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An upper bound on the Hausdorff distance between a Pareto set and its discretization in bi-objective convex quadratic optimization

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Abstract We provide upper bounds on the Hausdorff distances between the efficient set and its discretization in the decision space, and between the Pareto set (also called the Pareto front) and its discretization in the objective space, in the context of bi-objective convex quadratic optimization on a compact feasible set. Our results imply that if t is the dispersion of the sampled points in the discretized feasible set, then the Hausdorff distances in both the decision space and the objective space are $O(\sqrt{t})$ as t decreases to zero.

Keywords multi-objective optimization · convex quadratic · Pareto set approximation · Pareto front approximation

1 Introduction

Our aim is to find an upper bound on the Hausdorff distance between a Pareto set (also called the Pareto front) and its discretization, as a function of the dispersion of sampled points in the compact feasible set, in the context of bi-objective convex quadratic optimization. Consider the bi-objective optimization problem

$$\text{minimize} \quad \{f(x) = (f_1(x), f_2(x))\} \quad \text{s.t.} \quad x \in \mathcal{X}, \tag{Q}$$

where $f\colon \mathbb{R}^q \to \mathbb{R}^2$ is a vector-valued function comprised of convex quadratic functions $f_1, f_2\colon \mathbb{R}^q \to \mathbb{R}$; and $\mathcal{X} \subseteq \mathbb{R}^q, q \geq 2$ is a known, convex, compact feasible set. The solution to (Q) is the efficient set $\mathcal{E} \subset \mathcal{X}$, which the set of all feasible points whose images are non-dominated. Henceforth, we refer to the image of the efficient

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set as the Pareto set, $\mathcal{P}$. (We refer the reader to §1.3 for notation and terminology; we formalize our assumptions in §1.4.)

We approximate the solution to (Q) by observing the objective value at $m \geq 1$ distinct points and constructing a discretized Pareto set. Let the chosen set of m distinct points be denoted $\mathcal{X}_m := \{\tilde{x}_1, \ldots, \tilde{x}_m\} \subset \mathcal{X}$, where $\tilde{x}_i \neq \tilde{x}_{i'}$ for each $i, i' \in \{1, \ldots, m\}$. Then, solve the discretized problem

$$\text{minimize} \quad f(\tilde{x}) \quad \text{s.t.} \quad \tilde{x} \in \mathcal{X}_m \subset \mathcal{X}. \tag{Q_m}$$

Let $\mathcal{E}_m \subseteq \mathcal{X}_m$ be the solution to (Q_m) and let $\mathcal{P}_m$ be its image, where $|\mathcal{P}_m| = |\mathcal{E}_m| = m^*$.

We measure the quality of the discretization $\mathcal{X}_m$ in terms of its *dispersion* [23]

$$t := \text{dist}(\mathcal{X}, \mathcal{X}_m) = \sup_{x \in \mathcal{X}} \min_{1 \leq i \leq m} \|x - \tilde{x}_i\| > 0. \tag{1}$$

Letting $B(x,t)$ be a closed q-dimensional ball of radius t centered at $x \in \mathcal{X}$, (1) is also the smallest value of t such that the q-balls $B(\tilde{x}_1, t), \ldots, B(\tilde{x}_m, t)$ cover the feasible set $\mathcal{X}$ [23,35]. One can choose the points $\mathcal{X}_m$ in many ways, such as randomly or on a grid. We present a grid example from [35] and refer to [23] for other examples.

Example 1 ([35, p. 942]) Let $\mathcal{X} = [0,1]^q$ be the q-dimensional unit hypercube and set $\nu = \lfloor m^{1/q} \rfloor$. Choose $\mathcal{X}_m$ as $\{(x_1, \ldots, x_q) \colon x_j \in \{\frac{1}{2\nu}, \frac{3}{2\nu}, \ldots, \frac{2\nu-1}{2\nu}\}$ for each $j\} \cup \Psi$, where Ψ is an arbitrary set of $m - \nu^q$ points in $\mathcal{X}$. Then under the Euclidean norm, the dispersion of the grid $\mathcal{X}_m$ is $\text{dist}(\mathcal{X}, \mathcal{X}_m) = t = \sqrt{q}/(2\lfloor m^{1/q} \rfloor)$.

In what follows, any quantity that depends on the dispersion t also depends on m, in the sense that $t \to 0$ implies $m \to \infty$. For appropriately space-filling choices of $\{\mathcal{X}_m\}$, such as a sequence based on the gridding in Example 1, then $m \to \infty$ implies $t \to 0$.

1.1 Problem Statement

Let $\mathcal{X}_m$ be any configuration of $m \geq 1$ distinct points in the compact feasible set $\mathcal{X}$ such that $\text{dist}(\mathcal{X}, \mathcal{X}_m) = t > 0$ under the Euclidean norm; thus, t is given. Find least upper bounds on the Hausdorff distances $\mathbb{H}(\mathcal{E}, \mathcal{E}_m) = \max\{\text{dist}(\mathcal{E}, \mathcal{E}_m), \text{dist}(\mathcal{E}_m, \mathcal{E})\}$ and $\mathbb{H}(\mathcal{P}, \mathcal{P}_m) = \max\{\text{dist}(\mathcal{P}, \mathcal{P}_m), \text{dist}(\mathcal{P}_m, \mathcal{P})\}$ as a function of t.

1.2 Related Literature and Our Contributions

Many metrics exist for measuring the performance of discrete representations of Pareto sets [2,8,10,12,20], and a rich body of literature exists on Pareto set approximations [16,26]. Also, a variety of methods for designing the observed point set $\mathcal{X}_m$ have been investigated numerically [29], and [33] shows that a worst-case optimal algorithm for a multi-objective problem with Lipshitz objective functions corresponds to a uniform covering of the feasible region. In the context of convex vector optimization, [19] study well-posedness and Hausdorff convergence of solution sets. Despite this related work, to the best of our knowledge, a general upper bound on the Hausdorff distance as a function of the dispersion t has not yet appeared in the literature.

Perhaps most related to the present work are the bounds on the one-way distances $\mathrm{dist}(\mathcal{E},\mathcal{E}_m)$ and $\mathrm{dist}(\mathcal{P},\mathcal{P}_m)$ as a function of the dispersion t in [25], where the one-way distance $\mathrm{dist}(\mathcal{P},\mathcal{P}_m)$ is the *coverage error* [10,27]. The proofs hold for two or more Lipschitz objectives with distances measured under an L_p-norm for $p \geq 1$. The results in [25, Ch. 6] imply the following Lemma 1. First, define

$$h(t) := \sup_{z\in\mathcal{E}} \sup_{x\in B(z,t)\cap\mathcal{X}} \mathrm{dist}(f(x),\mathcal{P}) = \sup_{x\in B(\mathcal{E},t)\cap\mathcal{X}} \mathrm{dist}(f(x),\mathcal{P})$$

as the farthest point from the Pareto set across the images of all points in the t-expansion of $\mathcal{E}$. Then, Lemma 1 provides the one-way distances for our context.

Lemma 1 (See [25].) *For* (Q) *and* (Q_m), $\mathrm{dist}(\mathcal{E},\mathcal{E}_m) \leq t$ *and* $\mathrm{dist}(\mathcal{P},\mathcal{P}_m) \leq h(t)$.

In light of Lemma 1, our main contribution is to provide upper bounds on the distances $\mathrm{dist}(\mathcal{E}_m,\mathcal{E})$ and $\mathrm{dist}(\mathcal{P}_m,\mathcal{P})$ as a function of t for two convex quadratic objectives. That is, we answer the question, *Given the dispersion of the sampled points t, how far away from the efficient set can a point be and still be included in the discretized efficient set?* As we show, these distances determine the maximum in the expression for the Hausdorff distance and imply the main result in the decision space, that $\mathbb{H}(\mathcal{E},\mathcal{E}_m) = O(\sqrt{t})$; our results also imply $\mathbb{H}(\mathcal{P},\mathcal{P}_m) = O(\sqrt{t})$ in the objective space. In the special case of spherical level sets and given a fixed t, we find a least upper bound on the distance in the decision space, $\mathbb{H}(\mathcal{E},\mathcal{E}_m)$, calculated across all possible discretization choices $\mathcal{X}_m$.

Our results quantify the convergence rate discrepancy between the coverage error and the Hausdorff distance in the bi-objective convex quadratic context: the coverage error converges as $O(t)$, while the Hausdorff distance converges at the slower rate of $O(\sqrt{t})$. This rate discrepancy is consistent with the general preference for metrics other than the Hausdorff distance in the multi-objective optimization community, as the Hausdorff distance is sometimes considered overly conservative [8,28].

1.3 Notation and Terminology

The q-by-q identity matrix is I_q; the origin is $0_q \in \mathbb{R}^q$. A matrix $H \succeq 0$ is positive semidefinite. The Euclidean norm is $\|x\| := (x_1^2 + \ldots + x_q^2)^{1/2}$. For a square matrix H, $\|H\| := \sup\{\|Hx\|/\|x\| \colon x \neq 0_q\}$. For $x \in \mathbb{R}^q$, let $B(x,t) := \{x' \in \mathbb{R}^q \colon \|x' - x\| \leq t\}$. Let $\mathcal{A} \subseteq \mathbb{R}^q$ be a set. Its t-expansion is $B(\mathcal{A},t) := \cup_{x\in\mathcal{A}} B(x,t)$. Its complement is $\mathcal{A}^c$, convex hull is $\mathrm{conv}(\mathcal{A})$, boundary is $\mathrm{bd}(\mathcal{A})$, interior is $\mathrm{int}(\mathcal{A})$, and diameter is $\mathrm{diam}(\mathcal{A}) = \sup_{x,y\in\mathcal{A}} \|x-y\|$. $\mathcal{A}$ is bounded if $\exists c < \infty$ such that $\forall x,y \in \mathcal{A}$, $\|x - y\| \leq c$. Let $\mathcal{A} \subset \mathbb{R}^q, \mathcal{B} \subset \mathbb{R}^q$ be nonempty, bounded sets. For $x \in \mathbb{R}^q$, $\mathrm{dist}(x,\mathcal{B}) := \inf_{y\in\mathcal{B}} \|x-y\|$; $\mathrm{dist}(\mathcal{A},\mathcal{B}) := \sup_{x\in\mathcal{A}} \mathrm{dist}(x,\mathcal{B})$. Let $f \colon \mathcal{X} \subseteq \mathbb{R}^q \to \mathbb{R}^d$ be a vector-valued function. For a set $\mathcal{S} \subseteq \mathcal{X}$, $f(\mathcal{S}) := \{f(x) \colon x \in \mathcal{S}\}$. For $y,\tilde{y} \in \mathbb{R}^d$, $y \leqq \tilde{y}$ (y weakly dominates $\tilde{y}$) if $y_k \leq \tilde{y}_k$ for all $k = 1,2$; $y \leq \tilde{y}$ (y dominates $\tilde{y}$) if $y \leqq \tilde{y}$ and $y \neq \tilde{y}$; $y < \tilde{y}$ (y strictly dominates $\tilde{y}$) if $y_k < \tilde{y}_k$ for all $k = 1,2$. For sequences of reals $\{a_n\}$ and $\{b_n\}$, $a_n = O(1)$ if there exists $c \in (0,\infty)$ with $|a_n| < c$ for large enough n. For positive-valued sequences of reals $\{a_n\}$ and $\{b_n\}$, we say $a_n = O(b_n)$ if $a_n/b_n = O(1)$ as $n \to \infty$. See [17, p. 7] for commentary on our conventions regarding the terms *efficient* and *Pareto*. We use *Pareto set* instead of *Pareto front* for mathematical clarity.

1.4 Standing Assumptions and Preliminaries

Assumption 1 *We assume the following about the problem in* (Q)*:*
1. *For* $k \in \{1,2\}$, $f_k\colon \mathbb{R}^q \to \mathbb{R}$ *is a convex quadratic function with minimizer* x_k^*,

$$f_k(x) = (1/2)(x - x_k^*)^\intercal H_k (x - x_k^*) + b_k = (1/2)\|A_k(x - x_k^*)\|^2 + b_k$$

where $H_k = A_k^\intercal A_k$ *is symmetric and positive definite,* $b_k \in \mathbb{R}$, *and* $x_1^* \neq x_2^*$.
2. *The feasible set* $\mathcal{X} \subseteq \mathbb{R}^q$, $q \geq 2$ *is compact, convex, and* $\mathcal{E} \subset \mathrm{int}(\mathcal{X})$.
3. *The* t*-expansion of* $\mathcal{E}$ *is feasible, that is,* $B(\mathcal{E}, t) \subset \mathcal{X}$.

Assumption 1 Part 2 ensures that there exists a t such that the t-expansion of $\mathcal{E}$ becomes feasible as $t \to 0$, and Assumption 1 Part 3 is for convenience.

Consider implications of Assumption 1; see also [3,4,14,22,32]. First, the KKT conditions for the linear weighted sum scalarization of (Q) imply the efficient set is

$$\mathcal{E} = \{z \in \mathcal{X}\colon z(\beta) = (\beta H_1 + (1-\beta)H_2)^{-1}(\beta H_1 x_1^* + (1-\beta)H_2 x_2^*),\ 0 \leq \beta \leq 1\}. \quad (2)$$

The Pareto set is $\mathcal{P} = \{f(z)\colon z \in \mathcal{E}\}$. If f_1 and f_2 have proportional Hessians $a_1 I_q$ and $a_2 I_q$, respectively, then $\mathcal{E}$ is the line segment $\mathrm{conv}(\{x_1^*, x_2^*\})$ [32]. Henceforth, without loss of generality, let $x_2^* = 0_q$, although often, we explicitly write x_2^* when it aids intuition. Let $\ell := \|x_1^* - x_2^*\| = \|x_1^*\|$, and let $\ell_a \in [\ell, \infty)$ be the arc length of $\mathcal{E}$.

For each $k \in \{1,2\}$, by the spectral theorem, decompose the symmetric positive definite matrix $H_k = Q_k \Lambda_k Q_k^\intercal$, and label the eigenvalues on the diagonal of Λ_k so that $0 < \lambda_1(H_k) \leq \lambda_2(H_k) \leq \ldots \leq \lambda_q(H_k) < \infty$. Let $\kappa_k := \kappa(H_k) = \lambda_q(H_k)/\lambda_1(H_k) \in [1, \infty)$ be the condition number for objective $k \in \{1,2\}$, and let $\kappa^* := \max\{\kappa_1, \kappa_2\}$.

For a q-by-q Hermitian matrix H, the Rayleigh quotient is $R(H,x)$ where

$$\lambda_1(H) \leq R(H,x) := \frac{x^\intercal H x}{x^\intercal x} \leq \lambda_q(H) \ \text{ for all } x \in \mathbb{R}^q, x \neq 0_q. \quad (3)$$

Then for each $k \in \{1,2\}$,

$$\lambda_1(H_k)\|x - x_k^*\|^2 \leq 2(f_k(x) - b_k) = 2(\|A_k(x - x_k^*)\|^2 - b_k) \leq \lambda_q(H_k)\|x - x_k^*\|^2; \quad (4)$$

see, e.g., [30]. These results lead to the following Lemma 2 regarding $h(t)$.

Lemma 2 *Under Assumption 1,* $h(t) = O(t)$.

