To prove (4.46), we first validate the positions of the zero entries, then prove that the set of all remaining elements is generic. The positions of the zero entries in the first  $M-1$  rows follows from (4.35) and the fact that  $a_{i,j}^{(1)} = a_{i,j}, \forall i \in [M-1], j \in [MN]$ . To show that all remaining rows in (4.46) have zeros in the last  $N$  positions, consider the following equality that follows from (4.45),

$$\begin{aligned}
& \begin{bmatrix} a_{M,(M-1)N+1}^{(1)} & \dots & a_{M,MN}^{(1)} \\ \vdots & & \\ a_{x,(M-1)N+1}^{(1)} & \dots & a_{x,MN}^{(1)} \end{bmatrix} \stackrel{(a)}{=} \begin{bmatrix} a_{M,(M-1)N+1} & \dots & a_{M,MN} \\ \vdots & & \\ a_{x,(M-1)N+1} & \dots & a_{x,MN} \end{bmatrix} + CB \\
&= A + CB \\
&= A - AB^{-1}B \\
&= \begin{bmatrix} 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{bmatrix}, \tag{4.53}
\end{aligned}$$

where (a) follows as the matrix  $B$  is formed by taking the last  $N$  columns of the matrix  $D$ . In order to prove that the set of all elements in (4.46) that

are not identically zero is generic, we first note that

$$a_{i,j}^{(1)} = a_{i,j} + f(\{a_{k,q} : x < k \leq MN + 1 \text{ or } M(N-1) < q \leq MN\}),$$

$$\forall i \in [x], j \in [M(N-1)]. \quad (4.54)$$

In particular, for all  $i \in [x], j \in [M(N-1)]$ , the element  $a_{i,j}$  contributes only to  $a_{i,j}^{(1)}$  among the set  $\mathcal{S} = \{a_{i,j}^{(1)} : i \in [x], j \in [M(N-1)]\}$ . Since the set of elements that are not identical to zero in (4.46) is a subset of  $\mathcal{S}$ , it follows from (4.54) that the former set is generic as a result of the fact that the set  $\{a_{i,j} : i \in [MN+1], j \in [MN]\}$  is generic. The proof of the statement in (4.46) and (4.47) is now complete. Moreover, the same conclusions hold for the submatrix obtained by removing the last  $N$  columns in (4.46), i.e., for the matrix,

$$\begin{bmatrix} a_{1,1}^{(1)} & \cdots & a_{1,N}^{(1)} & 0 & \cdots & \cdots & \cdots & 0 \\ a_{2,1}^{(1)} & \cdots & \cdots & \cdots & a_{2,2N}^{(1)} & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ a_{M-1,1}^{(1)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{M-1,(M-1)N}^{(1)} \\ a_{M,1}^{(1)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{M,(M-1)N}^{(1)} \\ \vdots & & & & & & & \\ a_{x,1}^{(1)} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{x,(M-1)N}^{(1)} \end{bmatrix}, \quad (4.55)$$

the set of all elements not identically zero is generic, and,

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_x \end{bmatrix} \begin{bmatrix} a_{1,1}^{(1)} & \cdots & a_{1,(M-1)N}^{(1)} \\ \vdots & & \\ a_{x,1}^{(1)} & \cdots & a_{x,(M-1)N}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}. \quad (4.56)$$

We next show that  $\alpha_1, \dots, \alpha_x$  are not all zeros. Assume otherwise, then it follows that

$$\begin{bmatrix} \alpha_{x+1} & \cdots & \alpha_{MN+1} \end{bmatrix} \begin{bmatrix} a_{x+1,1} & \cdots & a_{x+1,MN} \\ \vdots & & \\ a_{MN+1,1} & \cdots & a_{MN+1,MN} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}. \quad (4.57)$$

Since  $\alpha_{i^*} = 0$ , then it follows that

$$ED = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}. \quad (4.58)$$

In particular, since  $B$  is formed by taking the last  $N$  columns of  $D$ , then  $EB$  is an all zero vector, which almost surely implies that  $E$  is an all zero vector, as  $B$  is full rank. We conclude from this argument that  $\alpha_1, \dots, \alpha_x$  cannot be all zeros as otherwise  $\alpha_i = 0, \forall i \in [MN + 1]$  which contradicts the assumption. We have proved so far that in case any  $MN \times MN$  submatrix of any matrix of the form in (4.35) is rank deficient, then it follows that a matrix of the form in (4.55) is rank deficient. Note also that  $\{(i, j) : a_{i,j}^{(1)} = 0\} \subseteq \{(i, j) : a_{i,j} = 0\}$ . We can then repeat the above argument by replacing the rows in (4.55) whose last  $N$  entries are not identically zero by  $N$  fewer rows whose last  $N$  entries are identically zero and then removing the last  $N$  columns, to finally show that the matrix in (4.59) is rank deficient,

$$\begin{bmatrix} a_{1,1}^{(M-1)} & \dots & a_{1,N}^{(M-1)} \\ \vdots & & \\ a_{N,1}^{(M-1)} & \dots & a_{N,N}^{(M-1)} \end{bmatrix}, \quad (4.59)$$

where the set of all elements in (4.59) is generic, which implies that any matrix in the form (4.59) is full rank with high probability, and hence, proving that the assumption that there exists a rank deficient  $MN \times MN$  submatrix of  $MN$  is not true with high probability.

# CHAPTER 5

## BACKHAUL LOAD CONSTRAINT

In this chapter, we first consider the linear interference channel model ( $L = 1$ ), and recall that the characterization we obtained in Chapter 4 for the asymptotic per user DoF under a maximum transmit set size constraint  $M$  is tight for linear interference channels,  $\tau_1(M) = \frac{2M}{2M+1}$ . We also note that maximum transmit set size constraint is not tightly met by the optimal message assignment strategy. In this section, we therefore consider a cooperation constraint that is more general and relevant to many scenarios of practical significance. In particular, we define the *backhaul load* constraint  $B$  as the ratio between the sum of the transmit set sizes for all the messages and the number of users. In other words, we allow the transmit set size constraints to vary across the messages, while maintaining a constraint on the average transmit set size of  $B$ . We establish in this chapter that the asymptotic per user DoF in this new setting is  $\frac{4B-1}{4B}$  for all integer values of the average transmit set size constraint  $B$ , which is larger than the per user DoF of  $\frac{2B}{2B+1}$  obtained with the more stringent per message transmit set size constraint of  $B$ .

Furthermore, we show that the scheme that achieves the optimal DoF of  $\frac{4B-1}{4B}$  uses only zero-forcing beam-forming at the transmitters, and assigns messages non-uniformly across the transmitters, with some messages being assigned to more than  $B$  transmitters and others being assigned to fewer than  $B$  transmitters. We show that these insights can apply to more general channel models than the simple linear model considered in this chapter.

More precisely, we assume that the average transmit set size is upper bounded by an integer valued backhaul load constraint  $B$ , as defined in the constraint (2.4),

$$\frac{\sum_{i=1}^K |\mathcal{T}_i|}{K} \leq B,$$

and recall that  $\eta_L^{(\text{avg})}(K, B)$  is used to denote the DoF of a  $K$ -user channel

with connectivity parameter  $L$ , and  $\tau_L^{(\text{avg})}(B)$  is used to denote the asymptotic per user DoF, under an average transmit set size constraint  $B$ .

## 5.1 Example: $B = 1$

Before introducing the characterization of  $\tau_1^{(\text{avg})}(B)$ , we illustrate through a simple example that the potential flexibility in the backhaul design according to the constraint in (2.4) can offer DoF gains over a traditional design where all messages are assigned to the same number of transmitters. We know from Section 4.5.2 that any asymptotic per user DoF greater than  $\frac{2}{3}$  cannot be achieved through assigning each message to one transmitter. We now show that  $\tau_1^{(\text{avg})}(B = 1) \geq \frac{3}{4}$ , by allowing few messages to be available at more than one transmitter at the cost of not transmitting other messages. Consider the following assignment of the first four messages,  $\mathcal{T}_1 = \{1, 2\}$ ,  $\mathcal{T}_2 = \{2\}$ ,  $\mathcal{T}_3 = \phi$ , and  $\mathcal{T}_4 = \{3\}$ . Message  $W_1$  is transmitted through  $X_1$  to  $Y_1$  without interference. Since the channel state information is known at the second transmitter, the transmit beam for  $W_1$  at  $X_2$  can be designed to cancel the interference caused by  $W_1$  at  $Y_2$ , and then  $W_2$  can be transmitted through  $X_2$  to  $Y_2$  without interference. Finally,  $W_4$  is transmitted through  $X_3$  to  $Y_4$  without interference. It follows that the sum DoF for the first four messages  $\sum_{i=1}^4 d_i \geq 3$ . Since the fourth transmitter is inactive, the subnetwork consisting of the first four users does not interfere with the rest of the network, and hence, we can see that  $\tau_1^{(\text{avg})}(B = 1) \geq \frac{3}{4}$  through similar assignment of messages in each consecutive four-user subnetwork.

## 5.2 Asymptotic per User DoF

We now characterize the asymptotic per user DoF  $\tau_1^{(\text{avg})}(B)$  for any integer value of the backhaul load constraint.

**Theorem 13.** *The asymptotic per user DoF  $\tau_1^{(\text{avg})}(B)$  is given by*

$$\tau_1^{(\text{avg})}(B) = \frac{4B - 1}{4B}, \forall B \in \mathbf{Z}^+. \quad (5.1)$$

*Proof.* We provide the proof for the inner and outer bounds in the Sections 5.2.1 and 5.2.2, respectively.  $\square$

### 5.2.1 Coding Scheme

We treat the network as a set of subnetworks, each consisting of consecutive  $4B$  transceivers. The last transmitter of each subnetwork is deactivated to eliminate *inter-subnetwork* interference. It then suffices to show that  $4B - 1$  DoF can be achieved in each subnetwork. Without loss of generality, consider the cluster of users with indices in the set  $[4B]$ . We define the following subsets of  $[4B]$ ,

$$\begin{aligned}\mathcal{S}_1 &= [2B] \\ \mathcal{S}_2 &= \{2B + 2, 2B + 3, \dots, 4B\}.\end{aligned}$$

We next show that each user in  $\mathcal{S}_1 \cup \mathcal{S}_2$  achieves one degree of freedom, while message  $W_{2B+1}$  is not transmitted. Let the message assignments be as follows,

$$\mathcal{T}_i = \begin{cases} \{i, i + 1, \dots, 2B\}, & \forall i \in \mathcal{S}_1, \\ \{i - 1, i - 2, \dots, 2B + 1\}, & \forall i \in \mathcal{S}_2, \end{cases}$$

and note that  $\frac{\sum_{i=1}^{4B} |\mathcal{T}_i|}{4B} = B$ , and hence, the constraint in (2.4) is satisfied. Now, due to the availability of channel state information at the transmitters, the transmit beams for message  $W_i$  can be designed to cancel its effect at receivers with indices in the set  $\mathcal{C}_i$ , where,

$$\mathcal{C}_i = \begin{cases} \{i + 1, i + 2, \dots, 2B\}, & \forall i \in \mathcal{S}_1 \\ \{i - 1, i - 2, \dots, 2B + 2\}, & \forall i \in \mathcal{S}_2 \end{cases}.$$

Note that both  $\mathcal{C}_{2B}$  and  $\mathcal{C}_{2B+2}$  equal the empty set, as both  $W_{2B}$  and  $W_{2B+2}$  do not contribute to interfering signals at receivers in the set  $Y_{\mathcal{S}_1} \cup Y_{\mathcal{S}_2}$ . The above scheme for  $B = 2$  is illustrated in Figure 5.1. We conclude that each receiver whose index is in the set  $\mathcal{S}_1 \cup \mathcal{S}_2$  suffers only from Gaussian noise, thereby enjoying one degree of freedom. Since  $|\mathcal{S}_1 \cup \mathcal{S}_2| = 4B - 1$ , it follows that  $\sum_{i=1}^{4B} d_i \geq 4B - 1$ . Using a similar argument for each following subnetwork, we establish that  $\tau_1^{(\text{avg})}(B) \geq \frac{4B-1}{4B}$ , thereby proving the lower

bound of Theorem 13.

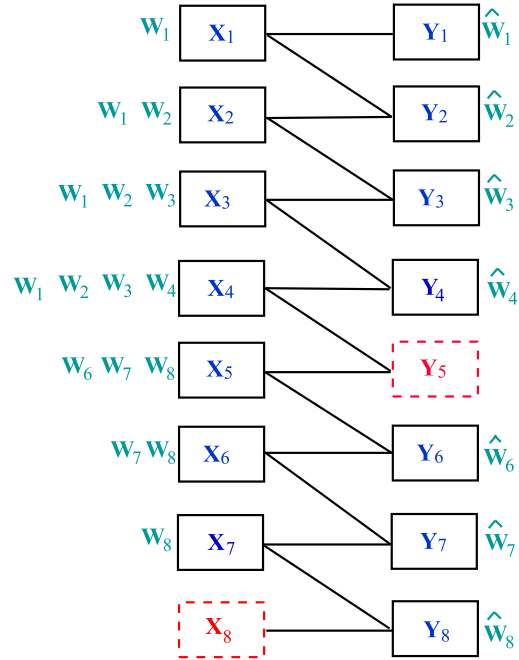


Figure 5.1: Achieving  $7/8$  per user DoF with a backhaul constraint  $B = 2$ . The figure shows only signals corresponding to the first subnetwork in a general  $K$ -user network. The signals in the dashed boxes are deactivated. Note that the deactivation of  $X_8$  splits this part of the network from the rest.

We note that the illustrated message assignment strategy satisfies the local cooperation constraint of (2.5). In other words, the network can be split into subnetworks, each of size  $4B$ , and the messages corresponding to users in a subnetwork can only be assigned to transmitters with indices in the same subnetwork.

## 5.2.2 Upper Bound

We prove the converse of Theorem 13 in two steps. First, we provide an information theoretic argument in Lemma 5 to prove an upper bound on the DoF of any network that has a subset of messages whose transmit set sizes are bounded. We then finalize the proof with a combinatorial argument that shows the existence of such a subset of messages in any assignment of messages satisfying the backhaul constraint of (2.4).



In order to prove the information theoretic argument in Lemma 5, we use Lemma 5 in Appendix A. Recall that for any set of receiver indices  $\mathcal{A} \subseteq [K]$ , we use  $U_{\mathcal{A}}$  as the set of indices of transmitters that exclusively carry the messages for the receivers in  $\mathcal{A}$ . We also need Corollary 4 in the proof of Lemma 5 to provide conditions on irreducible message assignments as defined in Section 4.4.

We now make the following definition to use in the proof of the following lemma. For any set  $\mathcal{S} \subseteq [K]$ , let  $g_{\mathcal{S}} : \mathcal{S} \rightarrow \{1, 2, \dots, |\mathcal{S}|\}$  be a function that returns the ascending order of any element in the set  $\mathcal{S}$ , e.g.,  $g_{\mathcal{S}}(\min\{i : i \in \mathcal{S}\}) = 1$  and  $g_{\mathcal{S}}(\max\{i : i \in \mathcal{S}\}) = |\mathcal{S}|$ .

**Lemma 5.** *For any  $K$ -user linear interference channel with DoF  $\eta$ , if there exists a subset of messages  $\mathcal{S} \subseteq [K]$  such that each message in  $\mathcal{S}$  is available at a maximum of  $M$  transmitters, i.e.,  $|\mathcal{T}_i| \leq M, \forall i \in \mathcal{S}$ , then the DoF is bounded by*

$$\eta \leq K - \frac{|\mathcal{S}|}{2M+1} + C_K, \quad (5.2)$$

where  $\lim_{K \rightarrow \infty} \frac{C_K}{K} = 0$ .

