Lecture 24 – Energy methods (cont.)

Reading assignment: Ch.16 lecturebook

PURDUE UNIVERSITY | Mechanical Engineering

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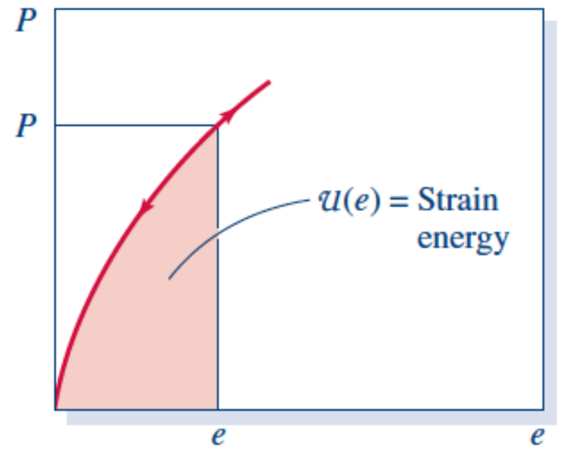
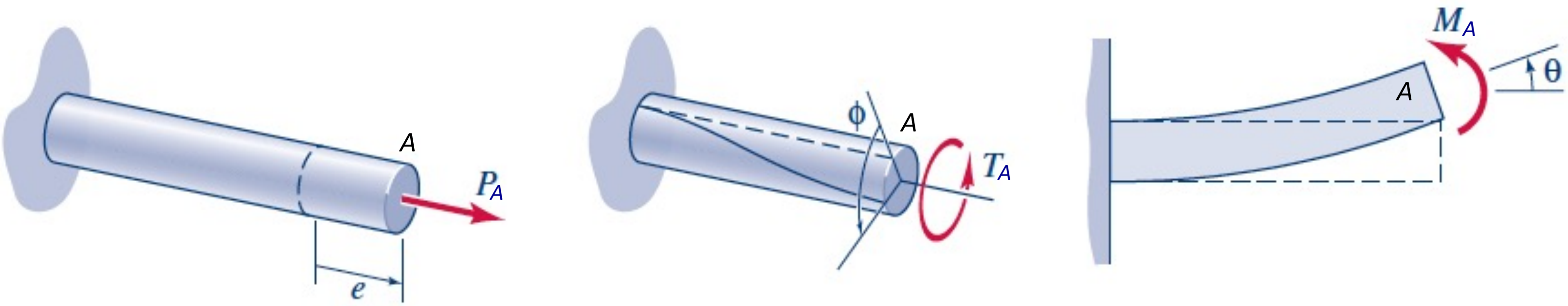
Energy methods

Theory of deformable bodies

- Geometry of the solid body
- Kinematic assumptions
- Material behavior
- Equilibrium



Work-energy principle: $U = W_{\text{ext}}$

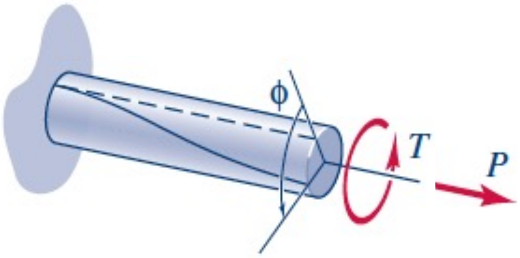


For an elastic body, the work, W_{ext}, done on the body by external loads (forces and moments) is stored as elastic strain energy, U.

Strain energy density

Strain energy density for linear elastic bodies

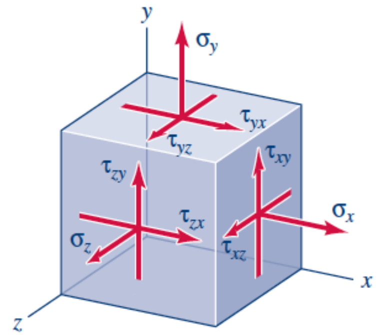
Under combined loading, each point in the body may experience a different state of stress, thus a different amount of elastic energy will be stored at different differential volumes in the body.