Proof Let $z \in \mathcal{E}, x \in \mathcal{X}$, and $d := z - x$. The mean value theorem [24, p. 629] implies that for each $k \in \{1,2\}$, $f_k(x+d) - f_k(x) = \nabla f_k(x)^\intercal d + \frac{1}{2} d^\intercal H_k d$. Thus,

$$\begin{aligned} |f_k(z) - f_k(x)| &= |(H_k(x - x_k^*))^\intercal d + \tfrac{1}{2} d^\intercal H_k d| \leq |\langle (x - x_k^*)^\intercal A_k^\intercal, A_k d\rangle| + \tfrac{1}{2}\|A_k d\|^2 \\ &\leq \|A_k(x - x_k^*)\|\|A_k d\| + \tfrac{1}{2}\|A_k d\|^2 \leq \lambda_q(H_k)\left(\|x - x_k^*\|\|d\| + \tfrac{1}{2}\|d\|^2\right), \quad (5) \end{aligned}$$

where we apply the Cauchy-Schwarz inequality in line (5). Then we have

$$\begin{aligned} h(t) &= \sup_{x \in B(\mathcal{E},t)} \inf_{z \in \mathcal{E}} \|f(x) - f(z)\| \leq \sup_{x \in B(\mathcal{E},t)} \inf_{z \in \mathcal{E}} \big(2 \max_{k \in \{1,2\}} \{|f_k(x) - f_k(z)|\}\big) \\ &\leq \sup_{x \in B(\mathcal{E},t)} \inf_{z \in \mathcal{E}} \big(2 \max_{k \in \{1,2\}} \big\{\lambda_q(H_k)\big(\|x - x_k^*\|\|z - x\| + \tfrac{1}{2}\|z - x\|^2\big)\big\}\big) \\ &\leq 2\max\big\{\lambda_q(H_1), \lambda_q(H_2)\big\}\big((\ell_a + t)t + t^2/2\big) = O(t). \end{aligned}$$

2 The Main Result

In this section, we provide a statement of the main theorem and its corollaries. We also discuss an overview of the proof technique and paper organization.

2.1 Statement of the Theorem and Corollaries

Theorem 1 provides an upper bound on the Hausdorff distance between the efficient set and its discretization for hyper-spherical ($\kappa^* = 1$) and hyper-ellipsoidal ($\kappa^* \geq 1$) (henceforth, spherical and ellipsoidal) level sets for the objectives under Assumption 1. When $\kappa^* = 1$, the bound is a least upper bound. Corollary 1 provides a corresponding result in the objective space.

Theorem 1 *Let Assumption 1 hold, and let $\mathcal{X}_m$ be any configuration of $m \geq 1$ distinct points in $\mathcal{X}$ such that* $\mathrm{dist}(\mathcal{X}, \mathcal{X}_m) = t > 0$. *Then for (Q) and ($Q_m$):*

1. *If $\kappa^* = 1$, then $\mathbb{H}(\mathcal{E}, \mathcal{E}_m) \leq \sqrt{t\ell + t^2}$ is a least upper bound across all possible discretizations specified by $\mathcal{X}_m$ having dispersion t;*
2. *If $\kappa^* \geq 1$, then $\mathbb{H}(\mathcal{E}, \mathcal{E}_m) = O\big(\sqrt{t}\big)$ as $t \to 0$.*

Corollary 1 *Under the postulates of Theorem 1, $\mathbb{H}(\mathcal{P}, \mathcal{P}_m) = O\big(\sqrt{t}\big)$ as $t \to 0$.*

Proof Lemma 1 implies $\mathbb{H}(\mathcal{P}, \mathcal{P}_m) \leq \max\{h(t), \mathrm{dist}(f(\mathcal{E}_m), f(\mathcal{E}))\}$ where if $\kappa^* = 1$,

$$\begin{aligned} \mathrm{dist}(f(\mathcal{E}_m), f(\mathcal{E})) &= \sup_{\tilde{x}^* \in \mathcal{E}_m} \inf_{z \in \mathcal{E}} \|f(\tilde{x}^*) - f(z)\| \leq \sup_{x \in B(\mathcal{E}, \sqrt{t\ell + t^2})} \inf_{z \in \mathcal{E}} \|f(x) - f(z)\| \\ &= h(\sqrt{t\ell + t^2}) = O(\sqrt{t\ell + t^2}) = O(\sqrt{t}). \end{aligned} \tag{6}$$

Line (6) follows from Theorem 1 and Lemma 2. Similar logic holds when $\kappa^* \geq 1$.

In Theorem 1 Part 1, we do not usually know the length of the efficient set ℓ. The following corollary presents a translation of this result into an upper bound based only on observable quantities t and $|\mathcal{E}_m| = |\mathcal{P}_m| = m^*$; it follows because $\mathrm{dist}(\mathcal{X}, \mathcal{X}_m) = t$ implies $\ell \leq 2tm^*$ in §4.

Corollary 2 *Under the postulates of Theorem 1, if $\kappa^* = 1$, $\mathbb{H}(\mathcal{E}, \mathcal{E}_m) \leq t\sqrt{2m^* + 1}$.*

2.2 Overview of the Proof Technique and Paper Organization

In the remainder of the paper, we provide a proof for the upper bounds on $\mathbb{H}(\mathcal{E}, \mathcal{E}_m)$ in Theorem 1. Broadly, given fixed $t > 0$, our strategy is to find an upper bound on the optimal value of the following optimization problem in m and $\mathcal{X}_m = \{\tilde{x}_1, \ldots, \tilde{x}_m\}$:

$$\text{maximize} \quad \mathbb{H}(\mathcal{E}, \mathcal{E}_m) \quad \text{s.t.} \quad m \geq 1,\ \mathcal{X}_m \subset \mathcal{X},\ \mathrm{dist}(\mathcal{X}, \mathcal{X}_m) = t. \tag{H}$$

That is, given a fixed value of t, we wish to choose a feasible number of points $m \geq 1$ and their locations $\tilde{x}_1, \ldots, \tilde{x}_m$ so that the dispersion is still t, but we create a maximal value for the Hausdorff distance $\mathbb{H}(\mathcal{E}, \mathcal{E}_m)$. In what follows, we consider a sequence of relaxations to (H) culminating in tractable upper bounds for the cases of spherical ($\kappa^* = 1$) and ellipsoidal ($\kappa^* \geq 1$) level sets.

We divide the proof of Theorem 1 into three parts: In Part 1 (§3), we relax the problem (H) and determine a strategy for finding a tractable formulation whose solution provides an upper bound. The results in Part 1 hold for all quadratics under Assumption 1. In Part 2 (§4), we solve the relaxed problem for the special case of spherical level sets and provide an example that demonstrates our bound is a least upper bound. In Part 3 (§5), we derive the general big O result for ellipsoidal level sets; supporting results for Part 3 appear in §A. Finally, §6 contains concluding remarks.

3 Proof Part 1: Relaxing and Simplifying the Optimization Problem (H)

In this section, we formulate a sequence of relaxations to the optimization problem (H) and determine a strategy for deriving the desired upper bound. To begin, in §3.1, we formulate the first relaxation of (H) based on a set we call the *good set*. Then, in §3.2, we reformulate the good set for better tractability. In §3.3, we provide an upper bound on the reformulated objective function that depends on finding the center and radius of a ball enclosing the intersection of two ellipsoids, which is a version of the Chebyshev center problem [5,6,7,11,34]. Finally, in §3.4, we outline a strategy for obtaining a closed-form upper bound by either solving or approximating the solution to the Chebyshev center problem, depending on whether the level sets are spherical ($\kappa^* = 1$) or ellipsoidal ($\kappa^* \geq 1$).

3.1 A Relaxation to Problem (H) Based on The Good Set

Our first relaxation of (H) involves both an upper bound on the objective function and a relaxation of the feasible set; the new feasible set removes variables that do not appear in the new objective function. To find an upper bound on the objective function of (H), first, notice that $\mathbb{H}(\mathcal{E}, \mathcal{E}_m) \leq \max\{t, \text{dist}(\mathcal{E}_m, \mathcal{E})\}$ by Lemma 1. In this section, we seek an upper bound on $\text{dist}(\mathcal{E}_m, \mathcal{E})$ that holds for each fixed $m \geq 1, \mathcal{E}_m \subset \mathcal{X}_m$. That is, we ask *how far away from the efficient set can a point be and still be included in the discretized efficient set?*

To answer this question, we want to find regions of $\mathcal{X}$ where points in $\mathcal{X}_m$ can be members of $\mathcal{E}_m$, but are not within distance t of $\mathcal{E}$. To this end, first, fix a feasible value of $m \geq 1, \mathcal{E}_m \subseteq \mathcal{X}_m$ and define

$$\mathcal{E}_{m,t} := \mathcal{E}_m \cap B(\mathcal{E}, t)$$

as the set of points in the discretized efficient set within distance t of $\mathcal{E}$. Let $|\mathcal{E}_{m,t}| = m_t^* \leq m$ be the cardinality; since $m \geq 1$, we also have $m^* \geq 1$, $m_t^* \geq 1$. Denote the points in $\mathcal{E}_{m,t}$ as $\{\tilde{x}_1^*, \ldots, \tilde{x}_{m_t^*}^*\}$. This set is important for the remainder of the paper; Lemma 3 provides a series of results relevant to $\mathcal{E}_{m,t}$.

Lemma 3 *If $m \geq 1$, $\mathcal{X}_m \subset \mathcal{X}$, and $\text{dist}(\mathcal{X}_m, \mathcal{X}) = t$, then:*
1. *The distance $\text{dist}(\mathcal{E}, \mathcal{E}_{m,t}) = \sup_{z \in \mathcal{E}} \inf_{\tilde{x}_i^* \in \mathcal{E}_{m,t}} \|z - \tilde{x}_i^*\| \leq t$;*

2. *$\mathcal{E}$ is contained in the union of t-radius balls, $\mathcal{E} \subset \cup_{\tilde{x}_i^* \in \mathcal{E}_{m,t}} B(\tilde{x}_i^*, t)$;*
3. *For each $\tilde{x}_i^* \in \mathcal{E}_{m,t}$, its t-radius ball intersects $\mathcal{E}$, that is, $\mathcal{E} \cap B(\tilde{x}_i^*, t) \neq \emptyset$;*
4. *The set $\cup_{\tilde{x}_i^* \in \mathcal{E}_{m,t}} B(\tilde{x}_i^*, t)$ is a connected set.*

Proof (Sketches) Part 1: By Lemma 1, $\mathrm{dist}(\mathcal{X}, \mathcal{X}_m) = t$ implies $\mathrm{dist}(\mathcal{E}, \mathcal{E}_m) \leq t$. Then $\mathrm{dist}(\mathcal{E}, \mathcal{E}_{m,t}) = \mathrm{dist}(\mathcal{E}, \mathcal{E}_m \cap B(\mathcal{E}, t)) \leq t$. *Part 2*: Let $z \in \mathcal{E}$. By Part 1, $\exists \tilde{x}_i^* \in \mathcal{E}_{m,t}$ such that $z \in B(\tilde{x}_i^*, t)$. *Part 3*: The result follows because $\mathcal{E}_{m,t} \subset B(\mathcal{E}, t)$. *Part 4*: By equation (2), $\mathcal{E}$ is a connected set. If $\cup_i B(\tilde{x}_i^*, t)$ is not connected, there exists a partition of $\mathcal{E}_{m,t}$ into two nonempty sets, indexed by j and n respectively, such that $\mathcal{E} \subset \cup_j B(\tilde{x}_j^*, t)$ or $\mathcal{E} \subset \cup_n B(\tilde{x}_n^*, t)$ but not both. Without loss of generality, let $\mathcal{E} \subset \cup_n B(\tilde{x}_n^*, t)$. Then $\mathcal{E} \cap \cup_j B(\tilde{x}_j^*, t) = \emptyset$, contradicting Part 3.

Each point $\tilde{x}_i^* \in \mathcal{E}_{m,t}, i \in \{1, \ldots, m_t^*\}$ is within distance t of $\mathcal{E}$ and disqualifies all points *outside* the union of its sublevel sets from entering $\mathcal{E}_m$. Under Assumption 1, the sublevel sets for each objective are q-dimensional ellipsoids; for $k = 1, 2$, define

$$\mathcal{L}_k(\tilde{x}) := \{x \in \mathbb{R}^q \colon f_k(x) \leq f_k(\tilde{x})\} = \{x \in \mathbb{R}^q \colon \|A_k(x - x_k^*)\|^2 \leq \|A_k(\tilde{x} - x_k^*)\|^2\}.$$

Then any point that belongs to $\mathcal{E}_m$ must also belong to the good set,

$$\mathcal{G}_t := \cap_{\tilde{x}_i^* \in \mathcal{E}_{m,t}} \mathcal{L}_1(\tilde{x}_i^*) \cup \mathcal{L}_2(\tilde{x}_i^*). \tag{7}$$

Since $\mathcal{E}_m \subset \mathcal{G}_t$, we have $\mathrm{dist}(\mathcal{E}_m, \mathcal{E}) \leq \mathrm{dist}(\mathcal{G}_t, \mathcal{E})$, which implies an upper bound on the objective function of (H), $\mathbb{H}(\mathcal{E}, \mathcal{E}_m) \leq \max\{t, \mathrm{dist}(\mathcal{G}_t, \mathcal{E})\}$.

We now formulate our first relaxation to (H) by replacing the objective with its upper bound and removing extra decision variables. First, define the feasible set $\mathfrak{E}_{m,t}(j)$ as a function of the total number of points $j = 1, 2, \ldots$ in $\mathcal{E}_{m,t}$,

$$\mathfrak{E}_{m,t}(j) := \{(m_t^*, \tilde{x}_1^*, \ldots, \tilde{x}_{m_t^*}^*) \colon m_t^* \geq j, \mathcal{E}_{m,t} = \{\tilde{x}_1^*, \ldots, \tilde{x}_{m_t^*}^*\} \subseteq \mathcal{E}_m, \ \mathrm{dist}(\mathcal{E}, \mathcal{E}_{m,t}) \leq t\}.$$

Then for fixed $t > 0$, we pose the following relaxed problem in $(m_t^*, \tilde{x}_1^*, \ldots, \tilde{x}_{m_t^*}^*)$,

$$\text{maximize} \quad \max\{t, \mathrm{dist}(\mathcal{G}_t, \mathcal{E})\} \quad \text{s.t.} \quad (m_t^*, \tilde{x}_1^*, \ldots, \tilde{x}_{m_t^*}^*) \in \mathfrak{E}_{m,t}(1). \tag{H.1}$$

To see that the feasible set in (H.1) is larger than the feasible set in (H), first, notice that any feasible value of m and arrangement of $\mathcal{X}_m$ implies a value of m_t^* and an arrangement of $\mathcal{E}_{m,t}$. Therefore, for each value of $m_t^* \geq 1$, we let points in $\mathcal{X}_m \setminus \mathcal{E}_{m,t}$ be free variables. Finally, $\mathrm{dist}(\mathcal{X}, \mathcal{X}_m) = t$ implies $\mathrm{dist}(\mathcal{E}, \mathcal{E}_{m,t}) \leq t$ by Lemma 3.

3.2 A Reformulation of the Good Set and the Objective Function of (H.1)

The good set expression in equation (7) is not tractable, since we first take a union over the level sets and then take an intersection across the points in $\mathcal{E}_{m,t}$. To make this expression tractable and facilitate an upper bound on the objective of (H.1), we reformulate $\mathcal{G}_t$ using the so-called brute force (bf) decomposition method of [1, 13]. That is, we decompose the good set using all possible ways a new point can enter the good set. Then, we re-write our optimization problem under this reformulation.