*Proof.* We use Lemma 5 in Appendix A with a set  $\mathcal{A}$  such that the size of the complement set  $|\bar{\mathcal{A}}| = \frac{|\mathcal{S}|}{2M+1} - o(K)$ . We define the set  $\mathcal{A}$  such that  $\bar{\mathcal{A}} = \{i : i \in \mathcal{S}, g_{\mathcal{S}}(i) = (2M+1)(j-1) + M+1, j \in \mathbf{Z}^+\}$ .

Now, we let  $s_1, s_2$  be the smallest two indices in  $\bar{\mathcal{A}}$ . We see that  $g_{\mathcal{S}}(s_1) = M+1, g_{\mathcal{S}}(s_2) = 3M+2$ . Note that  $X_1 + \frac{Z_1}{H_{1,1}} = \frac{Y_1}{H_{1,1}}$ , and

$$X_2 + \frac{Z_2 - \frac{H_{2,1}}{H_{1,1}}Z_1}{H_{2,2}} = \frac{Y_2 - \frac{H_{2,1}}{H_{1,1}}Y_1}{H_{2,2}}.$$

Similarly, it is clear how the first  $s_1 - 1$  transmit signals  $X_{[s_1-1]}$  can be recovered from the received signals  $Y_{[s_1-1]}$  and linear combinations of the noise signals  $Z_{[s_1-1]}$ . In what follows, we show how to reconstruct a noisy version of the signals  $\{X_{s_1}, X_{s_1+1}, \dots, X_{s_2-1}\}$ , where the reconstruction noise is a linear combination of the signals  $Z_{\mathcal{A}}$ . Then it will be clear by symmetry how the remaining transmit signals can be reconstructed.

We now notice that it follows from Corollary 4 that message  $W_{s_1}$  can be removed from any transmitter in  $\mathcal{T}_{s_1}$  whose index is greater than  $s_1 + M - 1$ , without affecting the sum rate. Similarly, there is no loss in generality in assuming that  $\forall s_i \in \mathcal{S}, s_i \neq s_1, \mathcal{T}_{s_i}$  does not have an element

with index less than  $s_i - M$ . Since  $s_i - s_1 \geq g_S(s_i) - g_S(s_1) \geq 2M + 1$ , it follows that  $X_{s_1+M} \in X_{U_A}$ . The signal  $X_{s_1+M+1} + \frac{Z_{s_1+M+1}}{H_{s_1+M+1, s_1+M+1}}$  can be reconstructed from  $Y_{s_1+M+1}$  and  $X_{s_1+M}$ . Then, it can be seen that the transmit signals  $\{X_{s_1+M+2}, X_{s_1+M+3}, \dots, X_{s_2-1}\}$  can be reconstructed from  $\{Y_{s_1+M+1}, Y_{s_1+M+2}, \dots, Y_{s_2-1}\}$ , and linear combinations of the noise signals  $\{Z_{s_1+M+1}, Z_{s_1+M+2}, \dots, Z_{s_2-1}\}$ . Similarly, since  $X_{s_1+M}$  is known, the transmit signals  $\{X_{s_1+M-1}, X_{s_1+M-2}, \dots, X_{s_1}\}$  can be reconstructed from  $\{Y_{s_1+M}, Y_{s_1+M-1}, \dots, Y_{s_1+1}\}$ , and linear combinations of the noise signals  $\{Z_{s_1+M}, Z_{s_1+M-1}, \dots, Z_{s_1+1}\}$ . By following a similar argument to reconstruct all transmit signals from the signals  $Y_A$ ,  $X_{U_A}$ , and linear combinations of the noise signals  $Z_A$ , we can show the existence of functions  $f_1$  and  $f_2$  of Lemma 5 to complete the proof.  $\square$

We now explain how Lemma 5 can be used to prove that  $\tau_1^{(\text{avg})}(B) \leq \frac{3}{4}$ . For any message assignment satisfying (2.4) for a  $K$ -user channel, let  $R_j$  be defined as follows for every  $j \in \{0, 1, \dots, K\}$ ,

$$R_j = \frac{|\{i : i \in [K], |\mathcal{T}_i| = j\}|}{K}. \quad (5.3)$$

$R_j$  is the fraction of users whose messages are available at exactly  $j$  transmitters. Now, if  $R_0 + R_1 \geq \frac{3}{4}$ , then Lemma 5 can be used directly to show that  $\eta \leq \frac{3K}{4} + o(K)$ . Otherwise, more than  $\frac{K}{4}$  users have their messages at two or more transmitters, and it follows from (2.4) that  $R_0 \geq \sum_{j=2}^K R_j \geq \frac{1}{4}$ , and hence,  $\eta \leq (1 - R_0)K \leq \frac{3K}{4}$ .

We generalize the above argument in the proof of Lemma 6 to complete the proof that  $\tau_1^{(\text{avg})}(B) \leq \frac{4B-1}{4B}, \forall B \in \mathbf{Z}^+$ .

**Lemma 6.** *For any message assignment satisfying (2.4) for a  $K$ -user channel with an average transmit set size constraint  $B$ , there exists an integer  $M \in \{0, 1, \dots, K\}$ , and a subset  $\mathcal{S} \subseteq [K]$  whose size  $|\mathcal{S}| \geq \frac{2M+1}{4B}K$ , such that each message in  $\mathcal{S}$  is available at a maximum of  $M$  transmitters, i.e.,  $|\mathcal{T}_i| \leq M, \forall i \in \mathcal{S}$ .*

*Proof.* Fix any message assignment satisfying (2.4) for a  $K$ -user channel with backhaul constraint  $B$ , and let  $R_j, j \in \{0, 1, \dots, K\}$  be defined as in (5.3). If  $\sum_{j=2B}^K R_j \leq \frac{1}{4B}$ , then more than  $\frac{4B-1}{4B}K$  users have a transmit set whose size is at most  $2B - 1$ , and the lemma follows with  $M = 2B - 1$ . It then suffices to assume that  $\sum_{j=2B}^K R_j > \frac{1}{4B}$  in the rest of the proof. We show

in the following that there exists an integer  $M \in \{0, \dots, 2B - 2\}$  such that  $\sum_{j=0}^M R_j > \frac{2M+1}{4B}$ , thereby completing the proof of the lemma.

Define  $R_j^*, j \in \{0, 1, \dots, 2B\}$  such that  $R_0^* = R_{2B}^* = \frac{1}{4B}$ , and  $R_j^* = \frac{1}{2B}, \forall j \in \{1, \dots, 2B - 1\}$ . Now, note that  $\sum_{j=0}^{2B} R_j^* = 1$ , and  $\sum_{j=0}^{2B} jR_j^* = B$ . It follows that if  $R_j = R_j^*, \forall j \in \{0, \dots, 2B\}$ , and  $R_j = 0, \forall j \geq 2B + 1$ , then the constraint in (2.4) is tightly met, i.e.,  $\frac{\sum_{i=1}^K |\mathcal{T}_i|}{K} = B$ . We will use this fact in the rest of the proof.

We prove the statement by contradiction. Assume that  $\sum_{j=2B}^K R_j > R_{2B}^* = \frac{1}{4B}$ , and that  $\forall M \in \{0, 1, \dots, 2B - 2\}, \sum_{j=0}^M R_j \leq \sum_{j=0}^M R_j^* = \frac{2M+1}{4B}$ . We know from (2.4) that  $\sum_{j=0}^K jR_j \leq \sum_{j=0}^{2B} jR_j^* = B$ . Also, since  $\sum_{j=0}^K R_j = \sum_{j=0}^{2B} R_j^* = 1$  and  $\sum_{j=2B}^K R_j > R_{2B}^*$ , it follows that there exists an integer  $M \in \{0, 1, \dots, 2B - 1\}$  such that  $R_M > R_M^*$ ; let  $m$  be the smallest such integer. Since  $\sum_{j=0}^m R_j \leq \sum_{j=0}^m R_j^*$ , and  $\forall j \in \{0, 1, \dots, m - 1\}, R_j \leq R_j^*$ , we can construct another message assignment by removing elements from some transmit sets whose size is  $m$ , such that the new assignment satisfies (2.4), and has transmit sets  $\mathcal{T}_i^*$  where  $\forall j \in \{0, 1, \dots, m\}, |\{i : i \in [K], |\mathcal{T}_i^*| = j\}| \leq R_j^*$ . By successive application of the above argument, we can construct a message assignment that satisfies (2.4), and has transmit sets  $\mathcal{T}_i^*$  where  $\forall j \in \{0, 1, \dots, 2B - 1\}, |\{i : i \in [K], |\mathcal{T}_i^*| = j\}| \leq R_j^*$  and  $|\{i : i \in [K], |\mathcal{T}_i^*| \geq 2B\}| \geq R_{2B}^*$ . Note that the new assignment has to violate (2.4) since  $\sum_{j=0}^{2B} jR_j^* = B$ , and we reach a contradiction.  $\square$

We now know from Lemmas 5 and 6 that under the backhaul load constraint of (2.4), the DoF for any  $K$ -user channel is upper bounded by  $\frac{4B-1}{4B}K + o(K)$ . It follows that the asymptotic per user DoF  $\tau_1^{(\text{avg})}(B) \leq \frac{4B-1}{4B}$ , thereby proving the upper bound of Theorem 13.

### 5.3 Discussion and Generalizations

We note that for the considered linear interference channel model an average transmit set size constraint  $B$ , the per user DoF  $\tau_1^{(\text{avg})}(B)$  can be achieved using a combination of the schemes that are characterized as optimal in Section 4.5.2 for the cases of  $M = 2B - 1$  and  $M = 2B$ . We note that even though the maximum transmit set size constraint may not reflect a physical constraint, the solutions obtained in Chapter 4 under this constraint

provide a useful toolset that can be used to achieve the optimal per user DoF value under the more natural constraint on the total backhaul load that is considered in this section.

### 5.3.1 General Values of the Connectivity Parameter $L$

Using a convex combination of the schemes that we derived under the maximum transmit set size constraint can also provide good coding schemes for the more general locally connected channel model, where each receiver can see interference from  $L$  neighbouring transmitters. We can use a convex combination of the schemes that are characterized in Section 4.3 to achieve the inner bounds stated in Table 5.1 for the case where  $B = 1$ .

Now, we note that the inner bounds stated in Table 5.1 can be achieved through the use of only zero-forcing transmit beam-forming. In other words, there is no need for the symbol extension idea required by the asymptotic interference alignment scheme of [11]. In Theorem 10, it is shown that for  $L \geq 2$ , by allowing each message to be available at one transmitter, the asymptotic per user DoF is  $\frac{1}{2}$ ; it is also shown in Theorem 8 that the  $\frac{1}{2}$  per user DoF value cannot be achieved through zero-forcing transmit forming for  $L \geq 3$ . In contrast, in Table 5.1 it can be seen that for  $L \leq 6$ , the  $\frac{1}{2}$  per user DoF value can be achieved through zero-forcing transmit beam-forming and a flexible design of the backhaul links, without incurring additional overall load on the backhaul ( $B = 1$ ).

### 5.3.2 Two-Dimensional Networks

The insights we have in this section on the backhaul design for linear interference networks, may apply in denser networks by treating the denser network as a set of interfering linear networks. For example, consider the

Table 5.1: Achievable per user DoF values for locally connected channels with a backhaul constraint  $\sum_{i=1}^K |\mathcal{T}_i| \leq K$ .

	$L = 2$	$L = 3$	$L = 4$	$L = 5$	$L = 6$
$\tau_L^{(\text{avg})}(B = 1) \geq$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{5}{9}$	$\frac{11}{21}$	$\frac{1}{2}$

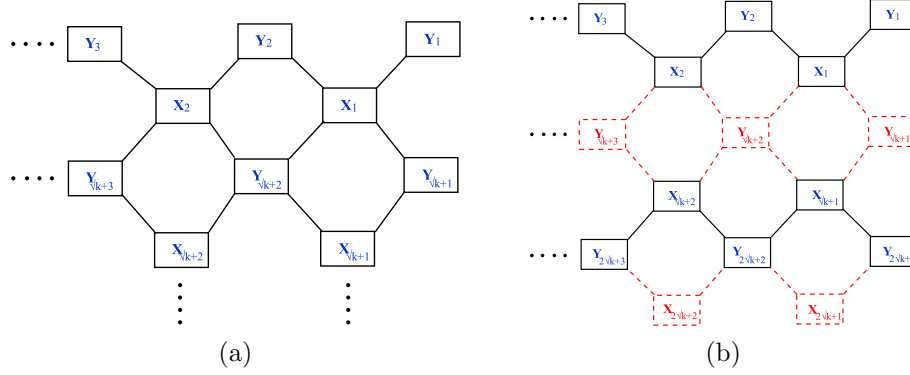


Figure 5.2: Two-dimensional interference network. In (a), we plot the channel model, with each transmitter being connected to four surrounding cell edge receivers. In (b), we show an example coding scheme where dashed red boxes and lines represent inactive nodes and edges. The signals  $\{X_1, \dots, X_{\sqrt{K}}\}$  and  $\{Y_1, \dots, Y_{\sqrt{K}}\}$  form a linear subnetwork. Similarly, the signals  $\{X_{\sqrt{K}+1}, \dots, X_{2\sqrt{K}}\}$  and  $\{Y_{2\sqrt{K}+1}, \dots, Y_{3\sqrt{K}}\}$  form a linear subnetwork

two-dimensional network depicted in Figure 5.2a where each transmitter is connected to four cell edge receivers. The precise channel model for a  $K$ -user channel is as follows,

$$\begin{aligned}
 &H_{i,j} \text{ is not identically 0, if and only if} \\
 &i \in \left\{ j, j+1, j + \lfloor \sqrt{K} \rfloor, j + \lfloor \sqrt{K} \rfloor + 1 \right\}.
 \end{aligned} \tag{5.4}$$

For this channel model, we can show that by assigning each message to one transmitter, i.e., imposing the constraint  $|\mathcal{T}_i| \leq 1, \forall i \in [K]$ , the asymptotic per user DoF is at most  $\frac{1}{2}$ , and the use of only zero-forcing transmit beam-forming can lead to at most  $\frac{4}{9}$  per user DoF. However, under the backhaul load constraint  $\frac{\sum_{i=1}^K |\mathcal{T}_i|}{K} \leq 1$ , a per user DoF value of  $\frac{5}{9}$  can be achieved using only zero-forcing transmit beam-forming. This can be done by deactivating every third row of transmitters, and splitting the rest of the network into non-interfering linear subnetworks (see Figure 5.2b). In each subnetwork, a backhaul load constraint of  $\frac{3}{2}$  is imposed. For example, the following constraint is imposed on the first row of users,  $\frac{\sum_{i=1}^{\lfloor \sqrt{K} \rfloor} |\mathcal{T}_i|}{\lfloor \sqrt{K} \rfloor} \leq \frac{3}{2}$ . A convex combination of the schemes that are characterized as optimal for linear in-

interference networks in Section 4.5.2 for the cases of maximum transmit set size constraints  $M = 2$  and  $M = 3$  is then used to achieve  $\frac{5}{6}$  per user DoF in each active subnetwork while satisfying a backhaul load constraint of  $\frac{3}{2}$ . Since  $\frac{2}{3}$  of the subnetworks are active, a per user DoF of  $\frac{5}{9}$  is achieved while satisfying a backhaul load constraint of unity.

# CHAPTER 6

## BLOCK ERASURE CHANNEL

We realize through the results presented in Chapters 3, 4, and 5 that conclusions related to the optimal assignment of messages to transmitters and the achievable DoF differ dramatically based on network topology. For example, under the maximum transmit set size constraint (2.3), local cooperation cannot lead to achieving a gain in the asymptotic per user DoF for the fully connected channel. However, local cooperation is optimal for locally connected channels and can lead to achieving scalable DoF gains, and the optimal assignment of messages to transmitters depends on the connectivity parameter  $L$ . In practice, the topology may change due to deep fading conditions (see e.g. [5]) or even intentionally to exploit spectrum opportunities (see e.g. [54]). In this chapter, we extend our results to dynamic interference networks where a fixed assignment of messages is selected to achieve average DoF optimal performance in networks with changing topology.