Q: Combined loading?

$$U = \int_V \bar{u} \mathrm{d}V$$

Strain energy density:

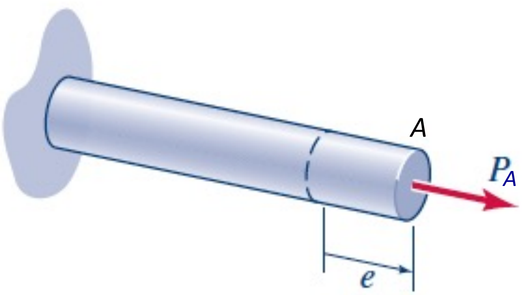


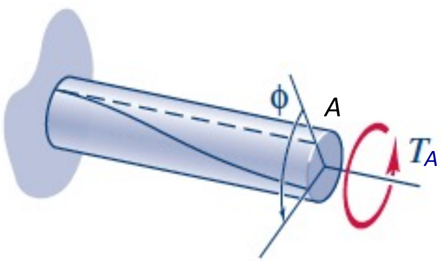
$$\bar{u} = \frac{1}{2}[\sigma_x(\epsilon_x - \alpha\Delta T) + \sigma_y(\epsilon_y - \alpha\Delta T) + \sigma_z(\epsilon_z - \alpha\Delta T) + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}]$$

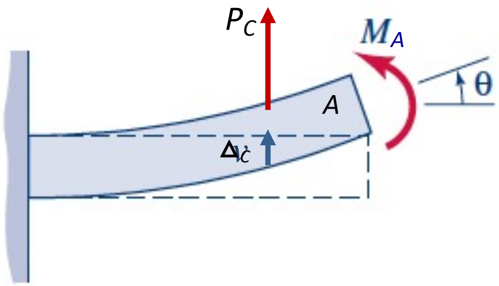
<u>Goal</u>: use the strain energy density to determine an expression for the energy stored in an elastic body under axial loads, torsional loads, and bending loads.

Energy methods

Work and elastic strain energy







Work done by the force:

$$W_{\text{ext}} = \frac{1}{2} P_A e_A$$

Stored elastic strain energy:

$$U = \frac{1}{2} \int_0^L \frac{P(x)^2}{E(x)A(x)} \mathrm{d}x$$

$$U = \frac{P^2 L}{2EA}$$

Work done by the torque:

$$W_{\text{ext}} = \frac{1}{2} T_A \phi_A$$

Stored elastic strain energy:

$$U = \frac{1}{2} \int_0^L \frac{T(x)^2}{G(x)I_p(x)} \mathrm{d}x$$

$$U = \frac{T^2 L}{2GI_p}$$

Work done by the moment:

$$W_{\text{ext}} = \frac{1}{2} M_A \theta_A$$

$$W_{\text{ext}} = \frac{1}{2} P_C \Delta_C$$

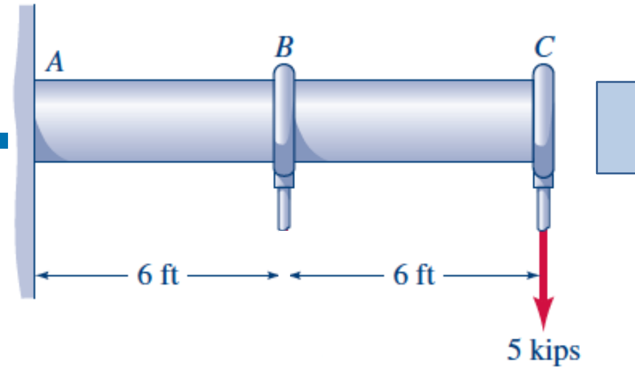
Stored elastic strain energy:

$$U_\sigma = \frac{1}{2} \int_0^L \frac{M(x)^2}{E(x)I(x)} \mathrm{d}x$$

$$U_\tau = \frac{1}{2} \int_0^L \frac{f_s V(x)^2}{G(x)A(x)} \mathrm{d}x$$

$$U_\tau << U_\sigma$$

Energy methods



Problem 44 (last lecture):

Determine the deflection at section C.
(Neglect the contribution of the the shear strain energy)

$$U_\sigma = \frac{1}{2}\int_0^L \frac{M(x)^2}{E(x)I(x)}\mathrm{d}x = \frac{P_C^2 L^3}{6EI}$$

$$W_{\text{ext}} = \frac{1}{2}P_C\Delta_C$$

Table with values of:

$$f_s = \frac{A(x)}{I(x)^2}\int_A \frac{Q(x,y)^2}{t(y)^2}\mathrm{d}A$$

We can solve it with the Work-Energy Principle!