To reformulate $\mathcal{G}_t$, suppose we want to add a point $x_{\text{new}} \in \mathcal{X}$ to the good set. Then for every $\tilde{x}_i^* \in \mathcal{E}_{m,t}$, there would need to be an objective $k(\tilde{x}_i^*)$ such that $f_{k(\tilde{x}_i^*)}(x_{\text{new}}) \leq f_{k(\tilde{x}_i^*)}(\tilde{x}_i^*)$ in (7). To simplify notation, first, define the objective index $v_i := k(\tilde{x}_i^*)$. Then let $v = (v_1, \ldots, v_{m_t^*})$ be a vector assigning the relevant objectives for each one of the m_t^* points in $\mathcal{E}_{m,t}$. Since we have two objectives, then $v_i \in \{1,2\}$ for each $i \in \{1, \ldots, m_t^*\}$. Let $\mathcal{V} := \{v \colon v \in \{1,2\}^{m_t^*}\}$ be the set containing all possible ways to *assign* the points in $\mathcal{E}_{m,t}$ an objective on which x_{new} must achieve a lower objective value. Now given $v \in \mathcal{V}$, we have specified sets of concentric level sets (ellipsoids) around one or both minimizers such that x_{new} must be a member of *all* the sublevel sets to enter the good set under this v; define

$$\mathcal{G}_t(v) := \begin{cases} \cap_{\tilde{x}_i^* \in \mathcal{E}_{m,t}} \mathcal{L}_k(\tilde{x}_i^*) \text{ if } v = (k, \ldots, k) \text{ for some } k \in \{1,2\} \\ \left(\cap_{\{\tilde{x}_i^* \in \mathcal{E}_{m,t} \colon v_i = 1\}} \mathcal{L}_1(\tilde{x}_i^*)\right) \cap \left(\cap_{\{\tilde{x}_i^* \in \mathcal{E}_{m,t} \colon v_i = 2\}} \mathcal{L}_2(\tilde{x}_i^*)\right) \text{ otherwise.} \end{cases} \tag{8}$$

Using this enumeration technique, construct the set

$$\mathcal{G}_t^{\text{bf}} := \cup_{v \in \mathcal{V}} \mathcal{G}_t(v) = \mathcal{G}_t. \tag{9}$$

Proving the right-side equivalence in (9) follows the same steps as the reformulation result in [13, p. 116]. Figure 1 shows regions of $\mathcal{G}_t(v)$ for a simple example.

Since the expression for $\mathcal{G}_t(v)$ in (8) is unwieldy, we define special notation for the points whose level sets determine the intersection in (8). To begin, given $v \in \mathcal{V}$, for each $k \in \{1,2\}$ such that there exists $v_i = k$ for some $i \in \{1, \ldots, m_t^*\}$, define

$$\mathcal{S}_k(v) := \begin{cases} \text{argmin}\{\text{diam}(\mathcal{L}_k(\tilde{x}_i^*)) \colon \tilde{x}_i^* \in \mathcal{E}_{m,t}\} \text{ if } v = (k, \ldots, k) \text{ for some } k \in \{1,2\} \\ \text{argmin}\{\text{diam}(\mathcal{L}_k(\tilde{x}_i^*)) \colon \tilde{x}_i^* \in \mathcal{E}_{m,t}, v_i = k\} \text{ otherwise} \end{cases}$$

as the set of closest points, whose cardinality may be greater than one if two or more points share a level set. Then, select a closest point as the one closest to $\mathcal{E}$,

$$y_k(v) := \text{argmin}\{\text{dist}(\tilde{x}_i^*, \mathcal{E}) \colon \tilde{x}_i^* \in \mathcal{S}_k(v)\} \tag{10}$$

where ties are broken arbitrarily, and we say $y_k(\nu)$ does not exist if no points are assigned to objective k through v. Finally, with this notation, re-write (8) as

$$\mathcal{G}_t(v) = \begin{cases} \mathcal{L}_k(y_k(v)) \text{ if } v = (k, \ldots, k) \text{ for some } k \in \{1,2\}, \\ \mathcal{L}_1(y_1(v)) \cap \mathcal{L}_2(y_2(v)) \text{ otherwise.} \end{cases} \tag{11}$$

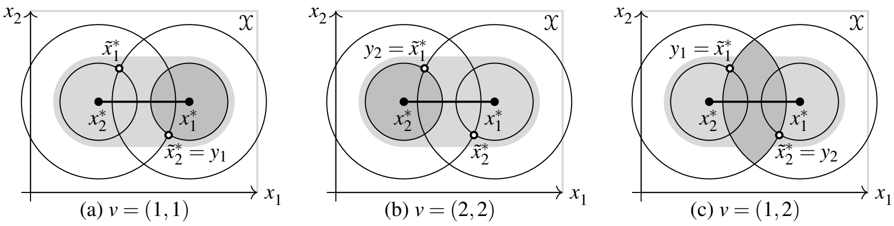


Figure 1 Let $\kappa^* = 1, m_t^* = 2$, and let the light gray region represent $B(\mathcal{E}, t)$. The dark gray regions show the sets $\mathcal{G}_t(v)$ for three different values of $v \in \mathcal{V}$, where $\mathcal{V} = \{(1,1), (1,2), (2,1), (2,2)\}$ and $v = (2,1)$ corresponds to $\mathcal{G}_t(v) = \emptyset$. The union of the dark gray sets across the v values equals $\mathcal{G}_t^{\text{bf}} = \mathcal{G}_t$. When they exist, $y_1(\nu)$ and $y_2(\nu)$ from (10) are shown. $\mathcal{V}_t$, defined in (14), equals $\{(1,2)\}$.

Often, we suppress the dependence of y_1, y_2 on v unless it is helpful for clarity. From (11), depending on v, $\mathcal{G}_t(v)$ is either an ellipsoidal sublevel set (henceforth, an ellipsoid), the intersection of two ellipsoidal sublevel sets (henceforth, the intersection of two ellipsoids), or the empty set; see Figure 1.

The definition of $y_k \in \mathcal{E}_{m,t}, k \in \{1,2\}$ in (10) and the requirement of feasibility in (H.1) directly imply the conditions in Lemma 4 on the remainder of the points in $\mathcal{E}_{m,t}$ which are relevant to later sections. (For an example of the conditions in Lemma 4, increase m_t^* to 3 in Figure 1(c) and add $\tilde{x}_3^*$ to $\mathcal{E}_{m,t}$ without moving y_1 or y_2.).

Lemma 4 *If* $(m_t^*, \tilde{x}_1^*, \ldots, \tilde{x}_{m_t^*}^*) \in \mathfrak{E}_{m,t}(3)$, $v \in \mathcal{V}$, y_1 *and* y_2 *exist, and the intersection of sublevel sets* $\mathcal{L}_1(y_1) \cap \mathcal{L}_2(y_2) \neq \emptyset$, *then*

1. *no other points in* $\mathcal{E}_{m,t}$ *lie on the interior of the intersection of sublevel sets, that is,* $(\mathcal{E}_{m,t} \setminus \{y_1, y_2\}) \cap \operatorname{int}(\mathcal{L}_1(y_1) \cap \mathcal{L}_2(y_2)) = \emptyset$; *and,*
2. *if a point in* $\mathcal{E}_{m,t}$ *lies on the boundary of the intersection of sublevel sets and shares a level set with* y_k, *then it must be assigned to objective* k *through* $v \in \mathcal{V}$ *and* y_k *must be closer to* $\mathcal{E}$; *that is, if* $\exists \tilde{x}_j^* \in \mathcal{E}_{m,t} \setminus \{y_1, y_2\}$ *such that* $\tilde{x}_j^* \in \mathcal{L}_k(y_k) \cap \operatorname{bd}(\mathcal{L}_1(y_1) \cap \mathcal{L}_2(y_2))$, *then* $v_j = k$ *and* $\operatorname{dist}(y_k, \mathcal{E}) < \operatorname{dist}(\tilde{x}_j^*, \mathcal{E})$.

Finally, using the reformulated good set from equation (9) in the objective of (H.1), the definitions of union, distance, and the definition of $\mathcal{G}_t(v)$ (in particular, that $\mathcal{G}_t(v)$ is bounded for each $v \in \mathcal{V}$), imply

$$\max\{t, \operatorname{dist}(\cup_{v\in\mathcal{V}} \mathcal{G}_t(v), \mathcal{E})\} = \max\{t, \max_{v\in\mathcal{V}} \operatorname{dist}(\mathcal{G}_t(v), \mathcal{E})\}$$

where, under the convention that $\max_{\emptyset} = -\infty$, we have $\operatorname{dist}(\emptyset, \mathcal{E}) = -\infty$. Then we re-write the problem (H.1) to yield its equivalent,

$$\text{maximize} \quad \max\{t, \max_{v\in\mathcal{V}} \operatorname{dist}(\mathcal{G}_t(v), \mathcal{E})\} \quad \text{s.t.} \quad (m_t^*, \tilde{x}_1^*, \ldots, \tilde{x}_{m_t^*}^*) \in \mathfrak{E}_{m,t}(1). \tag{H.2}$$

3.3 An Upper Bound on the Objective Function of (H.2) Based on Enclosing Balls

In this section, we simplify the objective of problem (H.2) by categorizing the different possible types of sets $\mathcal{G}_t(v)$ across the values of $v \in \mathcal{V}$. If $\mathcal{G}_t(v)$ is an ellipsoid, we find a closed-form upper bound. If $\mathcal{G}_t(v)$ is an intersection of two nonempty ellipsoids, we enclose it in a ball whose center is a member of the efficient set.

First, Lemma 5 presents an upper bound when $\mathcal{G}_t(v)$ is an ellipsoid.

Lemma 5 *Let* $(m_t^*, \tilde{x}_1^*, \ldots, \tilde{x}_{m_t^*}^*) \in \mathfrak{E}_{m,t}(1)$ *and* $k \in \{1,2\}$. *If* $v = (k, \ldots, k)$ *so that all points in* $\mathcal{E}_{m,t}$ *are assigned to* x_k^*, *then* $\operatorname{dist}(\mathcal{G}_t(v), \mathcal{E}) \leq (1/2)\operatorname{diam}(\mathcal{L}_k(y_k)) \leq t\sqrt{\kappa^*}$.

Proof We only show $k = 2$; $k = 1$ follows by a similar process. Recall $H_2 = Q_2 \Lambda_2 Q_2$ and let $\lambda_{2i} := \lambda_i(H_2)$, $i = 1, \ldots, q$. By Lemma 3, $y_2 \in B(x_2^*, t) = B(0_q, t)$. Consider

$$\begin{aligned} \max_{y_2 \in B(0_q,t)} \operatorname{diam}(\mathcal{L}_2(y_2)) &= \max_{y_2 \in B(0_q,t)} \operatorname{diam}(\{x \colon x^\intercal Q_2 \Lambda_2 Q_2^\intercal x \leq y_2^\intercal Q_2 \Lambda_2 Q_2^\intercal y_2\}) \\ &= \max_{a \in B(0_q,t)} \operatorname{diam}(\{w \colon w^\intercal \Lambda_2 w = a^\intercal \Lambda_2 a\}) \end{aligned} \tag{12}$$

where $w = Q_2^\intercal x$ and $a = Q_2^\intercal y_2$ rotate the axes in (12); since $y_2 \in B(0_q, t)$ is a ball, then $a \in B(0_q, t)$ also holds. The max in (12) occurs when $\|a\| = t$ and $a = (0, \ldots, 0, t)$, making the minor axis of the ellipsoid as large as possible. Then we have (12) = diam($\{w\colon \lambda_{21}w_1^2+\ldots+\lambda_{2q}w_q^2=\lambda_{2q}t^2\}$), so the largest value of w_1 is $t\sqrt{\lambda_{2q}/\lambda_{21}}$. Thus, the length of the semi-major axis is at most $t\sqrt{\kappa_2}$ [30, p. 358].

Now, consider the case that y_1, y_2 exist and $\mathcal{G}_t(v)$ is a nonempty intersection of two ellipsoids with separated centers. Then $\mathcal{G}_t(v)=\mathcal{L}_1(y_1)\cap\mathcal{L}_2(y_2)$, which implies that we seek an upper bound on $\mathrm{dist}(\mathcal{L}_1(y_1)\cap\mathcal{L}_2(y_2),\mathcal{E})$. We find an upper bound by enclosing $\mathcal{L}_1(y_1)\cap\mathcal{L}_2(y_2)$ in the smallest ball $B(z^*,\rho(v))$ such that the center is an efficient point, $z^*\in\mathcal{E}$. Then the squared radius of this enclosing ball is

$$\rho^2(v)=\min_{z\in\mathcal{E}}\ \max_{x\in\mathcal{L}_1(y_1)\cap\mathcal{L}_2(y_2)}\|z-x\|^2, \tag{C.1}$$

where the solution $z^*\in\mathcal{E}$ is the optimal center. By the max-min inequality [9, p. 238],

$$\begin{aligned}\mathrm{dist}^2(\mathcal{G}_t(v),\mathcal{E}) &\le \mathrm{dist}^2(\mathcal{L}_1(y_1)\cap\mathcal{L}_2(y_2),\mathcal{E})\\ &= \max_{x\in\mathcal{L}_1(y_1)\cap\mathcal{L}_2(y_2)}\ \min_{z\in\mathcal{E}}\|z-x\|^2\le\rho^2(v)\end{aligned} \tag{13}$$

for each $v\in\mathcal{V}$ such that $\mathcal{G}_t(v)$ is a nonempty intersection of two ellipsoids with separated centers. (Here again, both z^* and ρ depend on v, although we usually suppress this notational dependence unless it is helpful for clarity.)

Using Lemma 5 and (13), we formulate a problem whose optimal value is an upper bound on that of (H.2). Define the set of all assignments of points in $\mathcal{E}_{m,t}$ to minimizers that yields a distance larger than $t\sqrt{\kappa^*}$ (from Lemma 5) as

$$\mathcal{V}_t:=\{v\in\mathcal{V}\colon \mathrm{dist}(\mathcal{G}_t(v),\mathcal{E})>t\sqrt{\kappa^*}\}, \tag{14}$$

where $v\in\mathcal{V}_t$ implies $m_t^*\ge 2$ and $\mathcal{G}_t(v)$ is a nonempty intersection of ellipsoids. Then henceforth, we consider some version of the problem

$$\text{maximize}\ \ \max\left\{t\sqrt{\kappa^*},\max\nolimits_{v\in\mathcal{V}_t}\rho(v)\right\}\ \ \text{s.t.}\ \ (m_t^*,\tilde{x}_1^*,\ldots,\tilde{x}_{m_t^*}^*)\in\mathfrak{E}_{m,t}(2) \tag{H.3}$$

where the $m_t^*=1$ case is encompassed by the $t\sqrt{\kappa^*}$ term via Lemma 5. In Figure 1, notice that $\mathcal{V}_t=\{v\in\mathcal{V}\colon v=(1,2)\}$.

3.4 Closed-Form Upper Bounds from the Chebyshev Center Problem

To solve the optimization problem (H.3), ideally, we would find a closed-form solution for the enclosing ball problem in (C.1) that holds for each fixed value of $(m_t^*,\tilde{x}_1^*,\ldots,\tilde{x}_{m_t^*}^*)\in\mathfrak{E}_{m,t}(2)$, $v\in\mathcal{V}_t$. Toward this end, we note the similarity of the problem in (C.1) to the Chebyshev center problem, which is the problem of finding the minimum-radius ball enclosing the intersection of two q-balls [5,6,34] or two q-ellipsoids [7,11]. We write the Chebyshev center problem as

$$\rho_{\mathrm{C}}^2=\min_{x_{\mathrm{C}}}\ \max_{x\in\mathcal{L}_1(y_1)\cap\mathcal{L}_2(y_2)}\|x_{\mathrm{C}}-x\|^2, \tag{C.2}$$

where x_C^* is the optimal center and ρ_C^2 is the squared radius. Our (C.1) differs from (C.2) only in the requirement that our center must be an efficient point. Thus, (C.2) is a relaxation of (C.1), and $\rho_C^2 \le \rho^2$.

Despite (C.2) being a relaxation to our (C.1), the solution to (C.2) nevertheless provides significant insight into solving (C.1). If the level sets are spherical, $x_C^* \in \mathcal{E}$, which implies that x_C^* also solves (C.1). If the level sets are ellipsoidal, we can approximate the optimal center and radius using a center point in the efficient set, which provides the desired upper bound. Since the two cases of spherical and ellipsoidal level sets require different approaches, we address them separately in parts two and three of the proof, §4 and §5, respectively.