In [55], the authors analyzed the average capacity for a point-to-point channel model where slow changes result in varying severity of noise. In this chapter, we apply a similar concept to interference networks by assuming that slowly changing deep fading conditions result in link erasures. We consider the linear interference network ( $L = 1$ ), with the consideration of two fading effects. Long-term fluctuations that result in link erasures over a complete block of time slots, and short-term fluctuations that allow us to assume that any specific joint realization for the non-zero channel coefficients, will take place with zero probability. We study the problem of achieving the optimal average degrees of freedom (DoF) under a maximum transmit set size constraint (2.3). We note that the studied problem in Section 4.5.2 reduces here to the case of no erasures. In this chapter, we extend the schemes in Chapter 4 to consider the occurrence of link erasures, and propose new schemes that lead to achieving better average DoF at high probabilities of erasure.

## 6.1 Channel Model

Each transmitter can only be connected to its corresponding receiver as well as one following receiver, and the last transmitter can only be connected to its corresponding receiver. More precisely,

$$H_{i,j} \text{ is identically 0 iff } i \notin \{j, j + 1\}, \forall i, j \in [K]. \quad (6.1)$$

In order to consider the effect of long-term fluctuations (shadowing), we assume that communication takes place over blocks of time slots, and let  $p$  be the probability of block erasure. In each block, we assume that for each  $j$ , and each  $i \in \{j, j + 1\}$ ,  $H_{i,j} = 0$  with probability  $p$ . Moreover, short-term channel fluctuations allow us to assume that in each time slot, all non-zero channel coefficients are drawn independently from a continuous distribution. As in the previous chapters, we assume that global channel state information is available at all transmitters and receivers.

Recall that we use  $\eta_p(K, M)$  to denote the DoF of a  $K$ -user channel with block erasure probability  $p$  and a maximum transmit set size constraint  $M$ , and  $\tau_p(M)$  to denote the asymptotic per user DoF. We call a message assignment strategy *optimal* for a given erasure probability  $p$ , if there exists a sequence of coding schemes achieving  $\tau_p(M)$  using the transmit sets defined by the message assignment strategy. A message assignment strategy is *universally optimal* if it is optimal for all values of  $p$ . We characterize  $\tau_p(M = 1)$  in Section 6.2, and show that there is no universally optimal message assignment strategy for the case of  $M = 1$ .

## 6.2 Cell Association

We first consider the case where each receiver can be served by only one transmitter. This reflects the problem of associating mobile users with cells in a cellular downlink scenario. We start by discussing orthogonal schemes (TDMA-based) for this problem, and then show that the proposed schemes are optimal. It will be useful in the rest of this section to view each realization of the network where some links are erased, as a series of subnetworks that do not interfere. We say that a set of  $k$  users with successive indices  $\{i, i + 1, \dots, i + k - 1\}$  form a subnetwork if the following two conditions hold:



The first condition is that  $i = 1$  or it is the case that message  $W_{i-1}$  does not cause interference at  $Y_i$ , either because the direct link between the transmitter carrying  $W_{i-1}$  and receiver  $(i-1)$  is erased, or the transmitter carrying  $W_{i-1}$  is not connected to the  $i^{\text{th}}$  receiver. Second,  $i+k-1 = K$  or it is the case that message  $W_{i+k-1}$  does not cause interference at  $Y_{i+k}$ , because the carrying transmitter is not connected to one of the receivers  $(i+k-1)$  and  $(i+k)$ .

We say that the subnetwork is *atomic* if the transmitters carrying messages for users in the subnetwork have successive indices and for any transmitter  $t$  carrying a message for a user in the subnetwork, and receiver  $r$  such that  $r \in \{t, t+1\}$  and  $r \in \{i, i+1, \dots, i+k-1\}$ , the channel coefficient  $H_{r,t} \neq 0$ .

For  $i \in [K]$ , let  $N_i$  be the number of messages available at the  $i^{\text{th}}$  transmitter, and let  $\mathbf{N}^K = (N_1, N_2, \dots, N_K)$ . It is clear that the sequence  $\mathbf{N}^K$  can be obtained from the transmit sets  $\mathcal{T}_i, i \in [K]$ ; it is also true, as stated in Lemma 7, that the converse holds. We use the notion of *irreducible* message assignments from Section 4.4. For  $M = 1$ , an irreducible message assignment will have each message assigned to one of the two transmitters that can be connected to its designated receiver.

**Lemma 7.** *For any irreducible message assignment where each message is assigned to exactly one transmitter, i.e.,  $|\mathcal{T}_i| = 1, \forall i \in [K]$ , the transmit sets  $\mathcal{T}_i, i \in [K]$ , are uniquely characterized by the sequence  $\mathbf{N}^K$ .*

*Proof.* Since each message can only be available at one transmitter, then this transmitter has to be connected to the designated receiver. More precisely,  $\mathcal{T}_i \subset \{i-1, i\}, \forall i \in \{2, \dots, K\}$ , and  $\mathcal{T}_1 = \{1\}$ . It follows that each transmitter carries at most two messages and the first transmitter carries at least the message  $W_1$ , i.e.,  $N_i \in \{0, 1, 2\}, \forall i \in \{2, \dots, K\}$ , and  $N_1 \in \{1, 2\}$ . Assume that  $N_i = 1, \forall i \in [K]$ , then  $\mathcal{T}_i = \{i\}, \forall i \in [K]$ . For the remaining case, we know that there exists  $i \in \{2, \dots, K\}$  such that  $N_i = 0$ , since  $\sum_{i=1}^K N_i = K$ ; we handle this case in the rest of the proof.

Let  $x$  be the smallest index of a transmitter that carries no messages, i.e.,  $x = \min\{i : N_i = 0\}$ . We now show how to reconstruct the transmit sets  $\mathcal{T}_i, i \in \{1, \dots, x\}$  from the sequence  $(N_1, N_2, \dots, N_x)$ . We note that  $\mathcal{T}_i \subseteq [x], \forall i \in [x]$ , and since  $N_x = 0$ , it follows that  $\mathcal{T}_i \not\subseteq [x], \forall i \notin [x]$ . It follows that  $\sum_{i=1}^{x-1} N_i = x$ . Since  $\mathcal{T}_i \subset \{i-1, i\}, \forall i \in \{2, \dots, x\}$ , we know that at most one transmitter in the first  $x-1$  transmitters carries two messages.

Since  $\sum_{i=1}^{x-1} N_i = x$ , and  $N_i \in \{1, 2\}, \forall i \in [x-1]$ , it follows that there exists an index  $y \in [x-1]$  such that  $N_y = 2$ , and  $N_i = 1, \forall i \in [x-1] \setminus \{y\}$ . It is now clear that the  $y^{\text{th}}$  transmitter carries messages  $W_y$  and  $W_{y+1}$ , and each transmitter with an index  $j \in \{y+1, \dots, x-1\}$  is carrying message  $W_{j+1}$ , and each transmitter with an index  $j \in \{1, \dots, y\}$  is carrying message  $W_j$ . The transmit sets are then determined as follows.  $\mathcal{T}_i = \{i\}, \forall i \in [y]$  and  $\mathcal{T}_i = \{i-1\}, \forall i \in \{y+1, \dots, x\}$ .

We view the network as a series of subnetworks, where the last transmitter in each subnetwork is either inactive or it is the last transmitter in the network. If the last transmitter in a subnetwork is inactive, then the transmit sets in the subnetwork are determined in a similar fashion to the transmit sets  $\mathcal{T}_i, i \in [x]$ , in the above scenario. If the last transmitter in the subnetwork is the  $K^{\text{th}}$  transmitter, and  $N_K = 1$ , then each message in this subnetwork is available at the transmitter with the same index.  $\square$

We use Lemma 7 to describe message assignment strategies for large networks through repeating patterns of short ternary strings. Given a ternary string  $\mathbf{S} = (S_1, \dots, S_n)$  of fixed length  $n$  such that  $\sum_{i=1}^n S_i = n$ , we define  $\mathbf{N}^K, K \geq n$  as follows:

- $N_i = S_{i \bmod n}$  if  $i \in \{1, \dots, n \lfloor \frac{K}{n} \rfloor\}$ ,
- $N_i = 1$  if  $i \in \{n \lfloor \frac{K}{n} \rfloor + 1, \dots, K\}$ .

We now evaluate all possible message assignment strategies satisfying the cell association constraint using ternary strings through the above representation. We restrict our attention to irreducible message assignments, and note that if there are two transmitters with indices  $i$  and  $j$  such that  $i < j$  and each is carrying two messages, then there is a third transmitter with index  $k$  such that  $i < k < j$  that carries no messages. It follows that any string defining message assignment strategies that satisfy the cell association constraint has to have one of the following forms:

- $S^{(1)} = (1)$ ,
- $S^{(2)} = (2, 1, 1, \dots, 1, 0)$ ,
- $S^{(3)} = (1, 1, \dots, 1, 2, 0)$ ,
- $S^{(4)} = (1, 1, \dots, 1, 2, 1, 1, \dots, 1, 0)$ .

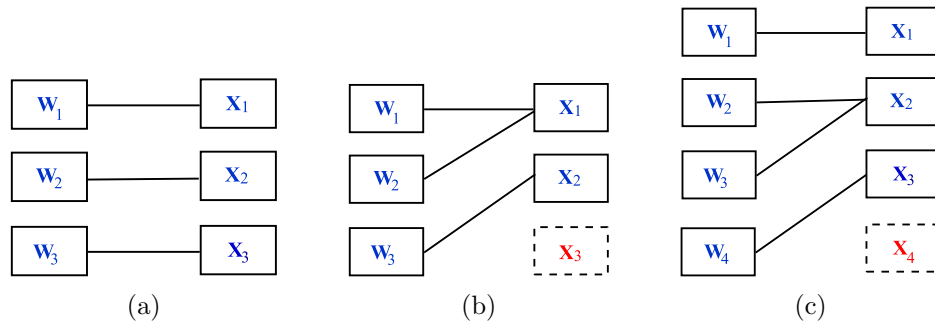


Figure 6.1: The optimal message assignment strategies for the cell association problem. The red dashed boxes represent transmit signals that are inactive in all network realizations. The strategies in (a), (b), and (c) are optimal at high, low, and middle values of the erasure probability  $p$ , respectively.

We now introduce the three candidate message assignment strategies illustrated in Figure 6.1, and we characterize the TDMA per user DoF achieved through each of them; we will show later that the optimal message assignment strategy at any value of  $p$  is given by one of the three introduced strategies. We first consider the message assignment strategy defined by the string having the form  $S^{(1)} = (1)$ . Here, each message is available at the transmitter having the same index.

**Lemma 8.** *Under the restriction to the message assignment strategy  $\mathcal{T}_{i,K} = \{i\}, \forall K \in \mathbf{Z}^+, i \in [K]$ , and orthogonal TDMA schemes, the average per user DoF is given by*

$$\begin{aligned} \tau_p^{(1)} &= \frac{1}{2} \left( 1 - p + (1 - p) (1 - (1 - p)^2)^2 \right) \\ &\quad + \sum_{i=1}^{\infty} \frac{1}{2} (1 - (1 - p)^2)^2 (1 - p)^{4i+1}. \end{aligned} \quad (6.2)$$

*Proof.* We will first show how  $\frac{1}{2} \left( 1 - p + (1 - p) (1 - (1 - p)^2)^2 \right)$  DoF can be achieved, and then modify the transmission scheme to show how to achieve  $\tau_p^{(1)}$ . For each user with an odd index  $i$ , message  $W_i$  is transmitted whenever the channel coefficient  $H_{i,i} \neq 0$ ; the rate achieved by these users contributes to the average per user DoF by  $\frac{1}{2}(1 - p)$ . For each user with an even index  $i$ , message  $W_i$  is transmitted whenever the following holds:  $H_{i,i} \neq 0$ ,  $W_{i-1}$  does not cause interference at  $Y_i$ , and the transmission of  $W_i$  will not disrupt the communication of  $W_{i+1}$  to its designated receiver; we note that this hap-

pens if and only if  $H_{i,i} \neq 0$  and  $(H_{i-1,i-1} = 0$  or  $H_{i,i-1} = 0)$  and  $(H_{i+1,i} = 0$  or  $H_{i+1,i+1} = 0)$ . It follows that the rate achieved by users with even indices contributes to the average per user DoF by  $\frac{1}{2}(1-p)(1-(1-p)^2)^2$ .

We now discuss a modification of the above scheme to achieve  $\tau_p^{(1)}$ . As above, users with odd indices have priority, i.e., their messages are delivered whenever their direct links exist, and users with even indices deliver their messages whenever their direct links exist and the channel connectivity allows for avoiding conflict with priority users. However, we make an exception to the priority setting in atomic subnetworks consisting of an odd number of users, and the first and last users have even indices; in these subnetworks, one extra DoF is achieved by allowing users with even indices to have priority and deliver their messages. The resulting extra term in the average per user DoF is calculated as follows. Fixing a user with an even index, the probability that this user is the first user in a subnetwork consisting of an odd number of users in a large network is  $\sum_{i=1}^{\infty} (1-(1-p)^2)^2 (1-p)^{4i+1}$ ; for each of these events, the sum DoF is increased by 1, and hence the added term to the average per user DoF is equal to half this value, since every other user has an even index.

The optimality of the above scheme within the class of orthogonal TDMA-based schemes follows directly from [48, Theorem 1] for each realization of the network.  $\square$

We will show later that the above scheme is optimal at high erasure probabilities. In Chapter 4, the optimal message assignment for the case of no erasures is characterized. The per user DoF is shown to be  $\frac{2}{3}$ , and is achieved by deactivating every third transmitter and achieving 1 DoF for each transmitted message. We now consider the extension of this message assignment illustrated in Figure 6.1b, which will be shown later to be optimal for low erasure probabilities.

**Lemma 9.** *Under the restriction to the message assignment strategy defined by the string  $S = (2, 1, 0)$ , and orthogonal TDMA schemes, the average per user DoF is given by*

$$\tau_p^{(2)} = \frac{2}{3}(1-p) + \frac{1}{3}p(1-p)(1-(1-p)^2). \quad (6.3)$$

*Proof.* For each user with an index  $i$  such that  $(i \bmod 3 = 0)$  or  $(i \bmod 3 = 1)$ ,

message  $W_i$  is transmitted whenever the link between the transmitter carrying  $W_i$  and the  $i^{\text{th}}$  receiver is not erased; these users contribute to the average per user DoF by a factor of  $\frac{2}{3}(1-p)$ . For each user with an index  $i$  such that  $(i \bmod 3 = 2)$ , message  $W_i$  is transmitted through  $X_{i-1}$  whenever the following holds:  $H_{i,i-1} \neq 0$ , message  $W_{i-1}$  is not transmitted because  $H_{i-1,i-1} = 0$ , and the transmission of  $W_i$  will not be disrupted by the communication of  $W_{i+1}$  through  $X_i$  because  $(H_{i,i} = 0)$  or  $(H_{i+1,i} = 0)$ ; these users contribute to the average per user DoF by a factor of  $\frac{1}{3}p(1-p)(1-(1-p)^2)$ . Using the considered message assignment strategy, the TDMA optimality of this scheme follows from [48, Theorem 1] for each network realization.  $\square$

We now consider the message assignment strategy illustrated in Figure 6.1c. We will show later that this strategy is optimal for a middle regime of erasure probabilities.