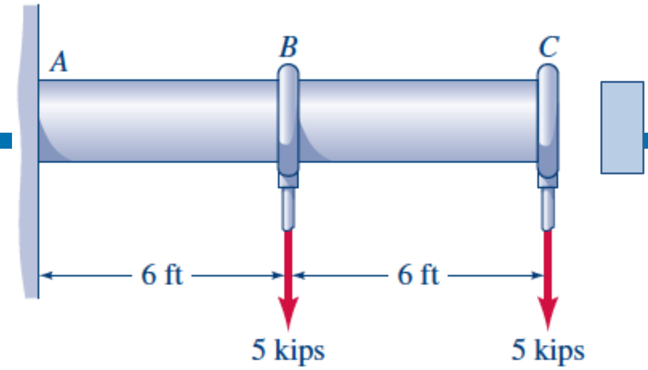
$$U = W_{\text{ext}}$$

$$v = \frac{Px^2}{6EI}(3L - x) \qquad v' = \frac{Px}{2EI}(2L - x)$$

$$\delta_B = \frac{PL^3}{3EI} \qquad \theta_B = \frac{PL^2}{2EI}$$

$$\Delta_C = \frac{P_C L^3}{3EI}$$

Energy methods



Problem 45 (last lecture):

Determine the deflections at sections B and C
(Neglect the contribution of the shear strain energy)

We cannot solve it with the Work-Energy Principle! $U = W_{\mathrm{ext}}$

Castigliano's Second Theorem

Consider a determinate linearly elastic deformable body acting upon by N_P forces P_i , N_M moments M_i , and N_T torques T_i . Among all possible equilibrium configurations of the body, the actual configuration is the one for which:

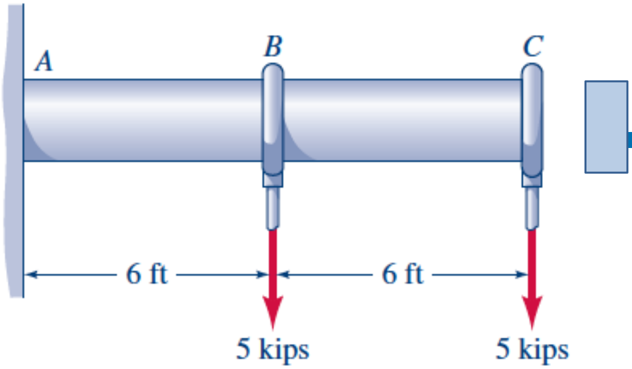
$$\Delta_i = \frac{\partial U}{\partial P_i} \qquad i = 1, 2, ...N_P \qquad \text{(displacement – force)}$$

$$\theta_i = \frac{\partial U}{\partial M_i} \qquad i = 1, 2, ...N_M \qquad \text{(slope – bending moment)}$$

$$\phi_i = \frac{\partial U}{\partial T_i} \qquad i = 1, 2, ...N_T \qquad \text{(angle of rotation – torque)}$$

where generalized displacements ($\Delta_i,\ \theta_i,\ \phi_i$) correspond to and are in the direction of the load ($P_i,\ M_i,\ T_i$).

Energy methods



Problem 45 (again):

Determine the deflections at sections B and C
(neglect the contribution of the the shear strain energy).

Castigliano's second theorem: $\Delta_i = \frac{\partial U}{\partial P_i}$ $\quad i = 1, 2, ...N_P$

$$\Delta_B = \int_0^L \frac{M(x)}{E(x)I(x)} \frac{\partial M}{\partial P_B} \mathrm{d}x = \frac{P_B L^3}{24EI} + \frac{5P_C L^3}{48EI}$$

$$\Delta_C = \int_0^L \frac{M(x)}{E(x)I(x)} \frac{\partial M}{\partial P_C} \mathrm{d}x = \frac{5P_B L^3}{48EI} + \frac{P_C L^3}{3EI}$$

- What if $P_B = 0$?