4 Proof Part 2: Spherical Level Sets, $\kappa^* = 1$

In the special case of spherical level sets, we build upon the results in §3 to provide a proof for the least upper bound for $\kappa^* = 1$ stated in Theorem 1.

Restricting ourselves to $\kappa^* = 1$ allows us to simplify our notation and the optimization problem (H.3). First, since the intersection of the sublevel sets can be written as the intersection of q-balls, define the radii

$$r_1(v) := \|y_1(v) - x_1^*\|, \quad r_2(v) := \|y_2(v) - x_2^*\|,$$

where $r(v) = (r_1(v), r_2(v))$ depends on $v \in \mathcal{V}_t$. Then the intersection of sublevel sets can be rewritten as the intersection of two q-balls whose centers are the minimizers,

$$\mathcal{L}_1(y_1) \cap \mathcal{L}_2(y_2) = B(x_1^*, r_1) \cap B(x_2^*, r_2).$$

Using this notation, in this section, we consider a simplified version of the problem in (H.3), which we write as

$$\begin{aligned} &\text{maximize} \;\; \max\left\{ t, \max_{v\in\mathcal{V}_t} \left\{ \rho(r(v)) = \sqrt{\min_{z\in\mathcal{E}} \max_{x\in B(x_1^*,r_1)\cap B(x_2^*,r_2)} \|z-x\|^2} \right\} \right\} \\ &\quad\text{s.t.} \;\; (m_t^*, \tilde{x}_1^*, \ldots, \tilde{x}_{m_t^*}^*) \in \mathfrak{E}_{m,t}(2), \end{aligned} \tag{H.4}$$

where the inner problem of finding $\rho(r)$ is problem (C.1), written for $\kappa^* = 1$. In §4.1, we find a closed-form expression for $\rho(r)$ that holds for each $(m_t^*, \tilde{x}_1^*, \ldots, \tilde{x}_{m_t^*}^*) \in \mathfrak{E}_{m,t}(2)$ and $v \in \mathcal{V}_t$ by solving the Chebychev center problem in (C.2). Then, in §4.2, we demonstrate that no feasible decision variable values in (H.4) can improve upon the upper bound $\sqrt{t\ell + t^2}$, which implies the bound in Theorem 1 Part 1. Finally, in §4.3, we provide a feasible example that achieves $\mathbb{H}(\mathcal{E}, \mathcal{E}_m)$ arbitrarily close to the bound $\sqrt{t\ell + t^2}$, which implies that our bound is a least upper bound.

4.1 A Solution to the Chebyshev Center Problem for $\kappa^* = 1$

The results in [5,6] imply the smallest-radius ball enclosing $B(x_1^*, r_1) \cap B(x_2^*, r_2)$ in (C.2) has center $x_C^* = \beta^* x_1^* + (1-\beta^*) x_2^*$ and squared radius

$$\rho_C^2 = \|x_C^*\|^2 - \beta^* \left(\|x_1^*\|^2 - r_1^2\right) - (1-\beta^*)\left(\|x_2^*\|^2 - r_2^2\right), \tag{15}$$

where β^* solves the convex quadratic minimization problem

$$\min\left\{\|\beta x_1^* + (1-\beta)x_2^*\|^2 - \beta\left(\|x_1^*\|^2 - r_1^2\right) - (1-\beta)\left(\|x_2^*\|^2 - r_2^2\right) : 0 \le \beta \le 1\right\}. \quad (16)$$

Since $x_C^* \in \mathcal{E}$ by (2), the solution to (C.2) also solves (C.1). Therefore, the smallest enclosing ball with its center in $\mathcal{E}$ has center $z^* = x_C^*$ and squared radius $\rho^2(r) = \rho_C^2$.

To simplify these expressions for use in our optimization problem, recall that the second minimizer is the origin, $x_2^* = 0_q$ and $\|x_1^*\| = \ell$. Then we can write (16) as $\min\left\{\beta^2\ell^2 - \beta\left(\ell^2 + r_2^2 - r_1^2\right) + r_2^2 : 0 \le \beta \le 1\right\}$, and solving for β^* yields

$$\beta^* = (\ell^2 + r_2^2 - r_1^2)/(2\ell^2), \quad \beta^* \in [0,1] \text{ if } \ell^2 + r_2^2 \ge r_1^2 \text{ and } \ell^2 + r_1^2 \ge r_2^2. \quad (17)$$

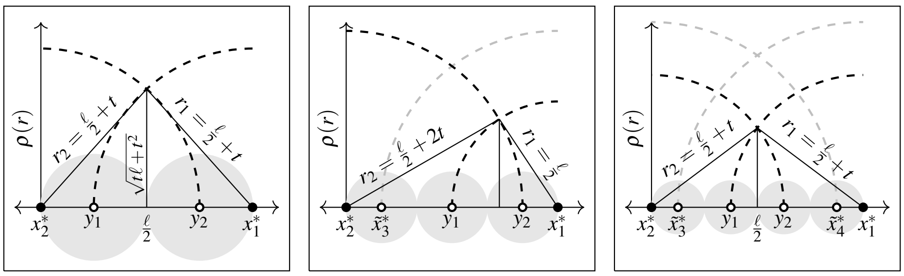
Using β^* in (15), algebraic simplifications yield $\rho^2(r) = r_2^2 - (\ell\beta^*)^2$. Thus, we use the following expression in the objective function of (H.4):

$$\rho^2(r) = r_2^2 - \left(\frac{\ell^2 + r_2^2 - r_1^2}{2\ell}\right)^2 = \|y_2\|^2 - \left(\frac{\ell^2 + \|y_2\|^2 - \|y_1 - x_1^*\|^2}{2\ell}\right)^2. \quad (18)$$

4.2 An Upper Bound Across All Possible Discretizations, $\kappa^* = 1$

We now demonstrate $\mathbb{H}(\mathcal{E}, \mathcal{E}_m) \le \sqrt{t\ell + t^2}$ by showing that $\sqrt{t\ell + t^2}$ is an upper bound on the objective function of (H.4) for all feasible decision variable values; that is, for all $(m_t^*, \tilde{x}_1^*, \ldots, \tilde{x}_{m_t^*}^*) \in \mathfrak{E}_{m,t}(2)$ and all $v \in \mathcal{V}_t$. To begin, suppose that in (H.4), we magically configure the points in $\mathcal{E}_{m,t}$ and their assignments to minimizers through the selection of $v \in \mathcal{V}_t$ so that $r^* = (r_1^*, r_2^*) := (\ell/2 + t, \ell/2 + t)$, which implies $\rho(r^*) = \sqrt{t\ell + t^2}$ from (18). Figures 2(a) and 2(c) demonstrate such configurations of points when $t = \ell/(2m_t^*)$; notice $\ell \le 2tm_t^*$ is necessary for $\text{dist}(\mathcal{E}, \mathcal{E}_{m,t}) \le t$ (thus, yielding Corollary 2). Figure 2(b) does not achieve the proposed bound.

For a contradiction to the upper bound in Theorem 1 Part 1, suppose that there exists a feasible number and configuration of points in $\mathcal{E}_{m,t}$ and an assignment to



(a) $m_t^* = 2, t = \ell/4$ (b) $m_t^* = 3, t = \ell/6$ (c) $m_t^* = 4, t = \ell/8$

Figure 2 The figure shows example configurations for points in $\mathcal{E}_{m,t}$ when $t = \ell/(2m_t^*)$ and the origin is $x_2^* = 0$. The gray t-radius balls around each $y_1, y_2, \tilde{x}_3^*$, and $\tilde{x}_4^*$ should appear as intervals but are shown in two dimensions for greater visibility. Only (a) and (c) demonstrate the bound; $\rho(r)$ in (b) is smaller.

the minimizers through $v \in \mathcal{V}_t$ such that $\mathbb{H}(\mathcal{E}, \mathcal{E}_m) > \sqrt{t\ell + t^2}$. Since $v \in \mathcal{V}_t$, then $r_1 + r_2 > \ell$. Further, for $\beta^* \in [0,1]$ from (17), we have

$$\nabla\rho(r) = (\partial\rho(r)/\partial r_1, \partial\rho(r)/\partial r_2)^\mathsf{T} = (r_1\beta^*/\rho(r), r_2(1-\beta^*)/\rho(r))^\mathsf{T} \geq (0,0)^\mathsf{T}$$

with strict inequality holding for $\ell^2 + r_2^2 > r_1^2$, $\ell^2 + r_1^2 > r_2^2$. Since we know $\rho^2(r^*) = t\ell + t^2$ and, based on the gradient information, $\rho^2(r)$ is elementwise nondecreasing for all feasible r, then at least one radius must be larger than $\ell/2 + t$. Without loss of generality, let it be r_2. Then there exists $\varepsilon > 0$ such that $r_2 = \ell/2 + \varepsilon + t$, which implies $\|y_2\| = \ell/2 + \varepsilon + t$. To be feasible, there must exist $\tilde{x}_j^* \in \mathcal{E}_{m,t}$ with $\|\tilde{x}_j^*\| < \|y_2\|$ and such that $B(\tilde{x}_j^*, t)$ covers the point $z_0 \in \mathcal{E}$ where $\|z_0\| = \ell/2 + \varepsilon$. Since $\|\tilde{x}_j^*\| < \|y_2\|$, $\tilde{x}_j^*$ must be assigned to x_1^* through v. To make r_1 as large as possible, set $\|\tilde{x}_j^*\| = \ell/2 + \varepsilon - t$, so that $r_1 = \ell/2 + t - \varepsilon$. However, algebraic simplification implies $\rho(r)^2 = t(t+\ell) - (2\varepsilon)^2(t/\ell)(t/\ell + 1) < t(t+\ell)$, which is a contradiction.

4.3 An Example that Demonstrates the $\kappa^* = 1$ Bound is a Least Upper Bound

From §4.2, no feasible configuration can achieve $\mathbb{H}(\mathcal{E}, \mathcal{E}_m) > \sqrt{t\ell + t^2}$; in this section, we provide an example such that $\mathbb{H}(\mathcal{E}, \mathcal{E}_m)$ is arbitrarily close to the bound.

Let $q = 2$ and $\mathcal{X}$ be a square, with minimizers x_1^*, x_2^* and efficient set $\mathcal{E}$ shown in Figure 3. In most of $\mathcal{X}$, we choose $\mathcal{X}_m$ on the grid in Example 1; this grid determines t. But, we do not grid for points near $\mathcal{E}$. First, we place points $\tilde{x}_1^*, \tilde{x}_2^* \in \mathcal{E}$ symmetrically around the center of $\mathcal{E}$ as in Figure 2(a), except we reduce the distance between them to $\|\tilde{x}_1^* - \tilde{x}_2^*\| < 2(t - \varepsilon)$ for $\varepsilon \in \big(0, (t\ell)/(2t+\ell)\big)$. Thus, $r_1 = r_2 = \ell/2 + t - \varepsilon$ and $\rho^2 = (\ell/2 + t - \varepsilon)^2 - (\ell/2)^2 = (t-\varepsilon)\ell + (t-\varepsilon)^2 > t^2$. Second, we introduce non-grid points to preserve t. In particular, we place five dominated points just outside $\mathcal{L}_1(\tilde{x}_1^*) \cap \mathcal{L}_2(\tilde{x}_2^*)$ to ensure its interior is covered. These two steps are key: since $B(\tilde{x}_1^*, t) \cap B(\tilde{x}_2^*, t)$ has positive Lebesgue measure in q dimensions, we can place points on or near $\mathrm{bd}(\mathcal{L}_1(\tilde{x}_1^*) \cap \mathcal{L}_2(\tilde{x}_2^*))$ so that $\mathcal{L}_1(\tilde{x}_1^*) \cap \mathcal{L}_2(\tilde{x}_2^*)$ is covered by the union of the

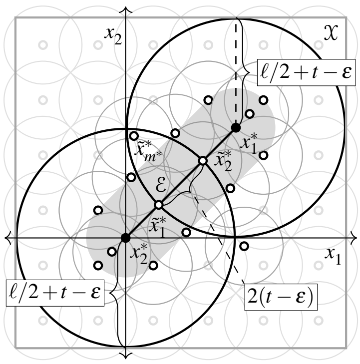


Figure 3 The figure shows a feasible example with $\mathbb{H}(\mathcal{E}, \mathcal{E}_m)$ arbitrarily close to the bound: for small $\varepsilon, \delta > 0$, $\mathbb{H}(\mathcal{E}, \mathcal{E}_m) = \mathrm{dist}(\tilde{x}_{m^*}^*, \mathcal{E}) = \sqrt{(t-\varepsilon)\ell + (t-\varepsilon)^2} - \delta$. $B(\mathcal{E}, t)$ is shaded in gray. The minimizers are represented by solid black dots. Points in $\mathcal{X}_m$ that do not necessarily lie on the grid are represented by dots with a black outline and white center.

points' t-radius balls, satisfying Lemma 4. (The example in Figure 2(a), for which $\|y_1 - y_2\| = 2t$, is infeasible for $q \geq 2$ by this reasoning.)

Now given $\delta \in (0, \rho - t)$, place $\tilde{x}^*_{m^*} \in \text{int}(\mathcal{L}_1(\tilde{x}^*_1) \cap \mathcal{L}_2(\tilde{x}^*_2))$ so that $\text{dist}(\tilde{x}^*_{m^*}, \mathcal{E}) = \rho - \delta > t$. As shown in Figure 3, the image of $\tilde{x}^*_{m^*}$ is non-dominated; thus, $\tilde{x}^*_{m^*} \in \mathcal{E}_m$. (Recall $m^* = |\mathcal{E}_m| \leq m^*_t$.) Now the Hausdorff distance is $\mathbb{H}(\mathcal{E}, \mathcal{E}_m) = \text{dist}(\mathcal{E}_m, \mathcal{E}) = \text{dist}(\tilde{x}^*_{m^*}, \mathcal{E}) = \sqrt{(t-\varepsilon)\ell + (t-\varepsilon)^2} - \delta$, which is arbitrarily close to the bound.

5 Proof Part 3: Ellipsoidal Level Sets, $\kappa^* \geq 1$

In this section, we build on the results in §3 and the intuition from the example in §4.3 to provide a proof for the general big O result in the second part of Theorem 1. Recall that our goal is to solve the optimization problem (H.3) from §3.3, where ρ is the radius of the smallest ball enclosing the intersection of ellipsoidal sublevel sets whose center is an efficient point. Unfortunately, when $\kappa^* > 1$, we cannot write a closed-form expression for the Chebyshev center x^*_{C} or the radius ρ_{C} in (C.2), as we did for $\kappa^* = 1$ in §4.1. Instead, we use the results in [7, 11] to formulate an approximation to the Chebyshev center that lies in the efficient set. This approximate Chebyshev center yields a radius ρ_{A} which is an upper bound on the desired radius ρ from (C.1). Then, we consider the problem

$$\text{maximize} \quad \{t\sqrt{\kappa^*},\ \max_{v \in \mathcal{V}_t} \rho_A(v)\} \quad \text{s.t.} \quad (m^*_t, \tilde{x}^*_1, \ldots, \tilde{x}^*_{m^*_t}) \in \mathfrak{E}_{m,t}(2) \tag{H.5}$$

where the optimal value of (H.5) is an upper bound on the optimal value of (H.3).

We derive an expression for $\rho_{\text{A}}(v)$ in §5.1. Then, in §5.2 we provide a sequence of lemmas that culminate in the proof of the big O result. Throughout this section, we require notation for the location where the level sets of y_1 and y_2 intersect $\mathcal{E}$. To define these points, for $k = 1, 2$, define the boundary of the sublevel set

$$\mathcal{J}_k(y_k) := \text{bd}(\mathcal{L}_k(y_k)),$$

and let E_k be the set of intersection points, $E_k := \mathcal{J}_k(y_k) \cap \mathcal{E}$. If $E_k \neq \emptyset$, then the structure of $\mathcal{E}$ implies E_k must be a singleton [4]. In this case, we say $\tilde{y}_k$ exists and define $\tilde{y}_k$ as the point in E_k. Otherwise, if $E_k = \emptyset$, then $\tilde{y}_k$ does not exist.