**Lemma 10.** *Under the restriction to the message assignment strategy defined by the string  $S = (1, 2, 1, 0)$ , and orthogonal TDMA schemes, the average per user DoF is given by*

$$\begin{aligned} \tau_p^{(3)} &= \frac{1}{2}(1-p) \\ &\quad + \frac{1}{4}(1-p)(1-(1-p)^2)(1+p+(1-p)^3). \end{aligned} \tag{6.4}$$

*Proof.* As in the proof of Lemma 8, we first introduce a transmission scheme achieving part of the desired rate, and then modify it to show how the extra term can be achieved. Let each message with an odd index be delivered whenever the link between the transmitter carrying the message and the designated receiver is not erased; these users contribute to the average per user DoF by a factor of  $\frac{1}{2}(1-p)$ . For each user with an even index  $i$ , if  $i \bmod 4 = 2$ , then  $W_i$  is transmitted through  $X_i$  whenever the following holds:  $H_{i,i} \neq 0$ , message  $W_{i+1}$  is not transmitted through  $X_i$  because  $H_{i+1,i} = 0$ , and the transmission of  $W_i$  will not be disrupted by the communication of  $W_{i-1}$  through  $X_{i-1}$  because either  $H_{i,i-1} = 0$  or  $H_{i-1,i-1} = 0$ ; these users contribute to the average per user DoF by a factor of  $\frac{1}{4}p(1-p)(1-(1-p)^2)$ . For each user with an even index  $i$  such that  $i$  is a multiple of 4,  $W_i$  is transmitted through  $X_{i-1}$  whenever  $H_{i,i-1} \neq 0$ , and the transmission of  $W_i$

will not disrupt the communication of  $W_{i-1}$  through  $X_{i-2}$  because either  $H_{i-1,i-1} = 0$  or  $H_{i-1,i-2} = 0$ ; these users contribute to the average per user DoF by a factor of  $\frac{1}{4} (1-p) (1 - (1-p)^2)$ .

We now modify the above scheme to show how  $\tau_p^{(3)}$  can be achieved. Since the  $i^{\text{th}}$  transmitter is inactive for every  $i$  that is a multiple of 4, users  $\{i-3, i-2, i-1, i\}$  are separated from the rest of the network for every  $i$  that is a multiple of 4, i.e., these users form a subnetwork. We explain the modification for the first four users, and it will be clear how to apply a similar modification for every following set of four users. Consider the event where message  $W_1$  does not cause interference at  $Y_2$ , because either  $H_{1,1} = 0$  or  $H_{2,1} = 0$ , and it is the case that  $H_{2,2} \neq 0$ ,  $H_{3,2} \neq 0$ ,  $H_{3,3} \neq 0$ , and  $H_{4,3} \neq 0$ ; this is the event that users  $\{2, 3, 4\}$  form an atomic subnetwork, and it happens with probability  $(1 - (1-p)^2) (1-p)^4$ . In this case, we let messages  $W_2$  and  $W_4$  have priority instead of message  $W_3$ , and hence the sum DoF for messages  $\{W_1, W_2, W_3, W_4\}$  is increased by 1. It follows that an extra term of  $\frac{1}{4} (1 - (1-p)^2) (1-p)^4$  is added to the average per user DoF.

The TDMA optimality of the illustrated scheme follows from [48, Theorem 1] for each network realization.  $\square$

In Figure 6.2, we plot the values of  $\frac{\tau_p^{(1)}}{1-p}$ ,  $\frac{\tau_p^{(2)}}{1-p}$ , and  $\frac{\tau_p^{(3)}}{1-p}$ , and note that  $\max \left\{ \tau_p^{(1)}, \tau_p^{(2)}, \tau_p^{(3)} \right\}$  equals  $\tau_p^{(1)}$  at high probabilities of erasure, and equals  $\tau_p^{(2)}$  at low probabilities of erasure, and equals  $\tau_p^{(3)}$  in a middle regime.

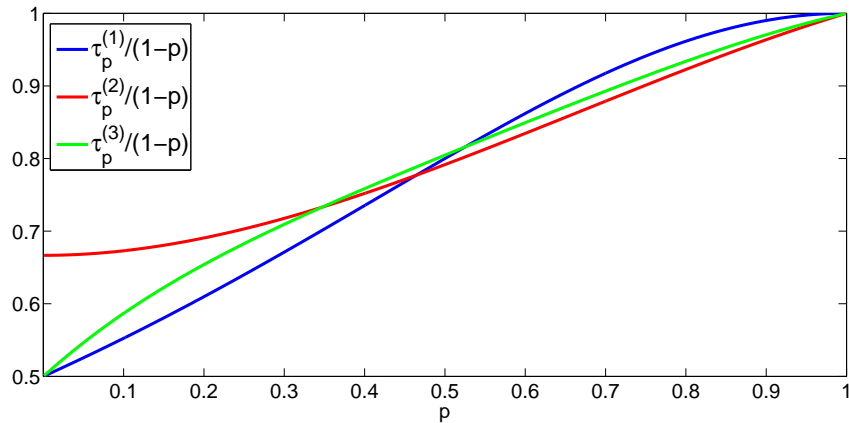


Figure 6.2: The average per user DoF achieved through the strategies in Lemmas 8, 9, and 10, normalized by  $(1-p)$ .

We now show that under the restriction to TDMA schemes, one of the

message assignment strategies illustrated in Lemmas 8, 9, and 10 is optimal at any value of  $p$ .

**Theorem 14.** *For a given erasure probability  $p$ , let  $\tau_p^{(TDMA)}$  be the average per user DoF under the restriction to orthogonal TDMA schemes, then at any value  $0 \leq p \leq 1$  the following holds,*

$$\tau_p^{(TDMA)} = \max \{ \tau_p^{(1)}, \tau_p^{(2)}, \tau_p^{(3)} \}, \quad (6.5)$$

where  $\tau_p^{(1)}$ ,  $\tau_p^{(2)}$ , and  $\tau_p^{(3)}$  are given in (6.2), (6.3), and (6.4), respectively.

*Proof.* The inner bound follows from Lemmas 8, 9, and 10. In order to prove the converse, we need to consider all irreducible message assignment strategies where each message is assigned to a single transmitter. We know from Lemma 8 that the TDMA average per user DoF achieved through the strategy defined by the string of all ones having the form  $S^{(1)} = (1)$  equals  $\tau_p^{(1)}$ , and hence the upper bound holds in this case.

We now show that the TDMA average per user DoF achieved through strategies defined by strings of the form  $S^{(2)} = (2, 1, \dots, 1, 0)$  is upper bounded by a convex combination of  $\tau_p^{(1)}$  and  $\tau_p^{(2)}$ , and hence, is upper bounded by  $\max \{ \tau_p^{(1)}, \tau_p^{(2)} \}$ . The considered message assignment strategy splits each network into subnetworks consisting of a transmitter carrying two messages followed by a number of transmitters, each is carrying one message, and the last transmitter in the subnetwork carries no messages. We first consider the case where the number of transmitters carrying single messages is odd. We consider the simple scenario of the message assignment strategy defined by the string  $(2, 1, 1, 1, 0)$ , and then the proof will be clear for strategies defined by strings of the form  $(2, 1, 1, \dots, 1, 0)$  that have an arbitrary odd number of ones. In this case, it suffices to show that the average per user DoF in the first subnetwork is upper bounded by a convex combination of  $\tau_p^{(1)}$  and  $\tau_p^{(2)}$ . The first subnetwork consists of the first five users;  $W_1$  and  $W_2$  can be transmitted through  $X_1$ .  $W_3$ ,  $W_4$  and  $W_5$  can be transmitted through  $X_2$ ,  $X_3$ , and  $X_4$ , respectively, and the transmit signal  $X_5$  is inactive.

We now explain the optimal TDMA scheme for the considered subnetwork. We first explain a simple scheme and then modify it to get the optimal scheme. Each of the messages  $W_1$ ,  $W_3$ , and  $W_5$  is delivered whenever the direct link between its carrying transmitter and its designated receiver is not

erased. Message  $W_2$  is delivered whenever message  $W_1$  is not transmitted, and message  $W_3$  is not causing interference at  $Y_2$ . Message  $W_4$  is transmitted whenever  $W_5$  is not causing interference at  $Y_4$ , and the transmission of  $W_4$  through  $X_3$  will not disrupt the communication of  $W_3$ . We now explain the modification. If there is an atomic subnetwork consisting of users  $\{2, 3, 4\}$ , then we switch the priority setting within this subnetwork, and messages  $W_2$  and  $W_4$  will be delivered instead of message  $W_3$ . The TDMA optimality of this scheme for each realization of the network follows from [48, Theorem 1]. Now, we note that the average sum DoF for messages  $\{W_1, \dots, W_5\}$  is equal to their sum DoF in the original scheme plus an extra term due to the modification. The average sum DoF for messages  $\{W_1, W_2, W_5\}$  in the original scheme equals  $3\tau_p^{(2)}$ , and the sum of the average sum DoF for messages  $\{W_3, W_4\}$  and the extra term is upper bounded by  $2\tau_p^{(1)}$ . It follows that the average per user DoF is upper bounded by  $\frac{2}{5}\tau_p^{(1)} + \frac{3}{5}\tau_p^{(2)}$ . The proof can be generalized to show that the average TDMA per user DoF for message assignment strategies defined by strings of the form  $S^{(2)}$  with an odd number of ones  $n$ , is upper bounded by  $\frac{n-1}{n+2}\tau_p^{(1)} + \frac{3}{n+2}\tau_p^{(2)}$ .

For message assignment strategies defined by a string of the form  $S^{(2)}$  with an even number of ones  $n$ , it can be shown in a similar fashion as above that the TDMA average per user DoF is upper bounded by  $\frac{n}{n+2}\tau_p^{(1)} + \frac{2}{n+2}\tau_p^{(2)}$ . Also, for strategies defined by a string of the form  $S^{(3)} = (1, 1, \dots, 1, 2, 0)$  with a number of ones  $n$ , the TDMA average per user DoF is the same as that of a strategy defined by a string of the form  $S^{(2)}$  with the same number of ones, and hence, is upper bounded by a convex combination of  $\tau_p^{(1)}$  and  $\tau_p^{(2)}$ . Finally, for strategies defined by a string of the form  $S^{(4)} = (1, 1, \dots, 1, 2, 1, 1, \dots, 1, 0)$  with a number of ones  $n$ , it can be shown in a similar fashion as above that the average per user DoF is upper bounded by  $\frac{n-2}{n+2}\tau_p^{(1)} + \frac{4}{n+2}\tau_p^{(3)}$ .  $\square$

We now characterize the average per user DoF for the cell association problem by proving that TDMA schemes are optimal for any candidate message assignment strategy. In order to prove an information theoretic upper bound on the per user DoF for each network realization, we use Lemma 5 from Appendix A. Recall that for any set of receiver indices  $\mathcal{A} \subseteq [K]$ , we use  $U_{\mathcal{A}}$  as the set of indices of transmitters that exclusively carry the messages for the receivers in  $\mathcal{A}$ .



**Theorem 15.** *The average per user DoF for the cell association problem is given by*

$$\tau_p(M=1) = \tau_p^{(TDMA)} = \max \{ \tau_p^{(1)}, \tau_p^{(2)}, \tau_p^{(3)} \}, \quad (6.6)$$

where  $\tau_p^{(1)}$ ,  $\tau_p^{(2)}$ , and  $\tau_p^{(3)}$  are given in (6.2), (6.3), and (6.4), respectively.

*Proof.* In order to prove the statement, we need to show that  $\tau_p(M=1) \leq \tau_p^{(TDMA)}$ ; we do so by using Lemma 5 to show that for any irreducible message assignment strategy satisfying the cell association constraint, and any network realization, the asymptotic per user DoF is given by that achieved through the optimal TDMA scheme.

Consider message assignment strategies defined by strings having one of the forms  $S^{(1)} = (1)$ ,  $S^{(2)} = (2, 1, 1, \dots, 1, 0)$ , and  $S^{(3)} = (1, 1, \dots, 1, 2, 0)$ . We view each network realization as a series of atomic subnetworks, and show that for each atomic subnetwork, the sum DoF is achieved by the optimal TDMA scheme. For an atomic subnetwork consisting of a number of users  $n$ , we note that  $\lfloor \frac{n+1}{2} \rfloor$  users are active in the optimal TDMA scheme; we now show in this case using Lemma 5 that the sum DoF for users in the subnetwork is bounded by  $\lfloor \frac{n+1}{2} \rfloor$ . Let the users in the atomic subnetwork have the indices  $\{i, i+1, \dots, i+n-1\}$ , then we use Lemma 5 with the set  $\mathcal{A} = \{i+2j : j \in \{0, 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor\}\}$ , except the cases of message assignment strategies defined by strings having one of the forms  $S^{(1)} = (1)$  and  $S^{(3)} = (1, 1, \dots, 1, 2, 0)$  with an even number of ones, where we use the set  $\mathcal{A} = \{i+1+2j : j \in \{0, 1, 2, \dots, \frac{n-2}{2}\}\}$ . We now note that each transmitter that carries a message for a user in the atomic subnetwork and has an index in  $\bar{U}_{\mathcal{A}}$ , is connected to a receiver in  $\mathcal{A}$ , and this receiver is connected to one more transmitter with an index in  $U_{\mathcal{A}}$ , and hence, the missing transmit signals  $X_{\bar{U}_{\mathcal{A}}}$  can be recovered from  $Y_{\mathcal{A}} - Z_{\mathcal{A}}$  and  $X_{U_{\mathcal{A}}}$ . The condition in the statement of Lemma 5 is then satisfied; allowing us to prove that the sum DoF for users in the atomic subnetwork is upper bounded by  $|\mathcal{A}| = \lfloor \frac{n+1}{2} \rfloor$ .

The proof is similar for message assignment strategies defined by strings that have the form  $S^{(4)} = \{1, 1, \dots, 1, 2, 1, 1, \dots, 1, 0\}$ . However, there is a difference in selecting the set  $\mathcal{A}$  for atomic subnetworks consisting of users with indices  $\{i, i+1, \dots, i+x, i+x+1, \dots, i+n-1\}$ , where  $1 \leq x \leq n-2$ , and messages  $W_{i+x}$  and  $W_{i+x+1}$  are both available at transmitter  $i+x$ . In this case, we apply Lemma 5 with the set  $\mathcal{A}$  defined as above, but including

indices  $\{i + x, i + x + 1\}$  and excluding indices  $\{i + x - 1, i + x + 2\}$ . It can be seen that the condition in Lemma 5 will be satisfied in this case, and the proved upper bound on the sum DoF for each atomic subnetwork, is achievable through TDMA.  $\square$

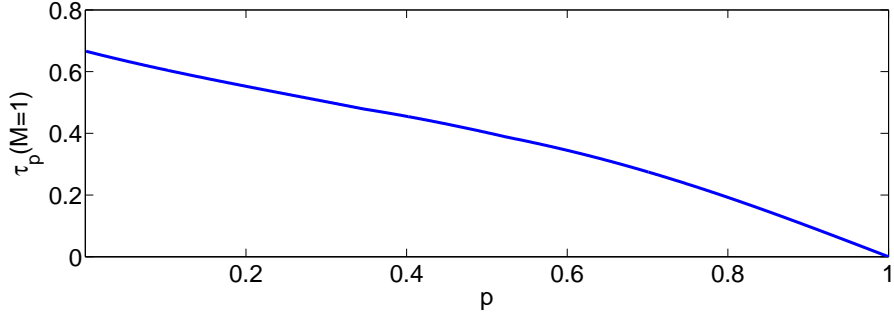


Figure 6.3: The average per user DoF for the cell association problem.