Energy methods

Castigliano's Second Theorem

Consider a <u>determinate</u> linearly elastic deformable body acting upon by N_P forces P_i , N_M moments M_i , and N_T torques T_i . Among all possible equilibrium configurations of the body, the actual configuration is the one for which:

$$\Delta_i = \frac{\partial U}{\partial P_i} \qquad i = 1, 2, ...N_P \qquad \text{(displacement – force)}$$

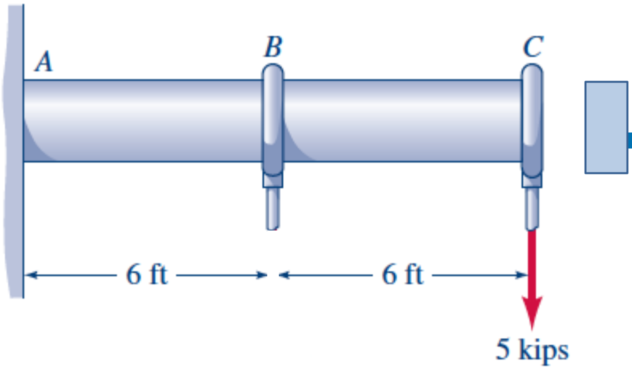
$$\theta_i = \frac{\partial U}{\partial M_i} \qquad i = 1, 2, ...N_M \qquad \text{(slope – bending moment)}$$

$$\phi_i = \frac{\partial U}{\partial T_i} \qquad i = 1, 2, ...N_T \qquad \text{(angle of rotation – torque)}$$

where generalized displacements ($\Delta_i,\ \theta_i,\ \phi_i$) correspond to and are in the direction of the load ($P_i,\ M_i,\ T_i$).

<u>Note</u>: some of these loads could be dummy loads, with value zero, that will facilitate the calculation of a generalized displacement at their point of application.

Energy methods



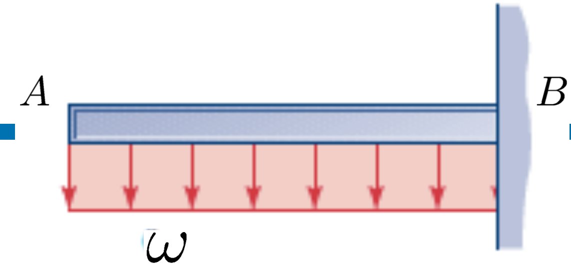
Problem 46:

Determine the deflection at section B.
(neglect the contribution of the the shear strain energy)

$$v = \frac{Px^2}{6EI}(3L - x) \qquad v' = \frac{Px}{2EI}(2L - x)$$

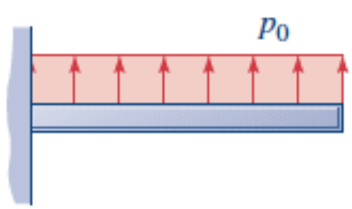
$$\delta_B = \frac{PL^3}{3EI} \qquad \theta_B = \frac{PL^2}{2EI}$$

Energy methods



Problem 47 (practice problem):

Determine the deflection and the rotation at section A.
(neglect the contribution of the the shear strain energy)



$$v = \frac{p_0 x^2}{24EI}(6L^2 - 4Lx + x^2)$$

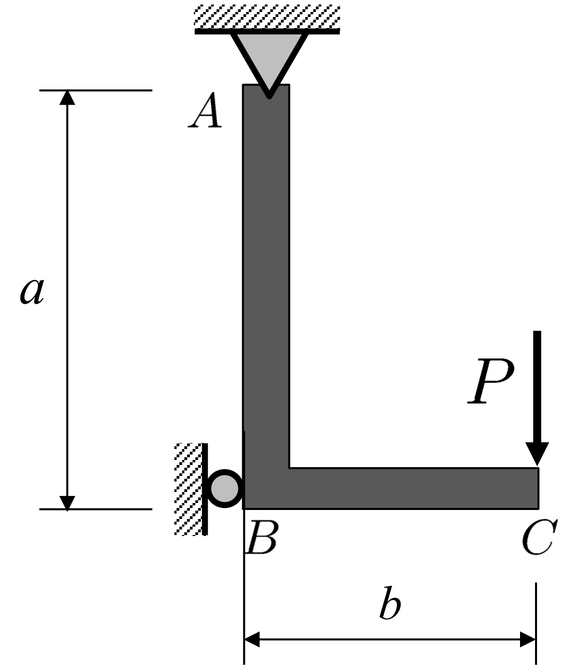
$$v' = \frac{p_0 x}{6EI}(3L^2 - 3Lx + x^2)$$

$$\delta_B = \frac{p_0 L^4}{8EI} \qquad \theta_B = \frac{p_0 L^3}{6EI}$$

Energy methods

Problem 48:

Determine the vertical displacement at point C.
(Neglect shear effects)



Any questions?