5.1 An Approximate Solution to the Chebyshev Center Problem for $\kappa^* \geq 1$

Given feasible decision variable values in (H.5), that is, $(m^*_t, \tilde{x}^*_1, \ldots, \tilde{x}^*_{m^*_t}) \in \mathfrak{E}_{m,t}(2)$ and $v \in \mathcal{V}_t$, in this section, we use the relaxed Chebyshev center (RCC) formulation from [7, 11] to determine an approximate Chebyshev center and an upper bound on ρ. The RCC, which we denote as z_{R}, is an efficient point for which the inequality $\rho^2_{\text{C}} \leq \rho^2_{\text{R}} := \max_{x \in \mathcal{L}_1(y_1) \cap \mathcal{L}_2(y_2)} \|z_{\text{R}} - x\|^2$ holds. Since $z_{\text{R}} \in \mathcal{E}$, we also have

$$\rho^2 = \min_{z \in \mathcal{E}} \max_{x \in \mathcal{L}_1(y_1) \cap \mathcal{L}_2(y_2)} \|z - x\|^2 \leq \max_{x \in \mathcal{L}_1(y_1) \cap \mathcal{L}_2(y_2)} \|z_{\text{R}} - x\|^2 = \rho^2_{\text{R}}. \tag{19}$$

However, we cannot calculate z_{R} in closed form; it is characterized as the solution to a semidefinite program [7, 11]. Thus, in the following sections, we find a closed-form upper bound on ρ^2_{R} which we use as the upper bound ρ^2_{A} in (H.5).

5.1.1 A Characterization of the RCC

First, we present the characterization of z_{R} and ρ_R^2 from [11]. For consistency with the notation in [11], recall that $x_2^* = 0_q$ and write the sublevel sets as

$$\mathcal{L}_1(y_1) = \left\{x\colon x^\intercal H_1 x + 2(-H_1 x_1^*)^\intercal x + \left(x_1^{*\intercal} H_1 x_1^* - (y_1 - x_1^*)^\intercal H_1 (y_1 - x_1^*)\right) \le 0\right\},$$
$$\mathcal{L}_2(y_2) = \left\{x\colon x^\intercal H_2 x - y_2^\intercal H_2 y_2 \le 0\right\}.$$

Then [11, Theorem III.1, p. 1390] implies that $z_{\mathrm{R}} = (\beta_1^* H_1 + \beta_2^* H_2)^{-1} \beta_1^* H_1 x_1^*$ where (β_1^*, β_2^*) is the solution to

$$\begin{aligned} \text{minimize} \quad & \{g(\beta_1,\beta_2) := \beta_1 (H_1 x_1^*)^\intercal (\beta_1 H_1 + \beta_2 H_2)^{-1} \beta_1 H_1 x_1^* + \beta_2 y_2^\intercal H_2 y_2 \\ & \qquad - \beta_1 (x_1^{*\intercal} H_1 x_1^* - (y_1 - x_1^*)^\intercal H_1 (y_1 - x_1^*))\} \\ \text{s.t.} \quad & \beta_1 H_1 + \beta_2 H_2 - I_q \succeq 0, \quad \beta_1 \ge 0, \beta_2 \ge 0. \end{aligned} \tag{20}$$

Further, $\rho_{\mathrm{R}}^2 = g(\beta_1^*, \beta_2^*)$, see [7, §3, pp. 611–614]. From (2), we see that $z_{\mathrm{R}} \in \mathcal{E}$.

5.1.2 An Approximation to the RCC

Since (19) implies $\rho^2 \le \rho_{\mathrm{R}}^2 = g(\beta_1^*, \beta_2^*) \le g(\beta_1, \beta_2)$ for suboptimal and feasible (β_1, β_2), we seek a closed-form upper bound ρ_{A}^2 by finding feasible values of (β_1, β_2). To find such values, we modify (20). First, reduce the feasible set to consider only $\beta_1 > 0$ and set $\alpha := \beta_2/\beta_1 \ge 0$. Let $z_{\mathrm{A}}(\alpha) = (H_1 + (\beta_2/\beta_1) H_2)^{-1} H_1 x_1^* = (H_1 + \alpha H_2)^{-1} H_1 x_1^*$, and define $\tilde{g}(\alpha) := g(\beta_1, \beta_2)/\beta_1$ so that

$$\tilde{g}(\alpha) = \alpha y_2^\intercal H_2 y_2 - x_1^{*\intercal} H_1 (x_1^* - z_{\mathrm{A}}(\alpha)) + (y_1 - x_1^*)^\intercal H_1 (y_1 - x_1^*).$$

Then consider the new problem

$$\text{minimize} \quad \beta_1 \tilde{g}(\alpha) \quad \text{s.t.} \quad H_1 + \alpha H_2 - I_q/\beta_1 \succeq 0,\ \alpha \ge 0,\ \beta_1 > 0. \tag{21}$$

We use (21) to find our upper bound ρ_{A}^2 as follows. First, we find a feasible α by setting $z_{\mathrm{A}}(\alpha)$ equal to a convenient point in the efficient set. We call our selected value α_{A}. Then, plugging α_{A} into (21), we find a value of $\beta_1 > 0$ for which the positive semidefinite constraint in (21) holds; call this value β_{A}. Then we obtain a bound ρ_{A}^2 such that

$$\rho^2 \le \rho_{\mathrm{R}}^2 = g(\beta_1^*, \beta_2^*) \le \beta_{\mathrm{A}} \tilde{g}(\alpha_{\mathrm{A}}) \le \rho_{\mathrm{A}}^2. \tag{22}$$

We now solve for α_{A} by setting $z_{\mathrm{A}}(\alpha) = z_0$ for some $z_0 \in \mathcal{E}, z_0 \ne 0_q$. Re-arranging terms yields $\alpha H_2 z_0 = H_1 x_1^* - H_1 z_0$. To get α alone, pre-multiply both sides by $z_0^\intercal$,

$$\alpha = \frac{z_0^\intercal H_1 x_1^* - z_0^\intercal H_1 x^*}{z_0^\intercal H_2 z_0} = \frac{z_0^\intercal H_1 (x_1^* - z_0)}{z_0^\intercal H_2 z_0}.$$

If the level set of y_2 on objective 2 intersects with $\mathcal{E}$ (as it does in Figure 1(c)), we select the intersection point $\tilde{y}_2$ as z_0. Then, for indicator variable $\mathbb{I}$, select

$$\alpha_{\mathrm{A}} := \frac{\tilde{y}_2^\intercal H_1 (x_1^* - \tilde{y}_2) \mathbb{I}\{E_2 \ne \emptyset\}}{\tilde{y}_2^\intercal H_2 \tilde{y}_2} = \frac{\langle \tilde{y}_2, -\nabla f_1(\tilde{y}_2)\rangle \mathbb{I}\{E_2 \ne \emptyset\}}{2(f_2(\tilde{y}_2) - b_2)}.$$

By the definition of $\tilde{y}_2$, we have $\tilde{y}_2^\intercal H_2 \tilde{y}_2 = y_2^\intercal H_2 y_2$. Plugging into $\tilde{g}(\alpha)$, if $E_2 \neq \emptyset$,

$$\begin{aligned}\tilde{g}(\alpha_{\mathrm{A}}) &= \alpha_{\mathrm{A}} y_2^\intercal H_2 y_2 - x_1^{*\intercal} H_1 (x_1^* - z_{\mathrm{A}}(\alpha_{\mathrm{A}})) + (y_1 - x_1^*)^\intercal H_1 (y_1 - x_1^*) \\ &= \tilde{y}_2^\intercal H_1 (x_1^* - \tilde{y}_2) - x_1^{*\intercal} H_1 (x_1^* - \tilde{y}_2) + (y_1 - x_1^*)^\intercal H_1 (y_1 - x_1^*) \\ &= -(\tilde{y}_2 - x_1^*)^\intercal H_1 (\tilde{y}_2 - x_1^*) + (y_1 - x_1^*)^\intercal H_1 (y_1 - x_1^*) = 2(f_1(y_1) - f_1(\tilde{y}_2)).\end{aligned}$$

Then since $E_2 = \emptyset$ implies $\alpha_A = 0$ and $z_0 = x_1^*$, it follows that

$$\tilde{g}(\alpha_{\mathrm{A}}) = 2(f_1(y_1) - f_1(\tilde{y}_2))\mathbb{I}\{E_2 \neq \emptyset\} + 2\,(f_1(y_1) - b_1)\,\mathbb{I}\{E_2 = \emptyset\}. \tag{23}$$

Now, we use α_{A} in the constraint of (21) to find β_{A}. First, we seek $\beta_1 > 0$ such that $H_1 + \alpha_{\mathrm{A}} H_2 - I_q/\beta_1 \succeq 0$. Equivalently, we seek $\beta_1 > 0$ such that $w^\intercal(H_1 + \alpha_{\mathrm{A}} H_2 - I_q/\beta_1)w \geq 0$ for all $w \neq 0_q$, which implies

$$1/\beta_1 \leq R(H_1, w) + \alpha_{\mathrm{A}} R(H_2, w) \leq \max\{\lambda_q(H_1), \lambda_q(H_2)\}(1 + \alpha_{\mathrm{A}})$$

by (3), (4). Therefore, $\beta_1 \geq (\max\{\lambda_q(H_1), \lambda_q(H_2)\}(1 + \alpha_{\mathrm{A}}))^{-1}$, and we select

$$\beta_{\mathrm{A}} := (\max\{\lambda_q(H_1), \lambda_q(H_2)\})^{-1} \geq (\max\{\lambda_q(H_1), \lambda_q(H_2)\}(1 + \alpha_{\mathrm{A}}))^{-1}. \tag{24}$$

Combining the results in (22), (23), and (24), let

$$\rho_{\mathrm{A}}^2 := \frac{2|f_1(y_1) - f_1(\tilde{y}_2)|\mathbb{I}\{E_2 \neq \emptyset\} + 2\,(f_1(y_1) - b_1)\,\mathbb{I}\{E_2 = \emptyset\}}{\max\{\lambda_q(H_1), \lambda_q(H_2)\}}. \tag{25}$$

We remark here that if $\tilde{y}_1 \in \mathcal{E}$ and $\tilde{y}_2 \in \mathcal{E}$ exist, since $f_1(y_1) = f_1(\tilde{y}_1)$, then we can determine an upper bound on (25) by working only with these efficient points.

5.2 An Upper Bound Across All Possible Discretizations, $\kappa^* \geq 1$

We now demonstrate that $\mathbb{H}(\mathcal{E}, \mathcal{E}_m) = O(\sqrt{t})$ by finding an upper bound on ρ_{A} from (25) that holds for all $(m_t^*, \tilde{x}_1^*, \ldots, \tilde{x}_{m_t^*}^*) \in \mathfrak{E}_{m,t}(2)$ and all $v \in \mathcal{V}_t$. There are three key steps to obtaining such an upper bound: First, in §5.2.1, we determine conditions on the placement of y_1 and y_2 that imply infeasibility via Lemma 4. That is, we determine conditions on y_1 and y_2 such that there exists $x_0 \in \mathcal{X}$ with $B(x_0, t) \subset \mathrm{int}(\mathcal{L}_1(y_1) \cap \mathcal{L}_2(y_2))$. In this case, x_0 cannot be covered by the t-radius balls of other points in $\mathcal{E}_{m,t}$ without violating the conditions in Lemma 4, which implies the arrangement is infeasible. (Intuition for these conditions comes from the example in §4.3.) Second, in §5.2.2, we determine the conditions for the existence of $\tilde{y}_1 \in \mathcal{E}$ and $\tilde{y}_2 \in \mathcal{E}$ and provide several implications of their non-existence. Third, in §5.2.3, we derive upper bounds on the terms in the numerator of ρ_{A}^2 in (25). After completing these three key steps, we combine the results to show $\mathbb{H}(\mathcal{E}, \mathcal{E}_m) = O(\sqrt{t})$ in §5.2.4.

Finally, we remark here that this section relies on supporting Lemmas 12–16, which we include in §A. Also, for points in the efficient set $z = z(\beta), \beta \in [0, 1]$, define $z_k := z(\beta_k)$ for any subscript k. From (2), recall that $z(0) = x_2^* = 0_q$ and $z(1) = x_1^*$.

5.2.1 Conditions that Imply Infeasible Decision Variables for (H.5)

First, in Lemma 6, we determine conditions on y_1 and y_2 such that the resulting arrangement of points is infeasible in (H.5). Simply stated, arrangements that allow y_1 and y_2 to be too far apart from each other are infeasible.

Lemma 6 *Let $v \in \mathcal{V}_t$, and let $\eta_0 \in [1,\infty)$ be a constant that depends on κ^*. If $y_1, y_2 \in \mathcal{E}_{m,t}$ are placed such that there exist $z_1, z_2 \in \mathcal{E}$ with $\beta_1 < \beta_2$, $\mathcal{L}_1(z_1) \cap \mathcal{L}_2(z_2) \subseteq \mathcal{L}_1(y_1) \cap \mathcal{L}_2(y_2)$, and $\|z_1 - z_2\| > 2t\kappa^*\eta_0$, then $(m_t^*, \tilde{x}_1^*, \ldots, \tilde{x}_{m_t^*}^*) \notin \mathfrak{E}_{m,t}(2)$.*

Proof For a contradiction, suppose the postulates hold and $(m_t^*, \tilde{x}_1^*, \ldots, \tilde{x}_{m_t^*}^*) \in \mathfrak{E}_{m,t}(2)$. Lemma 12 gives conditions on the positions of efficient points such that their level sets intersect; since $\beta_1 < \beta_2$, Lemma 12 implies $\mathcal{L}_1(z_1) \cap \mathcal{L}_2(z_2) \neq \emptyset$. Then since z_1 and z_2 are too far apart, that is, $\|z_1 - z_2\| > 2t\kappa^*\eta_0$, then Lemma 15 implies that there exists a point $x_0 \in \text{conv}(\{z_1, z_2\})$ such that $B(x_0, t) \subset \text{int}(\mathcal{L}_1(z_1) \cap \mathcal{L}_2(z_2))$. (In Lemma 15, $\kappa^* = 1$ implies $\eta_0 = 1$.) Then $x_0 \notin B(y_1, t) \cup B(y_2, t)$, thus, Lemma 3 implies $m_t^* \geq 3$. To cover x_0 with the t-radius ball of some point $\tilde{x}_i^* \in \mathcal{E}_{m,t} \setminus \{y_1, y_2\}$, we require $\tilde{x}_i^* \in \text{int}(\mathcal{L}_1(y_1) \cap \mathcal{L}_2(y_2))$, which contradicts Lemma 4.

5.2.2 On the Existence of $\tilde{y}_1$ and $\tilde{y}_2$

Recall that if they exist, $\tilde{y}_1, \tilde{y}_2 \in \mathcal{E}$ are the locations where the level sets of y_1, y_2 intersect $\mathcal{E}$. The expression for ρ_{A} depends on the existence of $\tilde{y}_2$; further, if $\tilde{y}_1$ exists, we can use it in place of y_1 in ρ_{A}. Therefore in this section, we state a sequence of lemmas that provide conditions for the existence of $\tilde{y}_1$ and $\tilde{y}_2$, as well as some key implications of their existence and nonexistence.