In Figure 6.3, we plot  $\tau_p(M = 1)$  at each value of  $p$ . The result of Theorem 15 implies that the message assignment strategies considered in Lemmas 8, 9, 10 are optimal at high, low, and middle values of the erasure probability  $p$ , respectively. We note that in densely connected networks at a low probability of erasure, the *interference-aware* message assignment strategy in Figure 6.1b is optimal; through this assignment, the maximum number of interference free communication links can be created for the case of no erasures. On the other hand, the linear nature of the channel connectivity does not affect the choice of optimal message assignment at high probability of erasure. As the effect of interference diminishes at high probability of erasure, assigning each message to a unique transmitter, as in the strategy in Figure 6.1a, becomes the only criterion of optimality. At middle values of  $p$ , the message assignment strategy in Figure 6.1c is optimal; in this assignment, the network is split into four user subnetworks. In the first subnetwork, the assignment is optimal as the maximum number of interference free communication links can be created for the two events where there is an atomic subnetwork consisting of users  $\{1, 2, 3\}$  or users  $\{2, 3, 4\}$ .

### 6.3 Coordinated Multi-Point Transmission

We have shown that there is no message assignment strategy for the cell association problem that is optimal for all values of  $p$ . We show in this section that this statement is true even for the case where each message can be available at more than one transmitter ( $M > 1$ ). Recall that for a given value of  $M$ , we say that a message assignment strategy is universally optimal if it can be used to achieve  $\tau_p(M)$  for all values of  $p$ .

**Theorem 16.** *For any value of the cooperation constraint  $M \in \mathbf{Z}^+$ , there does not exist a universally optimal message assignment strategy.*

*Proof.* The proof follows from Theorem 15 for the case where  $M = 1$ . We show that for any value of  $M > 1$ , any message assignment strategy that enables the achievability of  $\tau_p(M)$  at high probabilities of erasure, is not optimal for the case of no erasures, i.e., cannot be used to achieve  $\tau_p(M)$  for  $p = 0$ . For any message assignment strategy, consider the value of  $\lim_{p \rightarrow 1} \frac{\tau_p(M)}{1-p}$  and note this value equals the average number of transmitters in a transmit set that can be connected to the designated receiver. More precisely,

$$\lim_{p \rightarrow 1} \frac{\tau_p(M)}{1-p} = \frac{\sum_{i=1}^K |\mathcal{T}_i \cap \{i-1, i\}|}{K}, \quad (6.7)$$

where  $\mathcal{T}_i$  in (6.7) corresponds to an optimal message assignment strategy at high probabilities of erasure. It follows that there exists a value  $0 < \bar{p} < 1$  such that for any message assignment strategy that enables the achievability of  $\tau_p(M)$  for  $p \geq \bar{p}$ , almost all messages are assigned to the two transmitters that can be connected to the designated receiver, i.e., if we let  $S_K = \{i : \mathcal{T}_{i,K} = \{i-1, i\}\}$ , then  $\lim_{K \rightarrow \infty} \frac{|S_K|}{K} = 1$ .

We recall from Section 4.5.2 that for the case of no erasures, the average per user DoF equals  $\frac{2M}{2M+1}$ . We also note that following the same footsteps as in the proof of Theorem 9 in Section 4.5.2, we can show that for any message assignment strategy such that  $\lim_{K \rightarrow \infty} \frac{|S_K|}{K} = 1$ , the per user DoF for the case of no erasures is upper bounded by  $\frac{2M-2}{2M-1}$ ; we do so by using Lemma 5 in Appendix A for each  $K$ -user channel with the set  $\mathcal{A}$  defined such that the complement set  $\bar{\mathcal{A}} = \{i : i \in [K], i = (2M-1)(j-1) + M, j \in \mathbf{Z}^+\}$ .  $\square$

The condition of optimality identified in the proof of Theorem 16 for message assignment strategies at high probabilities of erasure suggest a new

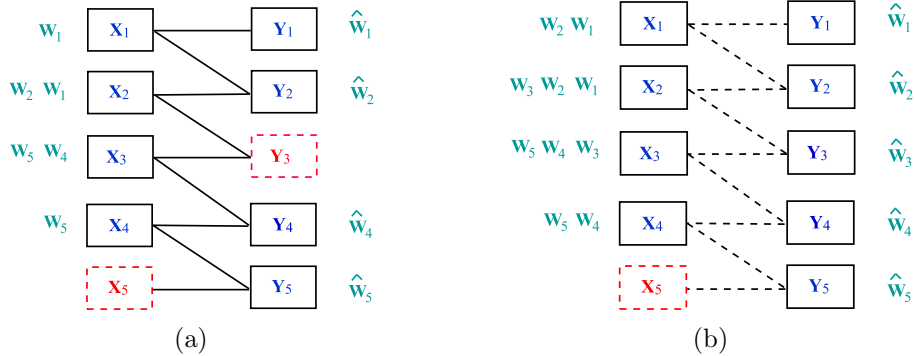


Figure 6.4: The message assignment in (a) is optimal for a linear network with no erasures ( $p = 0$ ). We extend this message assignment in (b) to consider non-zero erasure probabilities. In both figures, the red dashed boxes correspond to inactive signals.

role for cooperation in dynamic interference networks. The availability of a message at more than one transmitter may not only be used to cancel its interference at other receivers, but to increase the chances of connecting the message to its designated receiver. This new role leads to three effects at high erasure probability. The achieved DoF in the considered linear interference network becomes larger than that of  $K$  parallel channels, in particular,  $\lim_{p \rightarrow 1} \frac{\tau_p(M > 1)}{1-p} = 2$ . Secondly, as the effect of interference diminishes at high probabilities of erasures, all messages can simply be assigned to the two transmitters that may be connected to their designated receiver, and a simple interference avoidance scheme can be used in each network realization, as we show later in the scheme of Theorem 18. It follows that channel state information is no longer needed at transmitters, and only information about the slow changes in the network topology is needed to achieve the optimal average DoF. Finally, unlike the optimal scheme of Theorem 6 in Section 4.3 for the case of no erasures, where some transmitters are always inactive, achieving the optimal DoF at high probabilities of erasure requires all transmitters to be used in at least one network realization.

We now restrict our attention to the case where  $M = 2$ . Here, each message can be available at two transmitters, and transmitted jointly by both of them. We study two message assignment strategies that are optimal in the limits of  $p \rightarrow 0$  and  $p \rightarrow 1$ , and derive inner bounds on the average per user DoF  $\tau_p(M = 2)$  based on the considered strategies. In Chapter 4, the message assignment of Figure 6.4a was shown to be DoF optimal for the case

of no erasures ( $p = 0$ ). The network is split into subnetworks, each with five consecutive users. The last transmitter of each subnetwork is deactivated to eliminate inter-subnetwork interference. In the first subnetwork, message  $W_3$  is not transmitted, and each other message is received without interference at its designated receiver. Note that the transmit beams for messages  $W_1$  and  $W_5$  contributing to the transmit signals  $X_2$  and  $X_5$ , respectively, are designed to cancel the interference at receivers  $Y_2$  and  $Y_4$ , respectively. An analog scheme is used in each following subnetwork. The value of  $\tau_p(M = 2)$  is thus  $\frac{4}{5}$  for the case where  $p = 0$ . In order to prove the following result, we extend the message assignment of Figure 6.4a to consider the possible presence of block erasures.

**Theorem 17.** *For  $M = 2$ , the following average per user DoF is achievable,*

$$\tau_p(M = 2) \geq \frac{2}{5} (1 - p) (2 + A.p), \quad (6.8)$$

where

$$A = p + 1 - ((1 - p)^2 (1 - p(1 - p))) - \frac{1}{2}p(1 - p), \quad (6.9)$$

and is asymptotically optimal as  $p \rightarrow 0$ .

*Proof.* We know from Theorem 9 in Section 4.5.2 that  $\lim_{p \rightarrow 0} \tau_p(2) = \frac{4}{5}$ , and hence, it suffices to show that the inner bound in (6.8) is valid. For each  $i \in [K]$ , message  $W_i$  is assigned as follows,

$$\mathcal{T}_i = \begin{cases} \{i, i + 1\}, & \text{if } i \equiv 1 \pmod{5} \\ \{i - 1, i - 2\}, & \text{if } i \equiv 0 \pmod{5} \\ \{i - 1, i\}, & \text{otherwise.} \end{cases}$$

We illustrate this message assignment in Figure 6.4b. We note that the transmit signals  $\{X_i : i \equiv 0 \pmod{5}\}$  are inactive, and hence, we split the network into five user subnetworks with no interference between successive subnetworks. We explain the transmission scheme in the first subnetwork and note that a similar scheme applies to each following subnetwork. In the proposed transmission scheme, any receiver is either inactive or receives its desired message without interference, and any transmitter will not transmit more than one message for any network realization. It follows that one DoF is achieved for each message that is transmitted.

Messages  $W_1$ ,  $W_2$ ,  $W_4$ , and  $W_5$  are transmitted through  $X_1$ ,  $X_2$ ,  $X_3$ , and  $X_4$ , respectively, whenever the coefficients  $H_{1,1} \neq 0$ ,  $H_{2,2} \neq 0$ ,  $H_{4,3} \neq 0$ , and  $H_{5,4} \neq 0$ , respectively. Note that the transmit beam for message  $W_1$  contributing to  $X_2$  can be designed to cancel its interference at  $Y_2$ . Similarly, the interference caused by  $W_5$  at  $Y_4$  can be cancelled through  $X_3$ . It follows that  $(1-p)$  DoF is achieved for each of  $\{W_1, W_2, W_4, W_5\}$ , and hence,  $\tau_p(2) \geq \frac{4}{5}(1-p)$ . Also, message  $W_2$  is transmitted through  $X_1$  if it cannot be transmitted through  $X_2$  and message  $W_1$  is not transmitted through  $X_1$ . More precisely, message  $W_2$  is transmitted through  $X_1$  if  $H_{2,2} = 0$  and  $H_{2,1} \neq 0$  and  $H_{1,1} = 0$ , thereby achieving an extra  $p^2(1-p)$  DoF. Similarly, message  $W_4$  can be transmitted through  $X_4$  if  $H_{4,3} = 0$  and  $H_{4,4} \neq 0$  and  $H_{5,4} = 0$ . It follows that

$$\tau_p(2) \geq \frac{4}{5}(1-p) + \frac{2}{5}p^2(1-p). \quad (6.10)$$

Finally, message  $W_3$  will be transmitted through  $X_3$  if message  $W_4$  is not transmitted through  $X_3$ , and message  $W_2$  is not causing interference at  $Y_3$ . Message  $W_4$  is not transmitted through  $X_3$  whenever the coefficient  $H_{4,3} = 0$ , and message  $W_2$  does not cause interference at  $Y_3$  whenever the coefficient  $H_{2,2} = 0$  or the coefficient  $H_{3,2} = 0$  or  $W_2$  can be transmitted through  $X_1$ . More precisely, message  $W_3$  is transmitted through  $X_3$  if and only if all the following is true:

- $H_{3,3} \neq 0$ , and  $H_{4,3} = 0$ .
- $H_{2,2} = 0$ , or  $H_{3,2} = 0$ , or it is the case that  $H_{1,1} = 0$  and  $H_{2,1} \neq 0$ .

It follows that  $f(p)$  DoF is achieved for message  $W_3$ , where

$$f(p) = p(1-p) \left(1 - \left((1-p)^2 (1-p(1-p))\right)\right). \quad (6.11)$$

Similarly,  $W_3$  can be transmitted through  $X_2$  if and only if message  $W_2$  is not transmitted through  $X_2$  and message  $W_4$  is either not transmitted or can be transmitted without causing interference at  $Y_3$ , i.e., if and only if all the following is true:

- $H_{3,2} \neq 0$ , and  $H_{2,2} = 0$ .
- $H_{4,3} = 0$ , or  $H_{3,3} = 0$ , or it is the case that  $H_{5,4} = 0$  and  $H_{4,4} \neq 0$ .

The above conditions are satisfied with probability  $f(p)$ . Since we have counted twice the event that  $H_{3,3} \neq 0$  and  $H_{4,3} = 0$  and  $H_{3,2} \neq 0$  and  $H_{2,2} = 0$ , it follows that  $2f(p) - p^2(1-p)^2$  DoF is achieved for  $W_3$ . Summing the DoF achieved for other messages in (6.10), we conclude that

$$\tau_p(2) \geq \frac{4}{5}(1-p) + \frac{2}{5}p^2(1-p) + \frac{1}{5}(2f(p) - p^2(1-p)^2), \quad (6.12)$$

which is the same inequality as in (6.8).  $\square$

Although the scheme of Theorem 17 is optimal for the case of no erasures ( $p = 0$ ), we know from Theorem 16 that better schemes exist at high erasure probabilities. Since in each five-user subnet in the scheme of Theorem 17, only three users have their messages assigned to the two transmitters that can be connected to their receivers, and two users have only one of these transmitters carrying their messages, we get the asymptotic limit of  $\frac{8}{5}$  for the achieved average per user DoF normalized by  $(1-p)$  as  $p \rightarrow 1$ . This leads us to consider an alternative message assignment where the two transmitters carrying each message  $i$  are the two transmitters  $\{i-1, i\}$  that can be connected to its designated receiver. Such assignment would lead the ratio  $\frac{\tau_p(2)}{1-p} \rightarrow 2$  as  $p \rightarrow 1$ . In the following theorem, we analyze a transmission scheme based on this assignment.

**Theorem 18.** *For  $M = 2$ , the following average per user DoF is achievable,*

$$\tau_p(M = 2) \geq \frac{1}{3}(1-p)(1 + (1-p)^3 + B.p), \quad (6.13)$$

where

$$\begin{aligned} B &= 3 + (1 + (1-p)^3)(1 - (1-p)^2 + p(1-p)^3) \\ &+ p(1 + (1-p)^2), \end{aligned} \quad (6.14)$$

and

$$\lim_{p \rightarrow 1} \frac{\tau_p(2)}{1-p} = 2. \quad (6.15)$$

*Proof.* For any message assignment, no message can be transmitted if the links from both transmitters carrying the message to its designated receiver are absent, and hence, the average DoF achieved for each message is at most  $1-p^2$ . It follows that  $\lim_{p \rightarrow 1} \frac{\tau_p(2)}{1-p} \leq \lim_{p \rightarrow 1} \frac{(1-p)(1+p)}{1-p} = 2$ . We then need only

to prove that the inner bound in (6.13) is valid. In the achieving scheme, each message is assigned to the two transmitters that may be connected to its designated receiver, i.e.,  $\mathcal{T}_i = \{i-1, i\}, \forall i \in [K]$ . Also, in each network realization, each transmitter will transmit at most one message and any transmitted message will be received at its designated receiver without interference. It follows that 1 DoF is achieved for any message that is transmitted, and hence, the probability of transmission is the same as the average DoF achieved for each message.

Each message  $W_i$  such that  $i \equiv 0 \pmod{3}$  is transmitted through  $X_{i-1}$  whenever  $H_{i,i-1} \neq 0$ , and is transmitted through  $X_i$  whenever  $H_{i,i-1} = 0$  and  $H_{i,i} \neq 0$ . It follows that  $n_0$  DoF is achieved for each of these messages, where

$$n_0 = (1-p)(1+p). \quad (6.16)$$

We now consider messages  $W_i$  such that  $i \equiv 1 \pmod{3}$ . Any such message is transmitted through  $X_{i-1}$  whenever  $H_{i,i-1} \neq 0$  and  $H_{i-1,i-1} = 0$ . We note that whenever the channel coefficient  $H_{i-1,i-1} \neq 0$ , message  $W_i$  cannot be transmitted through  $X_{i-1}$  as the transmission of  $W_i$  through  $X_{i-1}$  in this case will prevent  $W_{i-1}$  from being transmitted due to either interference at  $Y_{i-1}$  or sharing the transmitter  $X_{i-1}$ . It follows that  $n_1^{(1)} = p(1-p)$  DoF is achieved for transmission of  $W_i$  through  $X_{i-1}$ . Also, message  $W_i$  is transmitted through  $X_i$  whenever it is not transmitted through  $X_{i-1}$  and  $H_{i,i} \neq 0$  and either  $H_{i,i-1} = 0$  or message  $W_{i-1}$  is transmitted through  $X_{i-2}$ . More precisely,  $W_i$  is transmitted through  $X_i$  whenever all the following is true:  $H_{i,i} \neq 0$ , and either  $H_{i,i-1} = 0$  or it is the case that  $H_{i,i-1} \neq 0$  and  $H_{i-1,i-1} \neq 0$  and  $H_{i-1,i-2} \neq 0$ . It follows that  $n_1^{(2)} = p(1-p) + (1-p)^4$  is achieved for transmission of  $W_i$  through  $X_i$ , and hence,  $n_1$  DoF is achieved for each message  $W_i$  such that  $i \equiv 1 \pmod{3}$ , where

$$n_1 = n_1^{(1)} + n_1^{(2)} = 2p(1-p) + (1-p)^4. \quad (6.17)$$

We now consider messages  $W_i$  such that  $i \equiv 2 \pmod{3}$ . Any such message is transmitted through  $X_{i-1}$  whenever all the following is true:

- $H_{i,i-1} \neq 0$ .
- Either  $H_{i-1,i-1} = 0$ , or  $W_{i-1}$  is not transmitted.



- $W_{i+1}$  is not causing interference at  $Y_i$ .

The first condition is satisfied with probability  $(1 - p)$ . In order to compute the probability of satisfying the second condition, we note that  $W_{i-1}$  is not transmitted for the case when  $H_{i-1,i-1} \neq 0$  only if  $W_{i-2}$  is transmitted through  $X_{i-2}$  and causing interference at  $Y_{i-1}$ , i.e., only if  $H_{i-2,i-3} = 0$  and  $H_{i-2,i-2} \neq 0$  and  $H_{i-1,i-2} \neq 0$ . It follows that the second condition is satisfied with probability  $p + p(1 - p)^3$ . The third condition is not satisfied only if  $H_{i,i} \neq 0$  and  $H_{i+1,i} \neq 0$ , and hence, will be satisfied with probability at least  $1 - (1 - p)^2$ . Moreover, even if  $H_{i,i} \neq 0$  and  $H_{i+1,i} \neq 0$ , the third condition can be satisfied if message  $W_{i+1}$  can be transmitted through  $X_{i+1}$  without causing interference at  $Y_{i+2}$ , i.e., if  $H_{i+1,i+1} \neq 0$  and  $H_{i+2,i+1} = 0$ . It follows that the third condition will be satisfied with probability  $1 - (1 - p)^2 + p(1 - p)^3$ , and  $n_2^{(1)}$  DoF is achieved by transmission of  $W_i$  through  $X_{i-1}$ , where

$$n_2^{(1)} = p(1 - p) \left( 1 + (1 - p)^3 \right) \left( 1 - (1 - p)^2 + p(1 - p)^3 \right). \quad (6.18)$$

Message  $W_i$  such that  $i \equiv 2 \pmod{3}$  is transmitted through  $X_i$  whenever  $H_{i,i} \neq 0$ , and  $H_{i+1,i} = 0$ , and either  $H_{i,i-1} = 0$  or  $W_{i-1}$  is transmitted through  $X_{i-2}$ . It follows that  $n_2^{(2)}$  DoF is achieved by transmission of  $W_i$  through  $X_i$ , where

$$n_2^{(2)} = p(1 - p) \left( p + d_1^{(1)}(1 - p) \right) \quad (6.19)$$

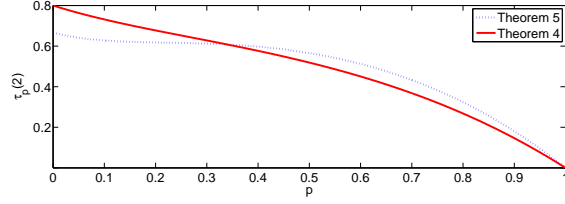
$$= p^2(1 - p) \left( 1 + (1 - p)^2 \right), \quad (6.20)$$

and hence,  $n_2 = n_2^{(1)} + n_2^{(2)}$  DoF is achieved for each message  $W_i$  such that  $i \equiv 2 \pmod{3}$ . We finally get

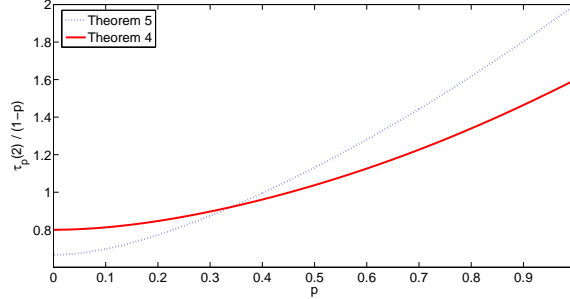
$$\tau_p(2) \geq \frac{n_0 + n_1 + n_2}{3}, \quad (6.21)$$

which is the same inequality as in (6.13).  $\square$

We plot the inner bounds of (6.8) and (6.13) in Figure 6.5. We note that below a threshold erasure probability  $p \approx 0.34$ , the scheme of Theorem 17 is better, and hence is proposed to be used in this case. For higher probabilities of erasure, the scheme of Theorem 18 should be used. It is worth mentioning that we also studied a scheme based on the message assignment  $\mathcal{T}_i = \{i, i + 1\}, \forall i \in [K - 1]$ , that is introduced in [19]. However, we did not include



(a)



(b)

Figure 6.5: Achieved inner bounds in Theorems 4 and 5. In (a) we plot the achieved per user DoF. In (b), we plot the achieved per user DoF normalized by  $(1 - p)$ .

it here as it does not increase the maximum of the bounds derived in (6.8) and (6.13) at any value of  $p$ .

Although the considered channel model allows for using the interference alignment scheme of [11] over multiple channel realizations (symbol extensions), all the proposed schemes require only coding over one channel realization because of the sparsity of the linear network. Finally, it is worth mentioning that while we have only focused on maximizing the sum rate of communication, it is natural to study the diversity-multiplexing tradeoff [56] for the channel model considered in this chapter.

# CHAPTER 7

## CONCLUSIONS AND FUTURE WORK

We studied the DoF gain achieved through CoMP transmission. In particular, it was of interest to know whether the achievable gain scales linearly with  $K$  as it goes to infinity, under a cooperation constraint that limits the number of transmitters at which any message can be available by a cooperation order  $M$ . In Chapter 3, we showed that the answer is negative for the fully connected channel where message assignment strategies satisfy the local cooperation constraint, as well as all possible message assignments for the case where  $M = 2$ . The problem is still open for fully connected channels and values of  $M \geq 3$ .

For locally connected channels where each transmitter is connected to the receiver carrying the same index as well as  $L$  neighboring receivers, we showed in Chapter 4 that the asymptotic per user DoF is lower bounded by  $\max\{\frac{1}{2}, \frac{2M}{2M+L}\}$ . The achieving coding scheme is simple as it relies only on zero-forcing transmit beam-forming. We showed that this lower bound is tight for the case where  $L = 1$ . In particular, the characterized asymptotic per user DoF for that case is  $\frac{2M}{2M+1}$ , and is higher than previous results in [19], and [47].

We also revealed insights on the optimal way of assigning messages to transmitters under a cooperation order constraint. For instance, we considered a local cooperation constraint, where each message can only be available at a neighborhood of transmitters whose size does not scale linearly with the number of users. While we showed that local cooperation does not achieve a scalable DoF gain for the fully connected channel, we also showed that local cooperation is optimal for locally connected channels. Furthermore, we have shed light on the intimate relation between the selection of message assignments and the design of transmit beams. We have shown that assigning messages to successive transmitters is beneficial for zero-forcing transmit beam-forming in locally connected channels as it minimizes the number of

receivers at which each message causes undesired interference. However, the same message assignment strategy can be an impediment to other techniques such as asymptotic interference alignment, because the overlap of sets of messages carried by transmit antennas is large for this assignment of messages.

In Chapter 5, we considered a backhaul load constraint that limits the average transmit set size across the users. We characterized the asymptotic per user DoF in linear interference channels, and showed that the backhaul constraint is satisfied in the optimal scheme by assigning some messages to more than  $B$  transmitters and others to fewer than  $B$  transmitters, where  $B$  is the average transmit set size. We showed that local cooperation is sufficient to achieve the DoF in large linear interference networks. We also noted that the characterized asymptotic per user DoF for linear interference networks can be achieved by using a convex combination of the coding schemes that are identified as optimal under the cooperation order constraint that limits the maximum size of a transmit set, as opposed to the average as we considered in Chapter 5. We then illustrated that these results hold in more general networks of practical relevance to achieve rate gains and simplify existing coding schemes. In particular, we showed that CoMP transmission can lead to significant DoF gains without incurring additional load on the backhaul link.

In Chapter 6, we considered the problem of assigning messages to transmitters in a linear interference network with link erasure probability  $p$ , under the cooperation order constraint that limits the number of transmitters  $M$  at which each message can be available. For the case where  $M = 1$ , we identified the optimal message assignment strategies at different values of  $p$ , and characterized the average per user DoF. For general values of  $M \geq 1$ , we proved that there is no message assignment strategy that is optimal for all values of  $p$ . We finally introduced message assignment strategies for the case where  $M = 2$ , and derived inner bounds on the average per user DoF that are asymptotically optimal as  $p \rightarrow 0$  and as  $p \rightarrow 1$ .

In [41], a CoMP reception model was studied, where base station receivers can share decoded messages in cellular uplink. We are considering an extension of the model introduced in [41] for future work, where both choices for the association of transmitters to receivers and the design of backhaul links can be optimized based on the channel connectivity. It is not difficult to find parallel schemes to the ZF transmit beam-forming coding schemes presented

in Chapters 4, 5, and 6 to achieve the same DoF in the CoMP reception model. These parallel schemes enjoy a practical advantage as there is no need for synchronization between transmitters or making the channel state information available at transmitters as in the presented CoMP transmission schemes. We believe that this line of work can lead to useful insights for the design of practical schemes for exploiting backhaul links to manage interference in cellular networks.

# APPENDIX A

## DEGREES OF FREEDOM UPPER BOUND

In order to characterize the DoF of any of the considered channel models under a constraint on the transmit set sizes, we need to consider all possible strategies for message assignments satisfying the considered cooperation constraint. In this appendix, we provide in Lemma 5 the key information theoretic argument that we use to upper bound the maximum achievable DoF for each such assignment, thereby, reducing the problem of finding a DoF upper bound for each considered channel model to a combinatorial problem.

Recall that for any set  $\mathcal{A} \subseteq [K]$ ,  $U_{\mathcal{A}}$  is the set of indices of transmitters that exclusively carry the messages for the receivers in  $\mathcal{A}$ , then we have the following lemma for any  $K$ -user Gaussian interference channel with a DoF number of  $\eta$ .

**Lemma 5.** *If there exists a set  $\mathcal{A} \subseteq [K]$ , a function  $f_1$ , and a function  $f_2$  whose definition does not depend on the transmit power constraint  $P$ , and  $f_1(Y_{\mathcal{A}}, X_{U_{\mathcal{A}}}) = X_{\bar{U}_{\mathcal{A}}} + f_2(Z_{\mathcal{A}})$ , then  $\eta \leq |\mathcal{A}|$ .*

*Proof.* We first provide a sketch of the proof. Recall that  $Y_{\mathcal{A}} = \{Y_i, i \in \mathcal{A}\}$ , and  $W_{\mathcal{A}} = \{W_i, i \in \mathcal{A}\}$ , and note that  $X_{U_{\mathcal{A}}}$  is the set of transmit signals that do not carry messages outside  $W_{\mathcal{A}}$ . Fix a reliable communication scheme for the considered  $K$ -user channel, and assume that there is only one centralized decoder that has access to the received signals  $Y_{\mathcal{A}}$ . We show that using the centralized decoder, the only uncertainty in recovering all the messages  $W_{[K]}$  is due to the Gaussian noise signals. In this case, the sum DoF is bounded by  $|\mathcal{A}|$ , as it is the number of received signals used for decoding.

Using  $Y_{\mathcal{A}}$ , the messages  $W_{\mathcal{A}}$  can be recovered reliably, and hence, the signals  $X_{U_{\mathcal{A}}}$  can be reconstructed. Using  $Y_{\mathcal{A}}$  and  $X_{U_{\mathcal{A}}}$ , the remaining transmit signals can be approximately reconstructed using the function  $f_1$  of the hypothesis. Finally, using all transmit signals, the received signals  $Y_{\bar{\mathcal{A}}}$  can be approximately reconstructed, and the messages  $W_{\bar{\mathcal{A}}}$  can then be recovered.

We now provide the proof. In any reliable  $n$ -block coding scheme,

$$\mathsf{H}(W_i|Y_i^n) \leq n\epsilon, \forall i \in [K].$$

Therefore,

$$\mathsf{H}(W_{\mathcal{A}}|Y_{\mathcal{A}}^n) \leq \sum_{i \in \mathcal{A}} \mathsf{H}(W_i|Y_i^n) \leq n|\mathcal{A}|\epsilon.$$

Now, the sum  $\sum_{i \in [K]} R_i = \sum_{i \in \bar{\mathcal{A}}} R_i + \sum_{i \in \mathcal{A}} R_i$  can be bounded as

$$\begin{aligned} n \left( \sum_{i \in \bar{\mathcal{A}}} R_i + \sum_{i \in \mathcal{A}} R_i \right) &= \mathsf{H}(W_{\bar{\mathcal{A}}}) + \mathsf{H}(W_{\mathcal{A}}) \\ &\leq I(W_{\bar{\mathcal{A}}}; Y_{\bar{\mathcal{A}}}^n) + I(W_{\mathcal{A}}; Y_{\mathcal{A}}^n) \\ &\quad + nK\epsilon, \end{aligned} \tag{A.1}$$

where  $\epsilon$  can be made arbitrarily small, by choosing  $n$  large enough. The two terms on the right-hand side of (A.1) can be bounded as

$$\begin{aligned} I(W_{\mathcal{A}}; Y_{\mathcal{A}}^n) &= \mathsf{h}(Y_{\mathcal{A}}^n) - \mathsf{h}(Y_{\mathcal{A}}^n|W_{\mathcal{A}}) \\ &\leq \sum_{i \in \mathcal{A}} \sum_{t=1}^n \mathsf{h}(Y_i(t)) - \mathsf{h}(Y_{\mathcal{A}}^n|W_{\mathcal{A}}) \\ &= |\mathcal{A}|n \log P + n(o(\log P)) - \mathsf{h}(Y_{\mathcal{A}}^n|W_{\mathcal{A}}), \end{aligned}$$

$$\begin{aligned} I(W_{\bar{\mathcal{A}}}; Y_{\bar{\mathcal{A}}}^n) &\leq I(W_{\bar{\mathcal{A}}}; Y_{\bar{\mathcal{A}}}^n, Y_{\mathcal{A}}^n, W_{\mathcal{A}}) \\ &= I(W_{\bar{\mathcal{A}}}; Y_{\bar{\mathcal{A}}}^n|W_{\mathcal{A}}) + I(W_{\bar{\mathcal{A}}}; Y_{\bar{\mathcal{A}}}^n|W_{\mathcal{A}}, Y_{\mathcal{A}}^n) \\ &\leq \mathsf{h}(Y_{\bar{\mathcal{A}}}^n|W_{\mathcal{A}}) - \mathsf{h}(Z_{\bar{\mathcal{A}}}^n) + \mathsf{h}(Y_{\bar{\mathcal{A}}}^n|W_{\mathcal{A}}, Y_{\mathcal{A}}^n) - \mathsf{h}(Z_{\bar{\mathcal{A}}}^n). \end{aligned}$$