Lemma 7 *Let $(m_t^*, \tilde{x}_1^*, \ldots, \tilde{x}_{m_t^*}^*) \in \mathfrak{E}_{m,t}(2)$, $v \in \mathcal{V}_t$. If $t < \ell/(2\kappa^*\eta_0)$, then at least one of $\tilde{y}_1$ or $\tilde{y}_2$ exists.*

Proof For a contradiction, suppose the postulates hold and $\tilde{y}_1$ and $\tilde{y}_2$ do not exist. Since $v \in \mathcal{V}_t$, it must be the case that $\mathcal{E} \subset \text{int}(\mathcal{L}_1(y_1) \cap \mathcal{L}_2(y_2))$ (see also [4]). Then $\mathcal{L}_1(x_2^*) \subset \mathcal{L}_1(y_1)$ and $\mathcal{L}_2(x_1^*) \subset \mathcal{L}_2(y_2)$, hence $\mathcal{L}_1(x_2^*) \cap \mathcal{L}_2(x_1^*) \subset \mathcal{L}_1(y_1) \cap \mathcal{L}_2(y_2)$. Then letting $z_1 = x_2^*$ and $z_2 = x_1^*$ in the postulates of Lemma 6, we have $\ell = \|x_2^* - x_1^*\| > 2t\kappa^*\eta_0$. Thus, the postulates of Lemma 6 hold, which contradicts feasibility.

Lemma 8 *Let $(m_t^*, \tilde{x}_1^*, \ldots, \tilde{x}_{m_t^*}^*) \in \mathfrak{E}_{m,t}(2)$, $v \in \mathcal{V}_t$, and $k, k' \in \{1,2\}$, $k \neq k'$.*
1. *If $\tilde{y}_k$ exists and $\|\tilde{y}_k - x_k^*\| > 2t\kappa^*\eta_0$, then $\tilde{y}_{k'}$ exists.*
2. *If $\tilde{y}_{k'}$ does not exist and $\tilde{y}_k$ exists, then $\|\tilde{y}_k - x_k^*\| \leq 2t\kappa^*\eta_0$.*

Proof For a contradiction to Part 1, suppose the postulates hold and $\tilde{y}_{k'}$ does not exist, hence $\mathcal{L}_{k'}(x_k^*) \subset \mathcal{L}_{k'}(y_{k'})$. Since $\tilde{y}_k \in \mathcal{E}$ exists, $\mathcal{L}_k(\tilde{y}_k) = \mathcal{L}_k(y_k)$. Thus, $\mathcal{L}_k(\tilde{y}_k) \cap \mathcal{L}_{k'}(x_k^*) \subset \mathcal{L}_k(y_k) \cap \mathcal{L}_{k'}(y_{k'})$. Then letting $z_k = \tilde{y}_k$ and $z_{k'} = x_k^*$ in the postulates of Lemma 6, $v \in \mathcal{V}_t$ implies $\beta_1 < \beta_2$. Since $\|\tilde{y}_k - x_k^*\| > 2t\kappa^*\eta_0$, the postulates of Lemma 6 hold, which contradicts feasibility. Part 2 follows from the contrapositive of Part 1.

Lemma 9 *Let $(m_t^*, \tilde{x}_1^*, \ldots, \tilde{x}_{m_t^*}^*) \in \mathfrak{E}_{m,t}(2)$, $v \in \mathcal{V}_t$, and let $k, k' \in \{1,2\}$, $k \neq k'$. If $\tilde{y}_{k'}$ does not exist, then $\exists t' > 0$ and a constant $\eta_1 \in [1,\infty)$, both dependent on κ^*, such that for all $t \leq t'$, $\|y_{k'} - x_k^*\| \leq t\eta_1$.*

Proof Suppose the postulates hold. Since $\tilde{y}_{k'}$ does not exist, $\mathcal{E} \subset \mathcal{L}_{k'}(x_k^*) \subset \mathcal{L}_{k'}(y_{k'})$. Let $z_{k'} := \arg\inf\{z \in \mathcal{E} \colon \|y_{k'} - z\| \leq t\}$ be the nearest efficient point to $y_{k'}$, which is within distance t by Lemma 3. Thus, $z_{k'} \in \mathcal{E} \cap \mathcal{L}_{k'}(x_k^*)$, while $y_{k'} \in B(\mathcal{E},t) \cap \mathcal{L}^c_{k'}(x_k^*)$. Since $\|y_{k'} - z_{k'}\| \leq t$, then as $t \to 0$, $y_{k'} \to z_{k'}$ implies $y_{k'} \to x_k^*$ and $z_{k'} \to x_k^*$.

Now, we find an upper bound on $\|y_{k'} - x_k^*\|$. Let $\tau(x_k^*)$ be a vector tangent to the efficient set at x_k^*, defined as $\tau(x_k^*) := -H_k^{-1}\nabla f_{k'}(x_k^*)/\|H_k^{-1}\nabla f_{k'}(x_k^*)\|$. Figure 4 depicts an example efficient set and its tangent vectors; see §A.2 for more details. The hyperplane for which $\tau(x_k^*)$ is the normal, called $\mathcal{T}_{\mathcal{E}}(x_k^*)$, creates two half spaces; let $\mathcal{N}(x_k^*) \supset \mathcal{E}$ be the closed half-space containing $\mathcal{E}$, and let $\mathcal{N}^c(x_k^*)$ be its complement. Then by the definitions of $\tau(x_k^*)$ and $\mathcal{N}(x_k^*)$, for all $x \in B(\mathcal{E},t) \cap \mathcal{N}^c(x_k^*)$, $\mathrm{dist}(x,\mathcal{E}) = \|x - x_k^*\| \leq t$. Therefore if $y_{k'} \in B(\mathcal{E},t) \cap \mathcal{N}^c(x_k^*)$, then $\|y_{k'} - x_k^*\| \leq t$.

Next, let $y_{k'} \in \mathcal{N}(x_k^*)$, and consider whether $\tilde{y}_{k'}$ is in the set of descent directions on objective k' at x_k^*, $\mathcal{H}_{k'}(x_k^*) = \{x \in \mathbb{R}^q \colon \langle x, \nabla f_{k'}(x_k^*)\rangle < 0\}$, where $\mathcal{E} \subset \mathcal{L}_{k'}(x_k^*) \subset \mathcal{H}_{k'}(x_k^*)$. In what follows, we consider two cases: first, when $y_{k'} \in \mathcal{H}^c_{k'}(x_k^*)$, and second, when $y_{k'} \in \mathcal{H}_{k'}(x_k^*)$. In both cases, since $z_{k'} \in \mathcal{E} \cap \mathcal{L}_{k'}(x_k^*)$, then $z_{k'} \in \mathcal{H}_{k'}(x_k^*)$.

First, let $y_{k'} \in B(\mathcal{E},t) \cap \mathcal{L}^c_{k'}(x_k^*) \cap \mathcal{H}^c_{k'}(x_k^*)$, and let $\mathcal{T}_{\mathcal{L}}(x_k^*) = \{x \colon \langle x, \nabla f_{k'}(x_k^*)\rangle = 0\}$ be the hyperplane normal to the gradient of objective k' at x_k^*, which is tangent to $\mathcal{L}_{k'}(x_k^*)$ at x_k^*. Let $x_{\mathcal{L}}(z_{k'})$ be the orthogonal projection of $z_{k'}$ onto the plane $\mathcal{T}_{\mathcal{L}}(x_k^*)$, and let θ_t be the angle between $z_{k'} - x_k^*$ and $-\nabla f_{k'}(x_k^*)$. Then

$$\|z_{k'} - x_k^*\| \sin(\pi/2 - \theta_t) = \|z_{k'} - x_k^*\| \cos\theta_t = \|z_{k'} - x_{\mathcal{L}}(z_{k'})\| \leq \|y_{k'} - z_{k'}\| \leq t. \quad (26)$$

By Lemma 13, $\exists c_0 \in [1,\infty)$, dependent on κ^*, such that $\cos\theta_t \geq c_0^{-1} > 0$ for all t. Intuitively, this result holds because under Assumption 1, $z_{k'} - x_k^*$ is strictly a descent direction on objective k' ($\kappa^* = 1$ implies $c_0 = 1$). Then using (26),

$$\|y_{k'} - x_k^*\| \leq \|y_{k'} - z_{k'}\| + \|z_{k'} - x_k^*\| \leq t(1 + 1/\cos\theta_t) \leq t(c_0 + 1). \quad (27)$$

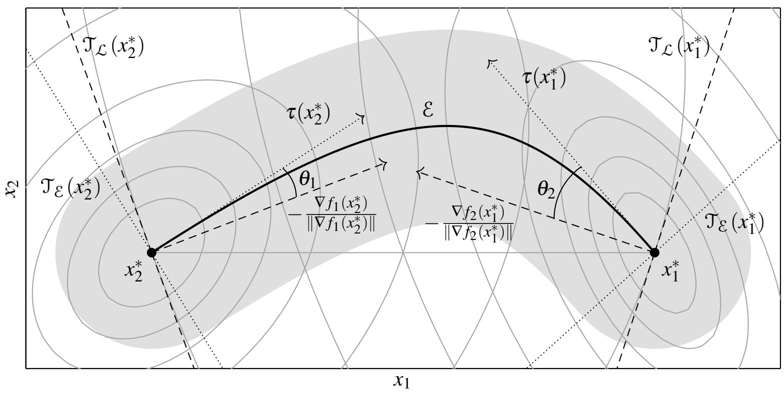


Figure 4 For $\mathcal{X} \subset \mathbb{R}^2$, the figure shows the efficient set $\mathcal{E} = \{z(\beta), \beta \in [0,1]\}$, the set $B(\mathcal{E},t)$ in light gray, contour plots of the objective functions $f_1(x)$ and $f_2(x)$ in dark gray, the unit tangent vectors τ at the minimizers, and the unit negative gradient vectors $-\nabla f_1, -\nabla f_2$ at the minimizers. The figure also shows the planes $\mathcal{T}_{\mathcal{L}}$ and $\mathcal{T}_{\mathcal{E}}$. Using notation from §A, the angle θ_1 is between $\tau(x_2^*)$ and $-\nabla f_1(x_2^*)$, and the angle θ_2 is between $\tau(x_1^*)$ and $-\nabla f_2(x_1^*)$.

Finally, let $y_{k'} \in B(\mathcal{E},t) \cap \mathcal{L}^c_{k'}(x^*_k) \cap \mathcal{H}_{k'}(x^*_k)$; to achieve a maximum distance, it is sufficient to consider $y_{k'} \in B(\mathcal{E},t) \cap \mathcal{J}_{k'}(x^*_k)$. Since $z_{k'} - x_{\mathcal{L}}(z_{k'})$ is parallel to $-\nabla f_{k'}(x^*_k)$, the angle between $x^*_k - z_{k'}$ and $x_{\mathcal{L}}(z_{k'}) - z_{k'}$ is θ_t, and $\|z_{k'} - x_{\mathcal{L}}(z_{k'})\| \sin\theta_t = \|x_{\mathcal{L}}(z_{k'}) - x^*_k\| \cos\theta_t$. (If $\kappa^* = 1$, then $\theta_t = 0$.) Then by Lemma 13,

$$\begin{aligned} \|y_{k'} - x^*_k\| &\le \|y_{k'} - z_{k'}\| + \|z_{k'} + x_{\mathcal{L}}(z_{k'})\| + \|x_{\mathcal{L}}(z_{k'}) - x^*_k\| \\ &\le t + \|z_{k'} - x_{\mathcal{L}}(z_{k'})\|(1 + \sin\theta_t/\cos\theta_t) \le t + (c_0+1)\|z_{k'} - x_{\mathcal{L}}(z_{k'})\|. \end{aligned} \tag{28}$$

Let γ_t denote the angle between $y_{k'} - x_{\mathcal{L}}(z_{k'})$ and the tangent plane $\mathcal{T}_{\mathcal{L}}(x^*_k)$. Using the law of cosines and the triangle inequality,

$$\begin{aligned} \|z_{k'} - x_{\mathcal{L}}(z_{k'})\|^2 &\le \|z_{k'} - x_{\mathcal{L}}(z_{k'})\|^2 + \|y_{k'} - x_{\mathcal{L}}(z_{k'})\|^2 \\ &= \|y_{k'} - z_{k'}\|^2 + 2\|z_{k'} - x_{\mathcal{L}}(z_{k'})\|\|y_{k'} - x_{\mathcal{L}}(z_{k'})\|\cos(\pi/2 - \gamma_t) \\ &\le t^2 + 2\|z_{k'} - x_{\mathcal{L}}(z_{k'})\|(\|z_{k'} - x_{\mathcal{L}}(z_{k'})\| + t)\cos(\pi/2 - \gamma_t) \\ &\le t^2 + 2\|z_{k'} - x_{\mathcal{L}}(z_{k'})\|^2 \sin\gamma_t + 2t(t + \|y_{k'} - x_{\mathcal{L}}(z_{k'})\|)\sin\gamma_t \\ &\le 3t^2 + 2\|z_{k'} - x_{\mathcal{L}}(z_{k'})\|^2 \sin\gamma_t + 2t\,\mathrm{dist}(y_{k'}, \mathcal{T}_{\mathcal{L}}(x^*_k)). \end{aligned} \tag{29}$$

By Lemma 16, $\exists t'_1 > 0$ and constant $c_1 \in [1,\infty)$, dependent on κ^*, such that if $t \le t'_1$, $\mathrm{dist}(y_{k'}, \mathcal{T}_{\mathcal{L}}(x^*_k)) \le tc_1$. Intuitively, this result holds because $\mathcal{T}_{\mathcal{L}}(x^*_k)$ is a linear approximation to the ellipsoid $\mathcal{L}_{k'}(x^*_k)$ at x^*_k. Using this result in (29), since $\gamma_t \to 0$, then for t small enough we have

$$\|z_{k'} - x_{\mathcal{L}}(z_{k'})\| \le 3t\sqrt{1+c_1}. \tag{30}$$

Using (30) in (28) and ensuring (27) holds, for all t small enough and constant $\eta_1 := (1 + (c_0+1)3\sqrt{1+c_1})$, then $\|y_{k'} - x^*_k\| \le t(1 + (c_0+1)3\sqrt{1+c_1}) \le t\eta_1$.

5.2.3 Upper Bounds on the Terms in the Numerator of ρ^2_{A}

Given the conditions for feasibility and the lemmas regarding $\tilde{y}_1$ and $\tilde{y}_2$ in the previous sections, we are now ready to provide upper bounds on the terms in the numerator of ρ^2_{A} in (25). We begin with Lemma 10, which provides a bound when $\tilde{y}_2$ does not exist. Then, Lemma 11 provides a bound when $\tilde{y}_2$ exists.

Lemma 10 *Let $(m^*_t, \tilde{x}^*_1, \ldots, \tilde{x}^*_{m^*_t}) \in \mathfrak{E}_{m,t}(2)$, $v \in \mathcal{V}_t$. If $t < \ell/(2\kappa^*\eta_0)$ and $\tilde{y}_2$ does not exist, then $2(f_1(y_1) - b_1)\mathbb{I}\{E_2 = \emptyset\} \le \lambda_q(H_1)(2t\kappa^*\eta_0)^2$.*

Proof Since $t < \ell/(2\kappa^*\eta_0)$ and $\tilde{y}_2$ does not exist, $\mathbb{I}\{E_2 = \emptyset\} = 1$ in (25) and by Lemma 7, $\tilde{y}_1$ exists. Since $f_1(y_1) = f_1(\tilde{y}_1)$, henceforth, we use $\tilde{y}_1 \in \mathcal{E}$. From (4),

$$2(f_1(\tilde{y}_1) - b_1) \le \lambda_q(H_1)\|\tilde{y}_1 - x^*_1\|^2, \tag{31}$$

thus, we seek an upper bound on $\|\tilde{y}_1 - x^*_1\|$. Since $\tilde{y}_1$ exists and $\tilde{y}_2$ does not exist, Lemma 8 implies $\|\tilde{y}_1 - x^*_1\| \le 2t\kappa^*\eta_0$. Using this bound in (31) yields the result.

Lemma 11 *Let $(m^*_t, \tilde{x}^*_1, \ldots, \tilde{x}^*_{m^*_t}) \in \mathfrak{E}_{m,t}(2)$, $v \in \mathcal{V}_t$, and let $\eta_2 := 2\kappa^*\eta_0 + \eta_1$. If $t < t'$ and $\tilde{y}_2$ exists, then $2|f_1(y_1) - f_1(\tilde{y}_2)|\mathbb{I}\{E_2 \neq \emptyset\} \le \lambda_q(H_1)\left(2\ell_a t\eta_2 + (t\eta_2)^2\right)$.*

Proof Suppose $\tilde{y}_2$ exists. Thus, $\mathbb{I}\{E_2 \neq \emptyset\} = 1$ in (25). Then use (4) and (5) to obtain

$$\begin{aligned} 2|f_1(y_1) - f_1(\tilde{y}_2)| &\leq 2\lambda_q(H_1)\left(\|\tilde{y}_2 - x_1^*\|\|\tilde{y}_2 - y_1\| + (1/2)\|\tilde{y}_2 - y_1\|^2\right) \\ &\leq \lambda_q(H_1)\left(2\ell_a\|\tilde{y}_2 - y_1\| + \|\tilde{y}_2 - y_1\|^2\right). \end{aligned} \quad (32)$$

Since $\tilde{y}_2$ exists, we use Lemma 8 to prove the lemma in two parts: first when $\tilde{y}_1$ does not exist, and second when $\|\tilde{y}_2 - x_2^*\| > 2t\kappa^*\eta_0$, which implies $\tilde{y}_1$ exists.

First, suppose $\tilde{y}_1$ does not exist, so that we seek an upper bound on $\|\tilde{y}_2 - y_1\|$ in (32). Since $\tilde{y}_1$ does not exist, Lemma 8 implies $\|\tilde{y}_2 - x_2^*\| \leq 2t\kappa^*\eta_0$, and Lemma 9 implies that if $t \leq t'$, then $\|y_1 - x_2^*\| \leq t\eta_1$. Therefore, for all $t \leq t'$,

$$\|\tilde{y}_2 - y_1\| \leq \|\tilde{y}_2 - x_2^*\| + \|y_1 - x_2^*\| \leq 2t\kappa^*\eta_0 + t\eta_1 = t(2\kappa^*\eta_0 + \eta_1) = t\eta_2. \quad (33)$$

Now suppose $\|\tilde{y}_2 - x_2^*\| > 2t\kappa^*\eta_0$, so that $\tilde{y}_1$ exists. Then $f_1(y_1) = f_1(\tilde{y}_1)$. Updating (32), we seek an upper bound on $\|\tilde{y}_2 - \tilde{y}_1\|$. Since feasibility holds, letting $z_1 = \tilde{y}_1$ and $z_2 = \tilde{y}_2$, the contrapositive of Lemma 6 implies $\|\tilde{y}_1 - \tilde{y}_2\| \leq 2t\kappa^*\eta_0$. Since this bound is smaller than (33), the result follows by using (33) in (32).

5.2.4 *An Upper Bound on the Optimal Value of* (H.5)

We now demonstrate $\mathbb{H}(\mathcal{E}, \mathcal{E}_m) = O(\sqrt{t})$. From (25) and using Lemmas 10 and 11, for all $(m_t^*, \tilde{x}_1^*, \ldots, \tilde{x}_{m_t^*}^*) \in \mathfrak{E}_{m,t}(2)$, $v \in \mathcal{V}_t$, and all $t < \min\{t', \ell/(2\kappa^*\eta_0)\}$,

$$\begin{aligned} \rho_{\mathrm{A}}^2(v) &= \frac{2|f_1(y_1) - f_1(\tilde{y}_2)|\mathbb{I}\{E_2 \neq \emptyset\} + 2\left(f_1(y_1) - b_1\right)\mathbb{I}\{E_2 = \emptyset\}}{\max\{\lambda_q(H_1), \lambda_q(H_2)\}} \\ &\leq \frac{\lambda_q(H_1)\left(2\ell_a t\eta_2 + (t\eta_2)^2\right)\mathbb{I}\{E_2 \neq \emptyset\} + \lambda_q(H_1)(2t\kappa^*\eta_0)^2\mathbb{I}\{E_2 = \emptyset\}}{\max\{\lambda_q(H_1), \lambda_q(H_2)\}} \\ &\leq 2\ell_a t\eta_2 + (t\eta_2)^2 = t(2\ell_a\eta_2) + t^2\eta_2^2. \end{aligned}$$

Thus, for all choices of feasible decision variables in (H.5), if $t < \min\{t', \ell/(2\kappa^*\eta_0)\}$, then $\mathbb{H}(\mathcal{E}, \mathcal{E}_m) \leq \max\left\{t\sqrt{\kappa^*}, \sqrt{t(2\ell_a\eta_2) + t^2\eta_2^2}\right\}$, and $\mathbb{H}(\mathcal{E}, \mathcal{E}_m) = O(\sqrt{t})$ as $t \to 0$.

6 Concluding Remarks

We show that for convex quadratic bi-objective optimization, the Hausdorff distances between the efficient set and its discretization in the decision space, and between the Pareto set and its discretization in the objective space, are $O(\sqrt{t})$ as $t \to 0$. Our results quantify the convergence rate discrepancy between the coverage error and the Hausdorff distance in this context: the coverage error converges as $O(t)$, while the Hausdorff distance converges at the slower rate of $O(\sqrt{t})$.

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A Supporting Results: Gradients, Tangents, and Inscribed q-Balls, $\kappa^* \geq 1$

In this section, we provide several supporting results regarding the geometry of Problem (Q) under Assumption 1. These results involve viewing the efficient set as a differentiable curve (§A.1); determining bounds on the angles between tangent vectors, gradients, and efficient points (§A.2); and whether a q-ball of certain radius fits inside an ellipsoid (§A.3).

A.1 The Efficient Set as a Differentiable Curve

To begin, from (2), the efficient set can be traced as the curve (see also [32])

$$z(\beta) := (\beta H_1 + (1-\beta)H_2)^{-1}(\beta H_1 x_1^* + (1-\beta)H_2 x_2^*), \quad \beta \in [0,1], \tag{34}$$

where $z(0) = x_2^* = 0_q$, $z(1) = x_1^*$, and for simplicity, define the shorthand notation $z_n := z(\beta_n)$ for any subscript n.

By definition, for any efficient point, there exists no direction in which we can improve both objectives at the same time. For each $k,k' \in \{1,2\}, k \neq k'$, let the set of descent directions on objective k at $z(\beta) \in \mathcal{E}$ be denoted by

$$\mathcal{H}_k(z(\beta)) := \{x \in \mathbb{R}^q : \langle x, \nabla f_k(z(\beta))\rangle < 0\} = \{x \in \mathbb{R}^q : \langle x, H_k(z(\beta) - x_k^*)\rangle < 0\},$$

where $\mathcal{L}_k(z(\beta)) \subset \mathcal{H}_k(z(\beta))$; the lack of common descent directions for efficient points implies $\mathcal{H}_k(z(\beta)) \cap \mathcal{H}_{k'}(z(\beta)) = \emptyset$. This fact leads to the following Lemma 12 regarding the relative positions of the points in the efficient set and whether or not their level sets intersect.

Lemma 12 *For $\beta_1, \beta_2 \in [0,1]$, $\mathcal{L}_1(z(\beta_1)) \cap \mathcal{L}_2(z(\beta_2)) = \emptyset$ if and only if $\beta_2 < \beta_1$.*

Proof First, we prove that $\mathcal{L}_1(z(\beta_1)) \cap \mathcal{L}_2(z(\beta_2)) = \emptyset$ implies $\beta_2 < \beta_1$ by contrapositive. Suppose $\beta_2 \geq \beta_1$. If $\beta_1 = \beta_2$, then $z_1 = z_2 \in \mathcal{L}_1(z_1) \cap \mathcal{L}_2(z_1)$. If $\beta_2 > \beta_1$, then $f_2(z_1) < f_2(z_2)$ and $f_1(z_1) > f_1(z_2)$. Then for $\beta_3 \in (\beta_1, \beta_2)$, $f_2(z_1) < f_2(z_3) < f_2(z_2)$ and $f_1(z_2) < f_1(z_3) < f_1(z_1)$, hence $z_3 \in \mathcal{L}_1(z_1) \cap \mathcal{L}_2(z_2)$. For the other direction, suppose $\beta_2 < \beta_1$. Then for $\beta_3 \in (\beta_2, \beta_1)$, $f_2(z_2) < f_2(z_3) < f_2(z_1)$ and $f_1(z_2) > f_1(z_3) > f_1(z_1)$. Since $z_3 \in \mathcal{E}$, $\mathcal{H}_1(z_3) \cap \mathcal{H}_2(z_3) = \emptyset$. Since $\mathcal{L}_2(z_2) \subset \mathcal{H}_2(z_3)$ and $\mathcal{L}_1(z_1) \subset \mathcal{H}_1(z_3)$, then $\mathcal{L}_1(z_1) \cap \mathcal{L}_2(z_2) = \emptyset$.

Further, for each $k \in \{1,2\}, k \neq k'$, [32] and the parameterization in (34) imply that for any $\beta \in [0,1]$,

$$\begin{aligned} -\beta \nabla f_1(z(\beta)) &= -\beta H_1(z(\beta) - x_1^*) \\ &= (1-\beta)H_2(z(\beta) - x_2^*) = (1-\beta)\nabla f_2(z(\beta)), \end{aligned} \tag{35}$$

and, for $H_\beta := \beta H_1 + (1-\beta)H_2$, the curve $z(\beta)$ is a differentiable curve in $\mathbb{R}^q$ with tangent vector (defined with respect to β)

$$z'(\beta) := \lim_{\delta \to 0} \frac{z(\beta+\delta) - z(\beta)}{\delta} = \left(\frac{\partial z_1(\beta)}{\partial \beta}, \dots, \frac{\partial z_q(\beta)}{\partial \beta}\right)^\intercal$$

$$
\begin{aligned}
&= H_\beta^{-1}(H_1x_1^* - H_2x_2^*) - H_\beta^{-1}(H_1 - H_2)H_\beta^{-1}(\beta H_1 x_1^* + (1-\beta)H_2x_2^*)\\
&= H_\beta^{-1}[H_2(z(\beta) - x_2^*) - H_1(z(\beta) - x_1^*)]\\
&= H_\beta^{-1}[\nabla f_2(z(\beta)) - \nabla f_1(z(\beta))] \text{ for } \beta \in [0,1], \quad (36)\\
&= \beta^{-1}H_\beta^{-1}\nabla f_2(z(\beta)) = -(1-\beta)^{-1}H_\beta^{-1}\nabla f_1(z(\beta)) \text{ for } \beta \in (0,1). \quad (37)
\end{aligned}
$$

Importantly, from (36) and (37), it follows that $\|z'(\beta)\| > 0$ for all $\beta \in [0,1]$.

A.2 Angles: Tangent Vectors and Gradients

In this section, we provide results on the angles between points in the efficient set, tangent vectors, and gradients. For $k, k' \in \{1,2\}, k \neq k'$, consider a point in the efficient set, $z_k = z(\beta_k) \in \mathcal{E}$, $\beta_k \in [0,1]$ and any sequence of efficient points

$$\{z_{k',n} = z(\beta_{k',n}), n \geq 0\} \quad (38)$$

such that $z_{k',n} \to z_k$ as $n \to \infty$, $z_{k',n} \neq z_k$ for all n, $\beta_{k',n} \in [0,1]$ for all n, and $z_{k',n}$ approaches z_k exclusively from one side; that is, $\beta_1 < \beta_{2,n}$ or $\beta_{1,n} > \beta_2$ for all n. Define the tangent vector at z_k with respect to the direction of approach as

$$\tau(z_k) := \lim_{n\to\infty} \frac{z_{k',n} - z_k}{\|z_{k'n} - z_k\|}, \quad (39)$$

and notice that $\|\tau(z_k)\| = 1$. Then from (36) and (37),

$$\tau(z_1) = \frac{z'(\beta_1)}{\|z'(\beta_1)\|} \quad \text{and} \quad \tau(z_2) = \frac{-z'(\beta_2)}{\|z'(\beta_2)\|}. \quad (40)$$

Therefore if $\beta_k \in (0,1)$, $-\nabla f_k(z_k)/\|\nabla f_k(z_k)\| = \nabla f_{k'}(z_k)/\|\nabla f_{k'}(z_k)\|$ and from (37) and (40), we have

$$\tau(z_k) = \frac{-H_\beta^{-1}\nabla f_k(z_k)}{\|H_\beta^{-1}\nabla f_k(z_k)\|} = \frac{H_\beta^{-1}\nabla f_{k'}(z_k)}{\|H_\beta^{-1}\nabla f_{k'}(z_k)\|}, \quad \beta_k \in (0,1). \quad (41)$$

If $\beta_k \in \{0,1\}$, so that $z_1 = x_2^*$ or $z_2 = x_1^*$, then from (40) and using $\beta = 0$ and $\beta = 1$ in (36), we have

$$\tau(z_k) = \tau(x_{k'}^*) = \frac{-H_{k'}^{-1}\nabla f_k(x_{k'}^*)}{\|H_{k'}^{-1}\nabla f_k(x_{k'}^*)\|}, \quad \beta_k \in \{0,1\}. \quad (42)$$

Figure 4 demonstrates the relative positions of the gradient and tangent vectors at the minimizers for an example in which $q = 2$. For general decision space dimension q, [4, Theorem 3.2, p. 369] asserts that $\mathcal{E} = \{z(\beta) \colon \beta \in [0,1]\}$ forms a finite arc of a hyperbola whose extreme points are the minimizers x_1^* and x_2^*.

Next, we present Lemma 13 which provides results on the angles between the efficient points, gradients, and tangent vectors at the minimizers.

Lemma 13 *Let $k, k' \in \{1,2\}, k \neq k'$ and let $\{z_{k',n} = z(\beta_{k',n}), n \geq 0\}$ be the sequence of efficient points converging to $z_k = z(\beta_k)$ defined in (38). The following hold:*

1. *If $\theta_{k,n}$ is the angle between $z_{k',n} - z_k$ and $-\nabla f_k(z_k)$, then $\cos\theta_{k,n} > 0$ for all n.*
2. *If θ_k is the angle between $\tau(z_k)$ and $-\nabla f_k(z_k)$, then $\cos\theta_k \geq (\kappa(H_\beta))^{-1}$.*
3. *There exists a constant $c_0 \in [1,\infty)$ such that $\cos\theta_{k,n} \geq c_0^{-1} > 0$ for all n.*

Proof *Part 1:* First, by the definition of $\{z_{k',n}, n \geq 0\}$ in (38), we have $z_{k',n} \notin \{z_k, x_k^*\}$ for all n and $z_k \neq x_k^*$. Then this result holds because under Assumption 1, for $k, k' \in \{1,2\}, k' \neq k$, $z_{k'} - z_k$ is strictly a descent direction on objective k by (34). Therefore, it must make an acute angle with the steepest descent direction. More formally,

$$\cos\theta_{k,n} = \frac{\langle z_{k',n} - z_k, -\nabla f_k(z_k)\rangle}{\|z_{k',n} - z_k\|\|\nabla f_k(z_k)\|} = \frac{(z_k - z_{k',n})^\intercal H_k(z_k - x_k^*)}{\|z_{k',n} - z_k\|\|\nabla f_k(z_k)\|} > 0 \text{ for all } n. \quad (43)$$

Part 2: From (43), by continuity of the relevant functions, (41), (42), and the facts that for a square positive definite matrix $H = A^\intercal A$, $\|Hx\| \leq \|H\|\|x\|$ for vector x, $\|H\| = \|A\|^2 = \|Q\Lambda Q^\intercal\| = \lambda_q(H)$, and $\|H^{-1}\| = \|Q\Lambda^{-1}Q^\intercal\| = \lambda_q(H^{-1}) = 1/\lambda_1(H)$,

$$\begin{aligned}\cos\theta_k = \lim_{n\to\infty}\cos\theta_{k,n} &= \frac{\langle \tau(z_k), -\nabla f_k(z_k)\rangle}{\|\nabla f_k(z_k)\|} = \frac{\|\nabla f_k(z_k)\|}{\|H_\beta^{-1}\nabla f_k(z_k)\|}\frac{\langle H_\beta^{-1}\nabla f_k(z_k), \nabla f_k(z_k)\rangle}{\|\nabla f_k(z_k)\|^2}\\ &= \frac{\|\nabla f_k(z_k)\|}{\|H_\beta^{-1}\nabla f_k(z_k)\|}R(H_\beta^{-1}, \nabla f_k(z_k)) \geq \frac{1}{\|H_\beta^{-1}\|}R(H_\beta^{-1}, \nabla f_k(z_k))\\ &\geq \lambda_1(H_\beta)/\lambda_q(H_\beta) \geq 1/\kappa(H_\beta) > 0.\end{aligned}$$

Part 3: This part follows from the first two.