Now, we have

$$\begin{aligned} I(W_{\mathcal{A}}; Y_{\mathcal{A}}^n) + I(W_{\bar{\mathcal{A}}}; Y_{\bar{\mathcal{A}}}^n) &\leq |\mathcal{A}|n \log P + \mathsf{h}(Y_{\bar{\mathcal{A}}}^n|W_{\mathcal{A}}, Y_{\mathcal{A}}^n) \\ &\quad + n(o(\log P)). \end{aligned}$$

Therefore, if we show that

$$\mathsf{h}(Y_{\bar{\mathcal{A}}}^n|W_{\mathcal{A}}, Y_{\mathcal{A}}^n) = n(o(\log P)),$$

then from (A.1), we have the required outer bound. Since  $W_{\mathcal{A}}$  contains all

the messages carried by transmitters with indices  $U_{\mathcal{A}}$ , they determine  $X_{U_{\mathcal{A}}}^n$ . Therefore,

$$\begin{aligned}
\mathbf{h}(Y_{\bar{\mathcal{A}}}^n | W_{\mathcal{A}}, Y_{\mathcal{A}}^n) &= \mathbf{h}(Y_{\bar{\mathcal{A}}}^n | W_{\mathcal{A}}, Y_{\mathcal{A}}^n, X_{U_{\mathcal{A}}}^n) \\
&\leq \mathbf{h}(Y_{\bar{\mathcal{A}}}^n | Y_{\mathcal{A}}^n, X_{U_{\mathcal{A}}}^n) \\
&\leq \sum_{t=1}^n \mathbf{h}(Y_{\bar{\mathcal{A}}}(t) | Y_{\mathcal{A}}(t), X_{U_{\mathcal{A}}}(t)) \\
&\stackrel{(a)}{\leq} \sum_{t=1}^n \mathbf{h}(Y_{\bar{\mathcal{A}}}(t) | X_{U_{\mathcal{A}}}(t), X_{\bar{U}_{\mathcal{A}}}(t) + f_2(Z_{\mathcal{A}}(t))) \\
&\stackrel{(b)}{\leq} n(o(\log P)),
\end{aligned}$$

where (a) follows from the existence of the function  $f_1$  such that  $f_1(Y_{\mathcal{A}}, X_{U_{\mathcal{A}}}) = X_{\bar{U}_{\mathcal{A}}} + f_2(Z_{\mathcal{A}})$ . Recall that for  $\mathcal{S}_1 \subseteq [K], \mathcal{S}_2 \subseteq [K]$ ,  $\mathbf{H}_{\mathcal{S}_1, \mathcal{S}_2}$  denotes the  $|\mathcal{S}_1| \times |\mathcal{S}_2|$  matrix of channel coefficients between  $X_{\mathcal{S}_2}$  and  $Y_{\mathcal{S}_1}$ , then (b) follows as,

$$\begin{aligned}
Y_{\bar{\mathcal{A}}} &= \mathbf{H}_{\bar{\mathcal{A}}, U_{\mathcal{A}}} X_{U_{\mathcal{A}}} + \mathbf{H}_{\bar{\mathcal{A}}, \bar{U}_{\mathcal{A}}} X_{\bar{U}_{\mathcal{A}}} + Z_{\bar{\mathcal{A}}} \\
&= \mathbf{H}_{\bar{\mathcal{A}}, U_{\mathcal{A}}} X_{U_{\mathcal{A}}} + \mathbf{H}_{\bar{\mathcal{A}}, \bar{U}_{\mathcal{A}}} (X_{\bar{U}_{\mathcal{A}}} + f_2(Z_{\mathcal{A}})) + Z_{\bar{\mathcal{A}}} - \mathbf{H}_{\bar{\mathcal{A}}, \bar{U}_{\mathcal{A}}} f_2(Z_{\mathcal{A}}),
\end{aligned}$$

and hence,

$$\begin{aligned}
\mathbf{h}(Y_{\bar{\mathcal{A}}} | X_{U_{\mathcal{A}}}, X_{\bar{U}_{\mathcal{A}}} + f_2(Z_{\mathcal{A}})) &\leq \mathbf{h}(Y_{\bar{\mathcal{A}}} | \mathbf{H}_{\bar{\mathcal{A}}, U_{\mathcal{A}}} X_{U_{\mathcal{A}}} + \mathbf{H}_{\bar{\mathcal{A}}, \bar{U}_{\mathcal{A}}} (X_{\bar{U}_{\mathcal{A}}} + f_2(Z_{\mathcal{A}}))) \\
&\leq \mathbf{h}(Z_{\bar{\mathcal{A}}} - \mathbf{H}_{\bar{\mathcal{A}}, \bar{U}_{\mathcal{A}}} f_2(Z_{\mathcal{A}})) \\
&= o(\log P).
\end{aligned}$$

□



# APPENDIX B

## ALGEBRAIC GEOMETRY

In this appendix, we present results in algebraic geometry that are essential in proving the achievability results in Chapter 3 for the fully connected channel. We start by recalling some basic terminology in algebraic geometry. We refer the reader to the book [57] for an excellent introduction.

### B.1 Varieties and Ideals

Let  $\mathbb{C}[t_1, t_2, \dots, t_n]$  and  $\mathbb{C}(t_1, t_2, \dots, t_n)$  denote the set of multivariate polynomials and rational functions, respectively, in the variables  $t_1, t_2, \dots, t_n$ . For any polynomials  $f_1, f_2, \dots, f_m \in \mathbb{C}[t_1, t_2, \dots, t_n]$ , the *affine variety* generated by  $f_1, f_2, \dots, f_m$  is defined as set of points at which the polynomials vanish:

$$V(\mathbf{f}) = \{\mathbf{t} \in \mathbb{C}^n : \mathbf{f}(\mathbf{t}) = \mathbf{0}\}.$$

Any subset  $I \subseteq \mathbb{C}[t_1, t_2, \dots, t_n]$  is called an *ideal* if it satisfies the following three properties:

- $0 \in I$ .
- If  $f_1, f_2 \in I$ , then  $f_1 + f_2 \in I$ .
- If  $f_1 \in I$  and  $f_2 \in \mathbb{C}[t_1, t_2, \dots, t_n]$ , then  $f_1 f_2 \in I$ .

For any set  $\mathcal{A} \subseteq \mathbb{C}^n$ , the ideal generated by  $\mathcal{A}$  is defined as

$$I(\mathcal{A}) = \{f \in \mathbb{C}[t_1, t_2, \dots, t_n] : f(\mathbf{t}) = 0 \forall \mathbf{t} \in \mathcal{A}\}.$$

For any ideal  $I$ , the affine variety generated by  $I$  is defined as

$$V(I) = \{\mathbf{t} \in \mathbb{C}^n : f(\mathbf{t}) = 0 \forall f \in I\}.$$

The *Zariski topology* on the affine space  $\mathbb{C}^n$  is obtained by taking the affine varieties as closed sets. For any set  $\mathcal{A} \subseteq \mathbb{C}^n$ , the Zariski closure  $\bar{\mathcal{A}}$  is defined as

$$\bar{\mathcal{A}} = V(I(\mathcal{A})).$$

A set  $\mathcal{A} \subseteq \mathbb{C}^n$  is said to be *constructible* if it is a finite union of locally closed sets of the form  $U \cap Z$  with  $U$  closed and  $Z$  open. If  $\mathcal{A} \subseteq \mathbb{C}^n$  is constructible and  $\bar{\mathcal{A}} = \mathbb{C}^n$ , then  $\mathcal{A}$  must be dense in  $\mathbb{C}^n$ , i.e.,  $\mathcal{A}^c \subseteq W$  for some non-trivial variety  $W \subsetneq \mathbb{C}^n$ .

## B.2 Algebraic Independence and Jacobian Criterion

The rational functions  $f_1, f_2, \dots, f_m \in \mathbb{C}(t_1, t_2, \dots, t_n)$  are called algebraically dependent (over  $\mathbb{C}$ ) if there exists a non-zero polynomial  $F \in \mathbb{C}[s_1, s_2, \dots, s_m]$  such that  $F(f_1, f_2, \dots, f_m) = 0$ . If there exists no such annihilating polynomial  $F$ , then  $f_1, f_2, \dots, f_m$  are algebraically independent.

**Lemma 6** (Theorem 3 on page 135 of [58]). *The rational functions  $f_1, f_2, \dots, f_m \in \mathbb{C}(t_1, t_2, \dots, t_n)$  are algebraically independent if and only if the Jacobian matrix*

$$\mathbf{J}_f = \left( \frac{\partial f_i}{\partial t_j} \right)_{1 \leq i \leq m, 1 \leq j \leq n} \quad (\text{B.1})$$

*has full row rank equal to  $m$ .*

The Jacobian matrix is a function of the variables  $t_1, t_2, \dots, t_n$ , and hence the Jacobian matrix can have different ranks at different points  $\mathbf{t} \in \mathbb{C}^n$ . Lemma 6 refers to the *structural rank* of the Jacobian matrix which is equal to  $m$  if and only if there exists at least one realization  $\mathbf{t} \in \mathbb{C}^n$  where the Jacobian matrix has full row rank.

## B.3 Dominant Maps and Generic Properties

A polynomial map  $\mathbf{f} : \mathbb{C}^n \rightarrow \mathbb{C}^m$  is said to be *dominant* if the Zariski closure of the image  $\mathbf{f}(\mathbb{C}^n)$  is equal to  $\mathbb{C}^m$ . The image of a polynomial map is constructible. Therefore, the image of a dominant polynomial map is dense, i.e., the complement of  $\mathbf{f}(\mathbb{C}^n)$  is contained in a non-trivial variety  $W \subsetneq \mathbb{C}^m$ .

The implication of this is that the system of polynomial equations

$$\begin{aligned}
 s_1 &= f_1(t_1, t_2, \dots, t_n) \\
 s_2 &= f_2(t_1, t_2, \dots, t_n) \\
 &\vdots \\
 s_m &= f_m(t_1, t_2, \dots, t_n)
 \end{aligned}
 \tag{B.2}$$

has a solution  $\mathbf{t} \in \mathbb{C}^n$  for generic  $\mathbf{s}$ , where the notion of a generic property is defined below.

**Definition 2.** *A property is said to be true for generic  $\mathbf{s} \in \mathbb{C}^m$  if the property holds true for all  $\mathbf{s} \in \mathbb{C}^m$  except on a non-trivial affine variety  $W \subsetneq \mathbb{C}^m$ . Such a property is said to be a generic property.*

For example, a generic square matrix  $\mathbf{A}$  has full rank because  $\mathbf{A}$  is rank deficient only when it lies on the affine variety generated by the polynomial  $f(\mathbf{A}) = \det \mathbf{A}$ . If the variables are generated randomly according to a continuous joint distribution, then any generic property holds true with probability 1.

Observe that the Zariski closure of the image  $\mathbf{f}(\mathbb{C}^n)$  is equal to  $\mathbb{C}^m$  if and only if the ideal  $I$  generated by the image set is equal to  $\{0\}$ . Since  $I$  is equal to the set of annihilating polynomials

$$\begin{aligned}
 I &= \{F \in \mathbb{C}[s_1, s_2, \dots, s_m] : F(\mathbf{s}) = 0 \forall \mathbf{s} \in \mathbf{f}(\mathbb{C}^n)\} \\
 &= \{F \in \mathbb{C}[s_1, s_2, \dots, s_m] : F(f_1, f_2, \dots, f_m) = 0\},
 \end{aligned}$$

the map  $\mathbf{f}$  is dominant if and only if the polynomials  $f_1, f_2, \dots, f_m$  are algebraically independent. Thus we obtain Lemma 7.

**Lemma 7.** *The system of polynomial equations (B.2) admits a solution for a generic  $\mathbf{s} \in \mathbb{C}^m$  if and only if the polynomials  $f_1, f_2, \dots, f_m$  are algebraically independent, i.e., if and only if the Jacobian matrix (B.1) has full row rank.*

## B.4 A Lemma on Full-Rankness of Certain Random Matrix

Let  $\mathbf{t} \in \mathbb{C}^n$  be a set of original variables, and let  $\mathbf{s} \in \mathbb{C}^m$  be a set of derived variables obtained through polynomial transformation  $\mathbf{s} = \mathbf{f}(\mathbf{t})$  for some rational map  $\mathbf{f}$ . Suppose we generate  $p$  instances of  $\mathbf{t}$

$$\mathbf{t}(1), \mathbf{t}(2), \dots, \mathbf{t}(p) \tag{B.3}$$

and the corresponding  $p$  instances of  $\mathbf{s}$

$$\mathbf{s}(1), \mathbf{s}(2), \dots, \mathbf{s}(p)$$

and generate the  $p \times q$  matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{s}(1)^{\mathbf{a}_1} & \mathbf{s}(1)^{\mathbf{a}_2} & \dots & \mathbf{s}(1)^{\mathbf{a}_q} \\ \mathbf{s}(2)^{\mathbf{a}_1} & \mathbf{s}(2)^{\mathbf{a}_2} & \dots & \mathbf{s}(2)^{\mathbf{a}_q} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{s}(p)^{\mathbf{a}_1} & \mathbf{s}(p)^{\mathbf{a}_2} & \dots & \mathbf{s}(p)^{\mathbf{a}_q} \end{bmatrix}$$

for some exponent vectors  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_q \in \mathbb{Z}_+^m$  and  $p \geq q$ . We are interested in determining the set of variables (B.3) such that the matrix  $\mathbf{M}$  has full column rank. If there exists an annihilating polynomial  $F \in \mathbb{C}[s_1, s_2, \dots, s_m]$  of the form

$$F(\mathbf{s}) = \sum_{i=1}^q c_i \mathbf{s}^{\mathbf{a}_i} \tag{B.4}$$

such that  $F(f_1, f_2, \dots, f_m) = 0$ , then the matrix  $\mathbf{M}$  satisfies  $\mathbf{M}\mathbf{c} = \mathbf{0}$ , and hence the matrix  $\mathbf{M}$  does not have full column rank for any realizations of the variables (B.3). Interestingly, even the converse holds true.

**Lemma 8.** *The matrix  $\mathbf{M}$  has full column rank for generic realizations of the variables (B.3) if and only if there does not exist an annihilating polynomial  $F$  of the form (B.4) satisfying  $F(f_1, f_2, \dots, f_m) = 0$ .*

*Proof.* We have already proved that  $\mathbf{M}$  does not have full column rank if there exists an annihilating polynomial  $F$  of the form (B.4). We now prove the converse; i.e., we assume that there does not exist an annihilating polynomial of the form (B.4), and prove that the matrix  $\mathbf{M}$  has full column rank for

generic realizations of the variables (B.3). Without any loss of generality, we assume that  $p = q$ . Otherwise, we can work with the  $q \times q$  submatrix obtained after deleting the last  $q - p$  rows.

Consider expanding the determinant  $\det \mathbf{M}$  in terms of the variables (B.3). Since the variables  $\mathbf{s}(1), \mathbf{s}(2), \dots, \mathbf{s}(q)$  are rational functions of  $\mathbf{t}(1), \mathbf{t}(2), \dots, \mathbf{t}(q)$ , respectively, the determinant is also a rational function; i.e.,

$$\det \mathbf{M} = \frac{d_1(\mathbf{t}(1), \mathbf{t}(2), \dots, \mathbf{t}(q))}{d_2(\mathbf{t}(1), \mathbf{t}(2), \dots, \mathbf{t}(q))}.$$

The determinant can either be identically equal to zero, or a nonzero function. If the determinant is a nonzero function, then  $\mathbf{M}$  has full column rank for generic realizations of the variables (B.3) because  $\mathbf{M}$  is rank deficient only when  $d_1(\mathbf{t}(1), \mathbf{t}(2), \dots, \mathbf{t}(q)) = 0$  or when  $(\mathbf{t}(1), \mathbf{t}(2), \dots, \mathbf{t}(q))$  belongs to the affine variety  $V(d_1) \subsetneq \mathbb{C}^{nq}$  generated by the polynomial  $d_1$ .