A.3 Osculating Spheres and Inscribed q-Balls

Finally, we complete the supporting results with Lemmas 14–16. First, Lemma 14 gives conditions on the lengths of the major axes of the sublevel sets for a t-radius ball touching the boundary to fit inside the sublevel set. Then, Lemma 15 gives conditions on the required distance between two efficient points for a t-radius ball to fit inside the intersection of the relevant sublevel sets. Finally, Lemma 16 uses several inscribed balls to determine the distance between a point in $B(\mathcal{E}, t) \cap \mathcal{J}_{k'}(x_k^*)$ and the tangent vector $\mathcal{T}_{\mathcal{L}}(x_k^*)$ for small enough t.

Lemma 14 *Let $k \in \{1,2\}$ and $x_0 \in \mathcal{X}$. If $\text{diam}(\mathcal{L}_k(x_0)) \geq 2\kappa^* t$ and $x_1 \in \mathcal{L}_k(x_0)$ is such that $x_1 = x_0 - t(\nabla f_k(x_0)/\|\nabla f_k(x_0)\|)$, then $B(x_1, t) \subset \mathcal{L}_k(x_0)$.*

Proof (Sketch) We show only $k = 2$; $k = 1$ holds by a similar process. Let $x_0 \in \mathcal{X}$, $\lambda_{2i} := \lambda_i(H_2)$, $i = 1, \ldots, q$, and notice $x_1 = x_0 - t(H_2 x_0/\|H_2 x_0\|)$ is a step of length t from x_0 in the direction of steepest descent on objective 2. Thus, $x_0 \in \text{bd}(B(x_1, t))$. To simplify calculations, for $\mathcal{L}_2(x_0)$, apply the rotation $w = Q_2^\intercal x, a_0 = Q_2^\intercal x_0$ from the proof of Lemma 5. Define $\mathcal{L}_{2r}(a_0) := \{w\colon\ w^\intercal \Lambda_2 w \leq a_0^\intercal \Lambda_2 a_0\}$, $\mathcal{J}_{2r}(a_0) := \text{bd}(\mathcal{L}_{2r}(a_0))$. Now, we consider $a_1 = a_0 - t(\Lambda_2 a_0)/\|\Lambda_2 a_0\|$, where $a_0 \in \text{bd}(B(a_1, t))$.

To ensure $B(a_1,t) \subset \mathcal{L}_{2r}(a_0)$, the maximum curvature at any point on $\mathcal{J}_{2r}(a_0)$ must be less than or equal to that at any point on a t-radius sphere. The maximum curvature on an ellipse is achieved at a vertex along the major axis; see, e.g., [31]. To determine the maximum curvature at a point on $\mathcal{J}_{2r}(a_0)$, it is sufficient to consider the intersection of $\mathcal{J}_{2r}(a_0)$ with the w_1-w_q plane (see [18, p. 43], [15, p. 33]) and let $a_0 = (a_{01},0,\ldots,0)$, resulting in the ellipse $\{(w_1,w_q)\colon \lambda_{21}w_1^2 + \lambda_{2q}w_q^2 = \lambda_{21}a_{01}^2\} = \{(w_1,w_q)\colon w_1^2/a_{01}^2 + w_q^2/(a_{01}^2/\kappa_2) = 1\}$. Write this ellipse as the plane curve $g(s) = (a_{01}\cos(s), (a_{01}/\sqrt{\kappa_2})\sin(s))$. The length of the semi-major axis is a_{01}, the length of the semi-minor axis is $(a_{01}/\sqrt{\kappa_2})$, and the maximum curvature is $a_{01}/(a_{01}^2/\kappa_2) = \kappa_2/a_{01}$ at $s=0$ [31, p. 77]. Then for an osculating circle at $(a_{11},a_{1q}) = (a_{01}-t,0)$ having radius t, curvature $1/t$, and passing through $(a_{01},0)$ to fit inside the ellipse, we require $(1/t) \geq \kappa_2/a_{01}$, hence $a_{01} \geq t\kappa_2$. Now since $a_{01} = \mathrm{diam}(\mathcal{L}_{2r}(a_0))/2 = \mathrm{diam}(\mathcal{L}_2(x_0))/2 \geq t\kappa_2$, then $B(a_1,t) \subset \mathcal{L}_{2r}(a_0)$ and $B(x_1,t) \subset \mathcal{L}_2(x_0)$.

Lemma 15 *Let $z_1,z_2 \in \mathcal{E}, \beta_1,\beta_2 \in [0,1], \beta_1 < \beta_2$ be efficient points. There exists a constant $\eta_0 \in [1,\infty)$, dependent on κ^*, such that if $\|z_1 - z_2\| > 2t\kappa^*\eta_0$, then there exists $x_0 \in \mathrm{conv}(\{z_1,z_2\})$ such that $B(x_0,t) \subset \mathrm{int}(\mathcal{L}_1(z_1) \cap \mathcal{L}_2(z_2))$.*

Proof The main idea for the proof is that for each level set $\mathcal{L}_k(z_k)$, $k \in \{1,2\}$, we can use Lemma 14 to fit a (relatively large) osculating ball at z_k. Then we fit a smaller t-radius ball, centered at a point w_k along $\mathrm{conv}(\{z_1,z_2\})$, inside this larger osculating ball. As long as $\|z_1 - z_2\|$ is sufficiently large and the respective centers w_1 and w_2 of the t-radius balls are appropriately spaced along $\mathrm{conv}(\{z_1,z_2\})$, the lemma holds.

To begin, let the postulates hold. Since $\beta_1 < \beta_2$, $\mathcal{L}_1(z_1) \cap \mathcal{L}_2(z_2) \neq \emptyset$ by Lemma 12. To fit the osculating balls inside the level sets, first, for each $k \in \{1,2\}$, let x_k be the point resulting from a step of size $u_k(t)$ from z_k in the direction of steepest descent on objective k, so that

$$x_k = z_k - u_k(t)(\nabla f_k(z_k)/\|\nabla f_k(z_k)\|),$$

and $\|x_k - z_k\| = u_k(t)$. By Lemma 14, if

$$\mathrm{diam}(\mathcal{L}_k(z_k)) \geq 2\kappa^* u_k(t) \tag{44}$$

then $B(x_k,u_k(t)) \subset \mathcal{L}_k(z_k)$. Let w_k be the orthogonal projection of x_k onto the line segment $\mathrm{conv}(\{z_1,z_2\})$, and let ζ_k be the angle between $w_k - z_k, x_k - z_k$. Since ζ_1,ζ_2 are also the angles between $z_2 - z_1, -\nabla f_1(z_1)$ and between $z_1 - z_2, -\nabla f_2(z_2)$, respectively, the results in Lemma 13 apply. Thus, $\cos\zeta_k = \sin(\pi/2 - \zeta_k) \geq c_0^{-1} > 0$ implies $\zeta_k \in [0,\pi/2)$, and hence there exists $\tilde{c}_0 \in [1,\infty)$ such that $1 - \sin\zeta_k \geq (\tilde{c}_0)^{-1} > 0$ regardless of t and the choice of z_1,z_2. To ensure $B(w_k,t) \subset B(x_k,u_k(t))$, we require

$$u_k(t) - \|w_k - x_k\| = u_k(t) - \|x_k - z_k\|\sin\zeta_k = u_k(t)(1 - \sin\zeta_k) \geq t. \tag{45}$$

Finally, for each $k,k' \in \{1,2\}, k' \neq k$, by the definition of w_k, we have $\|z_k - w_k\| + \|w_k - z_{k'}\| = \|z_1 - z_2\|$. Then to ensure $\|z_k - w_k\| \leq \|w_k - z_{k'}\|$, we require

$$2\|z_k - w_k\| = 2\|x_k - z_k\|\cos\zeta_k = 2u_k(t)\cos\zeta_k \leq \|z_1 - z_2\|. \tag{46}$$

Combining the requirements of (44), (45), and (46), it follows that

$$\frac{t}{1-\sin\zeta_k} \le u_k(t) \le \min\left\{\frac{\operatorname{diam}(\mathcal{L}_k(z_k))}{2\kappa^*}, \frac{\|z_1-z_2\|}{2\cos\zeta_k}\right\}. \tag{47}$$

If we can find values of $u_1(t), u_2(t)$, and a requirement on the distance $\|z_1 - z_2\|$ such that (47) is satisfied, then for each $k, k' \in \{1,2\}, k' \neq k$, we have $B(w_k,t) \subseteq B(x_k, u_k(t)) \subset \mathcal{L}_k(z_k)$ and $\|z_1 - z_2\| = \|z_1 - w_1\| + \|w_1 - w_2\| + \|w_2 - z_2\|$, which implies that $B(w_k,t) \subset \mathcal{L}_{k'}(z_{k'})$. Since $\mathcal{L}_1(z_1) \cap \mathcal{L}_2(z_2)$ is convex, then $\operatorname{conv}(B(w_1,t) \cup B(w_2,t)) \subset \mathcal{L}_1(z_1) \cap \mathcal{L}_2(z_2)$. Thus, $\exists x_0 \in \operatorname{conv}(\{w_1, w_2\})$ such that the result holds.

Finally, we find $u_1(t), u_2(t)$, and a requirement on $\|z_1 - z_2\|$ such that (47) is satisfied. First, for each $k \in \{1,2\}$, $\beta_1 < \beta_2$ and the triangle inequality imply

$$\|z_1 - z_2\| \le \|z_1 - x_k^*\| + \|z_2 - x_k^*\| \le 2\|z_k - x_k^*\| \le \operatorname{diam}(\mathcal{L}_k(z_k)).$$

Since $\kappa^* \ge 1$ while $\cos\zeta_k \le 1$, and since $1 - \sin\zeta_k \ge (\tilde{c}_0)^{-1}$, we satisfy (47) if

$$\frac{t}{1-\sin\zeta_k} \le t\tilde{c}_0 \le u_k(t) \le \frac{\|z_1 - z_2\|}{2\kappa^*}. \tag{48}$$

Now select $u_k(t) = t\tilde{c}_0$ for each $k \in \{1,2\}$ and let $\eta_0 := \tilde{c}_0$. Since $\|z_1 - z_2\| > 2t\kappa^*\eta_0$, the result holds.

Lemma 16 *Let $k \in \{1,2\}$, and let y be a point in the intersection of the t-expansion of $\mathcal{E}$ with the boundary of the sublevel set $\mathcal{L}_{k'}(x_k^*)$; that is, $y \in B(\mathcal{E},t) \cap \mathcal{J}_{k'}(x_k^*)$. Then there exists $t_1' > 0$ and a constant $c_1 \in [1,\infty)$, both dependent on κ^*, such that for all $t \le t_1'$, $\operatorname{dist}(y, \mathcal{T}_{\mathcal{L}}(x_k^*)) \le tc_1$.*

Proof We find an upper bound on $\operatorname{dist}(y, \mathcal{T}_{\mathcal{L}}(x_k^*))$ by fitting a series of three balls inside the sublevel set $\mathcal{L}_{k'}(x_k^*)$. The first ball is the largest ball, constructed as follows. Take a step of size $\ell/\kappa_{k'}$ from x_k^* in the steepest descent direction,

$$x_1 = x_k^* - (\ell/\kappa_{k'})\nabla f_{k'}(x_k^*)/\|\nabla f_{k'}(x_k^*)\|.$$

Then Lemma 14 implies $B(x_1, \ell/\kappa_{k'}) \subset \mathcal{L}_{k'}(x_k^*)$. Next, we construct the second ball by taking a step $s_k(t) < \|x_1 - x_k^*\| = \ell/\kappa_{k'}$ from x_k^* in the steepest descent direction,

$$x_2 = x_k^* - s_k(t)\nabla f_{k'}(x_k^*)/\|\nabla f_{k'}(x_k^*)\|.$$

Let $z_3(x_2)$ be the efficient point such that x_2 is its orthogonal projection onto the line $\operatorname{conv}(\{x_1, x_k^*\})$; such a point exists for small enough $s_k(t)$. We choose $s_k(t)$ so that the third ball, $B(z_3(x_2),t)$, is a subset of the first ball. That is, we find $s_k(t)$ such that $B(z_3(x_2),t) \subset B(x_1, \ell/\kappa_{k'}) \subset \mathcal{L}_{k'}(x_k^*)$. To find $s_k(t)$, first, notice that if we ensure

$$\ell/\kappa_{k'} - \|z_3(x_2) - x_1\| \ge t, \tag{49}$$

it follows that $B(z_3(x_2),t) \subset B(x_1, \ell/\kappa_{k'})$. Let $\theta_{3,t}$ be the angle between $z_3(x_2) - x_k^*$ and $x_2 - x_k^*$. By Lemma 13, $\cos\theta_{3,t} \ge c_0^{-1}$ for all t. Then to ensure (49) holds, we need $s_k(t) > t$ such that $\|z_3(x_2) - x_1\|^2 = (\ell/\kappa_{k'} - s_k(t))^2 + (s_k(t)\tan\theta_{3,t})^2 \le (\ell/\kappa_{k'} - t)^2$; re-arranging terms implies

$$\frac{s_k(t)^2(1+\tan^2\theta_{3,t}) - t^2}{s_k(t) - t} = \frac{s_k(t)^2/\cos^2\theta_{3,t} - t^2}{s_k(t) - t} \le 2\ell/\kappa_{k'}. \tag{50}$$

Choose $s_k(t) = t(\kappa_{k'}+1)$ and plug into (50) to see that (49) holds for all

$$t \leq \frac{2\ell c_0^{-2}}{(\kappa^*+1)^2-1} \leq \frac{2\ell \cos^2\theta_{3,t}}{(\kappa_{k'}+1)^2-\cos^2\theta_{3,t}}.$$

Now for t small enough and $s_k(t) = t(\kappa_{k'}+1)$, we have $B(z_3(x_2),t) \subset B(x_1,\ell/\kappa_{k'}) \subset \mathcal{L}_{k'}(x_k^*)$. Let y_3 be any point in $B(z_3(x_2),t) \subset B(\mathcal{E},t)$ such that $\text{dist}(y_3,\mathcal{T}_{\mathcal{L}}(x_k^*)) \geq \text{dist}(y,\mathcal{T}_{\mathcal{L}}(x_k^*))$; such a point exists by construction. Now, the result holds since for all t small enough and $c_1 := (1+c_0(\kappa^*+1))$, we have

$$\begin{aligned}\text{dist}(y,\mathcal{T}_{\mathcal{L}}(x_k^*)) &\leq \text{dist}(y_3,\mathcal{T}_{\mathcal{L}}(x_k^*)) \leq t + \|z_3(x_2)-x_k^*\| \\ &\leq t + s_k(t)/\cos\theta_{3,t} \leq t + c_0(\kappa_{k'}+1)t \leq tc_1.\end{aligned}$$

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