Therefore, it remains to prove that  $\det M$  is not identically equal to zero under the assumption that no annihilating polynomial  $F$  of the form (B.4) exists. We prove this claim by induction on  $q$ . The claim is trivial to check for  $q = 1$ . We now prove the induction step. We may assume that the determinant of the  $(q - 1) \times (q - 1)$  submatrix  $\tilde{\mathbf{M}}$ , obtained after deleting the last row and column, is a nonzero function in  $(\mathbf{t}(1), \mathbf{t}(2), \dots, \mathbf{t}(q - 1))$ . Therefore, there must exist specific realizations

$$(\mathbf{t}(1), \mathbf{t}(2), \dots, \mathbf{t}(q - 1)) = (\mathbf{a}(1), \mathbf{a}(2), \dots, \mathbf{a}(q - 1)) \quad (\text{B.5})$$

such that  $\tilde{\mathbf{M}}$  has full rank. Consider the matrix  $\mathbf{M}^*(\mathbf{t})$  obtained from  $\tilde{\mathbf{M}}$  by setting  $\mathbf{t}(q) = \mathbf{t}$  for each  $\mathbf{t} \in \mathbb{C}^n$ . If  $\det \mathbf{M}$  is identically equal to zero, then the matrix  $\mathbf{M}^*(\mathbf{t})$  must be rank deficient for all  $\mathbf{t}$ ; i.e., there must exist  $\mathbf{c}(\mathbf{t}) \neq \mathbf{0}$  such that  $\mathbf{M}^*(\mathbf{t})\mathbf{c}(\mathbf{t}) = \mathbf{0}$  for each  $\mathbf{t} \in \mathbb{C}^n$ . Since the first  $q - 1$  rows are linearly independent and do not depend on  $\mathbf{t}$ , the vector  $\mathbf{c}(\mathbf{t}) = \mathbf{c}^*$  is unique (up to a scaling factor) and is determined by (B.5). Therefore, we have that  $\mathbf{M}^*(\mathbf{t})\mathbf{c}^* = \mathbf{0}$  for each  $\mathbf{t} \in \mathbb{C}^n$ . By expanding the last row of  $\mathbf{M}^*(\mathbf{t})\mathbf{c}^* = \mathbf{0}$ , we obtain

$$\sum_{i=1}^q c_i^* \mathbf{f}(\mathbf{t})^{\mathbf{a}_i} = 0.$$

This is a contradiction since we assumed that no annihilating polynomial of the form (B.4) exists. Therefore,  $\det \mathbf{M}$  is not identically equal to zero and

hence  $\mathbf{M}$  has full rank for generic realizations of the variables (B.3).  $\square$

If the rational functions  $f_1, f_2, \dots, f_m$  are algebraically independent, then there cannot exist an annihilating polynomial  $F$  (of any form) satisfying  $F(f_1, f_2, \dots, f_m) = 0$ . Thus, we immediately have the following corollary.

**Corollary 1.** *The matrix  $\mathbf{M}$  has full column rank for generic realizations of the variables (B.3) if the rational functions  $f_1, f_2, \dots, f_m$  are algebraically independent, i.e., if the Jacobian matrix (B.1) has full row rank.*

## REFERENCES

- [1] “3GPP TR 36.814 v9.0.0. Further advancements for E-UTRA - physical layer aspects,” Mar. 2010.
- [2] “3GPP TR 36.902 v9.2.0. E-UTRAN - Self-configuring and self-optimizing network (SON) use cases and solutions (Release 9),” Jun. 2010.
- [3] “3GPP TR 36.300 v9.3.0. E-UTRAN - Overall description (Release 9),” Mar. 2010.
- [4] H. Holma and A. Toskala, *LTE for UMTS-OFDMA and SC-FDMA Based Radio Access*. Hoboken, NJ: Wiley, 2009.
- [5] D. Tse and P. Viswanath, *Fundamentals of Wireless Communications*. New York, NY: Cambridge University Press, 2005.
- [6] A. B. Carleial, “A case where interference does not reduce capacity,” *IEEE Trans. Inf. Theory*, vol. 21, pp. 294-304, May 1977.
- [7] X. Shang, G. Kramer, and B. Chen “A new outer bound and noisy-interference sum-rate capacity for the Gaussian interference channel,” *IEEE Trans. Inf. Theory*, vol. 55, no. 2, pp. 689-699, Feb. 2009.
- [8] A. S. Motahari and A. K. Khandani “Capacity bounds for the Gaussian interference channel,” *IEEE Trans. Inf. Theory*, vol. 55, no. 2, pp. 620-643, Feb. 2009.
- [9] V. S. Annapureddy and V. V. Veeravalli “Gaussian interference networks: Sum capacity in the low interference regime and new outerbounds on the capacity region,” *IEEE Trans. Inf. Theory*, vol. 55, no. 6, pp. 3032-3050, Jun. 2009.
- [10] A. Host-Madsen and A. Nosratinia, “The multiplexing gain of wireless networks,” in *Proc. IEEE International Symposium on Information Theory (ISIT)*, pp. 2065-2069, Nice, Jun. 2007.
- [11] V. Cadambe and S. A. Jafar, “Interference alignment and degrees of freedom of the K-user interference channel,” *IEEE Trans. Inf. Theory*, vol. 54, no. 8, pp. 3425-3441, Aug. 2008.

- [12] P. Marsch and G. P. Fettweis, *Coordinated Multi-Point in Mobile Communications: From Theory to Practice*, 1st ed. New York, NY: Cambridge, 2011.
- [13] H. Viswanathan, S. Venkatesan, and H. Huang, "Downlink capacity evaluation of cellular network with known-interference cancellation," *IEEE Journal on Selected Areas in Comms.*, vol. 21, no. 5, pp. 802-811, Jun. 2003.
- [14] S. Shamai and B. Zaidel "Enhancing the cellular downlink capacity via co-processing at the transmitter end," in *Proc. Vehicular Technology Conference*, vol. 3, pp. 1745-1749, May 2001.
- [15] S. Shamai, O. Simeone, O. Somekh, and H. V. Poor "Joint multi-cell processing for downlink channels with limited-capacity backhaul," in *Proc. Information Theory and Applications Workshop*, pp. 345-349, Jan. 2008.
- [16] A. El Gamal, V. S. Annapureddy, and V. V. Veeravalli. (2012, Nov.). Interference channels with coordinated multi-point transmission: Degrees of freedom, message assignment, and fractional reuse. Accepted for Publication in *IEEE Trans. Inf. Theory*. [Online]. Available: <http://arxiv.org/abs/1211.2897>
- [17] V. S. Annapureddy, A. El Gamal, and V. V. Veeravalli, "Degrees of freedom of interference channels with CoMP transmission and reception," *IEEE Trans. Inf. Theory*, vol. 58, no. 9, pp. 5740-5760, Sep. 2012.
- [18] A. Wyner, "Shannon-theoretic approach to a Gaussian cellular multiple-access channel," *IEEE Trans. Inf. Theory*, vol. 40, no. 5, pp. 1713-1727, Nov. 1994.
- [19] A. Lapidath, S. Shamai (Shitz) and M. A. Wigger, "A linear interference network with local side-information," in *Proc. IEEE International Symposium on Information Theory (ISIT)*, pp. 2201-2205, Nice, Jun. 2007.
- [20] N. Palleit and T. Weber, "Obtaining transmitter side channel state information in MIMO FDD systems," in *Proc. IEEE International Symposium on Personal, Indoor and Mobile Radio Communications*, pp. 2439-2443, Sept. 2009.
- [21] N. Palleit and T. Weber, "Frequency prediction of the channel transfer function in multiple antenna systems," in *Proc. International ITG/IEEE Workshop on Smart Antennas*, pp. 244-250, Feb. 2010.



- [22] T. Weber, M. Meurer, and W. Zirwas “Improved channel estimation exploiting long term channel properties,” in *Proc. IEEE International Conference on Telecommunications*, May 2005.
- [23] L. Li, H. Jafarkhani, and S. A. Jafar, “Towards the feasibility conditions for linear interference alignment with symbol extensions: A diversity constraint,” in *Proc. IEEE Global Communications Conference*, pp. 2328-2333, Dec. 2012.
- [24] K. Gomadam, V. Cadambe, and S. A. Jafar “Approaching the capacity of wireless networks through distributed interference alignment,” *IEEE Trans. Inf. Theory*, vol. 57, no. 6, pp. 3309-3322, Jun. 2011.
- [25] I. H. Wang and D. Tse “Interference mitigation through limited transmitter cooperation,” *IEEE Trans. Inf. Theory*, vol. 57, no. 5, pp. 2941-2965, May 2011.
- [26] I. Shomorony and A. S. Avestimehr. (2012, Oct.). Degrees of freedom of two-hop wireless networks: Everyone gets the entire cake. Submitted to *IEEE Trans. Inf. Theory*. [Online]. Available: <http://arxiv.org/abs/1210.2143>
- [27] B. Zarikoff, “Frequency synchronization techniques for coordinated multibase MIMO wireless communication systems,” Ph.D. dissertation, Simon Fraser University, British Columbia, Canada, 2008.
- [28] N. Devroye, P. Mitran, and V. Tarokh “Achievable rates in cognitive radio channels,” *IEEE Trans. Inf. Theory*, vol. 52, no. 5, pp. 1813-1827, May 2006.
- [29] A. Lapidoth, S. Shamai (Shitz) and M. A. Wigger, “On cognitive interference networks,” in *Proc. IEEE Information Theory Workshop (ITW)*, pp. 325-330, Lake Tahoe, California, Sept. 2007.
- [30] A. Jovicic and P. Viswanath “Cognitive radio: An information theoretic perspective,” *IEEE Trans. Inf. Theory*, vol. 55, no. 9, pp. 3945-3958, 2009.
- [31] A. Lapidoth, N. Levy, S. Shamai (Shitz) and M. A. Wigger, “A cognitive network with clustered decoding,” in *Proc. IEEE International Symposium on Information Theory (ISIT)*, pp. 596-600, Seoul, Jul. 2009.
- [32] A. Lapidoth, N. Levy, S. Shamai (Shitz) and M. A. Wigger. (2012, Mar.). Cognitive Wyner networks with clustered decoding. Submitted to *IEEE Trans. Inf. Theory*. [Online]. Available: <http://arxiv.org/abs/1203.3659>
- [33] D. Maamari, D. Tuninetti, and N. Devroye. (2012, Jan.). Approximate sum-capacity of K-user cognitive interference channels with cumulative message sharing. [Online]. Available: <http://arxiv.org/abs/1301.6198>

- [34] N. Devroye and M. Sharif, "The multiplexing gain of MIMO x-channels with partial transmit side information," in *Proc. IEEE International Symposium on Information Theory (ISIT)*, Nice, Jun. 2007.
- [35] I-Hsiang Wang and D. Tse "Interference mitigation through limited transmitter cooperation," in *Proc. IEEE International Symposium on Information Theory (ISIT)*, Austin, Jun. 2010.
- [36] P. Prabhakaran and P. Viswanath "Interference channels with source cooperation," *IEEE Trans. Inf. Theory*, vol. 57, no. 1, pp. 156-186, 2011.
- [37] M. Maddah-Ali, A. Motahari, and A. Khandani "Communication over MIMO X channels: Interference alignment, decompositions, and performance analysis," *IEEE Trans. Inf. Theory*, vol. 54, no. 8, pp. 3457-3470, 2008.
- [38] S. A. Jafar and S. Shamai (Shitz) "Degrees of freedom region of the MIMO X channel," *IEEE Trans. Inf. Theory*, vol. 54, no. 1, pp. 151-170, 2008.
- [39] V. S. Annapureddy, A. El Gamal, and V. V. Veeravalli, "Degrees of freedom of cooperative interference networks," in *Proc. IEEE International Symposium on Information Theory (ISIT)*, pp. 1377-1381, Saint Petersburg, Aug. 2011.
- [40] R. Zhang and S. Cui "Cooperative interference management with MISO beamforming," *IEEE Transactions on Signal Processing*, vol. 58, pp. 5450-5458, Oct. 2010.
- [41] V. Ntranos, M. Maddah-Ali, and G. Caire. (2014, Feb.). Cellular interference alignment. [Online]. Available: <http://arxiv.org/abs/1402.3119>
- [42] A. El Gamal and Y. Kim, *Network Information Theory*. New York, NY: Cambridge University Press, 2012.
- [43] A. El Gamal and V. V. Veeravalli, "Dynamic interference management," in *Proc. Asilomar Conference on Signals, Systems, and Computers*, Pacific Grove, Nov. 2013.
- [44] G. Bresler, D. Catwright, and D. Tse. (2011, Apr.). Settling the feasibility of interference alignment for the MIMO interference channel: The symmetric square case. [Online]. Available: <http://arxiv.org/abs/1104.0888>
- [45] A. El Gamal, V. S. Annapureddy, and V. V. Veeravalli, "On optimal message assignments for interference channel with CoMP transmission," in *Proc. 46<sup>th</sup> Annual Conference on Information Sciences and Systems*, pp. 1-6, Princeton, NJ, Mar. 2012.

- [46] M. Pinsker, “On the complexity of a concentrator,” in *Proc. 7<sup>th</sup> International Teletraffic Conference*, pp. 318/1-318/4, 1973.
- [47] S. Shamai (Shitz) and M. A. Wigger, “Rate-limited transmitter-cooperation in Wyner’s asymmetric interference network,” in *Proc. IEEE International Symposium on Information Theory (ISIT)*, pp. 425-429, Saint Petersburg, Aug. 2011.
- [48] H. Maleki and S. A. Jafar. (2013, May). Optimality of orthogonal access for one-dimensional convex cellular networks. [Online]. Available: <http://arxiv.org/abs/1305.2173>
- [49] T. Yoo and A. Goldsmith, “Optimality of zero-forcing beamforming with multiuser diversity,” in *Proc. IEEE International Conference on Communications*, vol. 1, pp. 542-546, May 2005.
- [50] T. M. Cover, and J. A. Thomas, *Elements of Information Theory*, 2nd ed. Hoboken, NJ: Wiley, 2006.
- [51] A. El Gamal, V. S. Annapureddy, and V. V. Veeravalli, “Degrees of freedom (DoF) of locally connected interference channels with coordinated multi-point (CoMP) transmission,” in *Proc. IEEE International Conference on Communications (ICC)*, Ottawa, Jun. 2012.
- [52] E. Telatar, “Capacity of multi-antenna Gaussian channels,” *European Trans. on Telecomms.*, vol. 10, no. 6, pp. 585-595, Nov. 1999.
- [53] A. El Gamal, V. S. Annapureddy, and V. V. Veeravalli, “Degrees of freedom (DoF) of locally connected interference channels with cooperating multiple-antenna transmitters,” in *Proc. IEEE International Symposium on Information Theory*, Cambridge, MA, Jul. 2012.
- [54] M. Lin and T. La Porta, “Dynamic interference management in femto-cells,” in *IEEE International Conference on Computer Communications and Networks*, pp. 1-9, Jul. 2012.
- [55] R. J. McEliece and W. E. Stark, “Channels with block interference,” *IEEE Trans. Inf. Theory*, vol. 30, no. 1, pp. 44-53, Jan. 1984.
- [56] D. Tse and L. Zheng, “Diversity and multiplexing: A fundamental trade-off in multiple-antenna channels,” *IEEE Trans. Inf. Theory*, vol. 49, no. 50, pp. 1073-1096, May 2003.
- [57] D. Cox, J. Little, and D. O’Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. New York, NY: Springer Verlag, 2007.
- [58] W. Hodge and D. Pedoe, *Methods of Algebraic Geometry*. New York, NY: Cambridge University Press, 1953.