Impulse and Momentum

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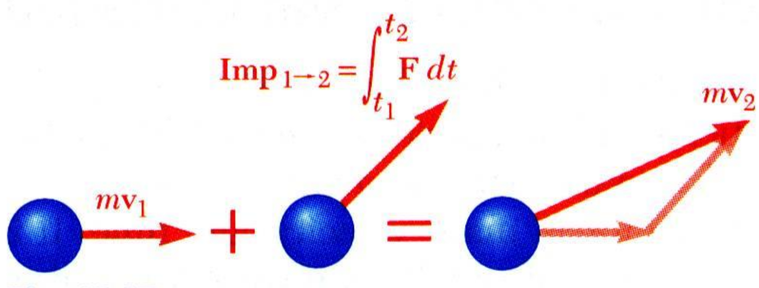
February 13, 2016

Principle of Impulse and Momentum

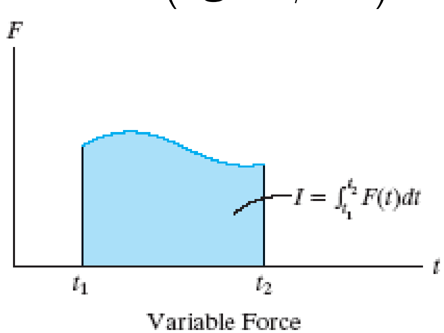


- Dimensions of the impulse of a force are

  *force * time.*

- Units for the impulse of a force are

$$\mathrm{N \cdot s} = \left(\mathrm{kg \cdot m/s^2}\right) \cdot \mathrm{s} = \mathrm{kg \cdot m/s}$$



- From Newton's second law,

$$\vec{F} = \frac{d}{dt}(m\vec{v}) \qquad m\vec{v} = \text{linear momentum}$$

$$\vec{F}dt = d(m\vec{v})$$

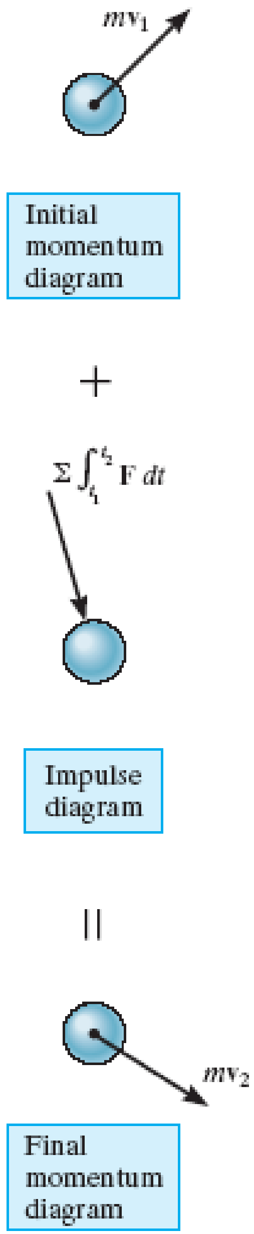
$$\int_{t_1}^{t_2} \vec{F}dt = m\vec{v}_2 - m\vec{v}_1$$

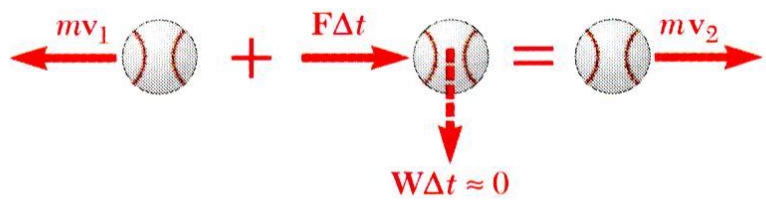
$$\int_{t_1}^{t_2} \vec{F}dt = \mathbf{Imp}_{1\to 2} = \text{impulse of the force } \vec{F}$$

$$m\vec{v}_1 + \mathbf{Imp}_{1\to 2} = m\vec{v}_2$$

- The final momentum of the particle can be obtained by adding vectorially its initial momentum and ***the impulse of the force during the time interval.***

Scalar Equations for a System of Particles





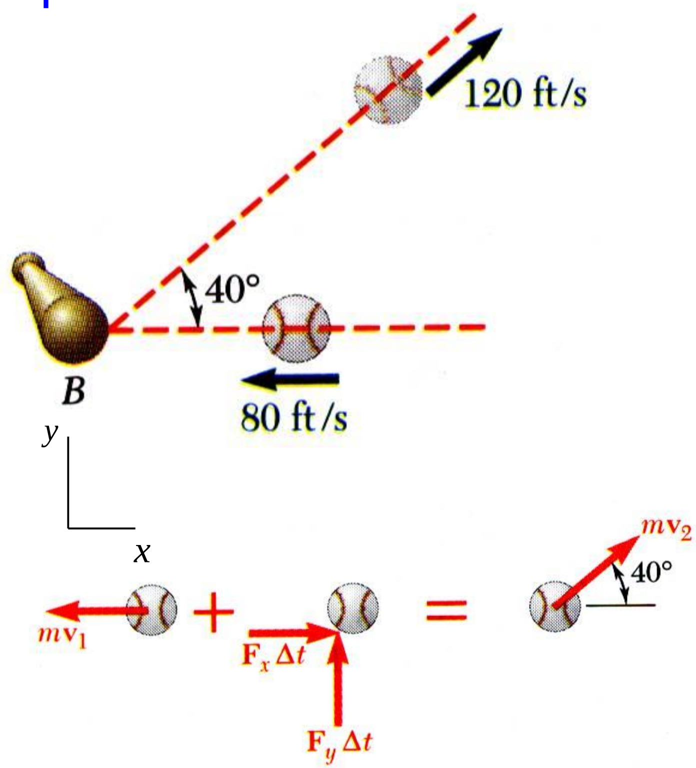
$$m(\upsilon_x)_1 + \sum \int_{t_1}^{t_2} F_x \, dt = m(\upsilon_x)_2$$

$$m(\upsilon_y)_1 + \sum \int_{t_1}^{t_2} F_y \, dt = m(\upsilon_y)_2$$

$$m(\upsilon_z)_1 + \sum \int_{t_1}^{t_2} F_z \, dt = m(\upsilon_z)_2$$

Sample Problem: Impact

Ball: 4 oz
Bat and ball
in contact for 0.015 s
Determine the average impulsive force



1 oz = 1/16 lbm

$$1\,lbf\,(=1\,lb)=1\,lbm\times\left(32.2\frac{ft}{\sec^2}\right)$$

Apply the principle of impulse and momentum in terms of horizontal and vertical component equations.

$$m\vec{v}_1+\mathbf{Imp}_{1\to 2}=m\vec{v}_2$$

x component equation:

$$-mv_1+F_x\Delta t=mv_2\cos 40°$$

$$-\frac{4/16}{32.2}(80)+F_x(0.15)=\frac{4/16}{32.2}(120\cos 40°)$$

$$F_x=89\,\text{lb}$$

y component equation:

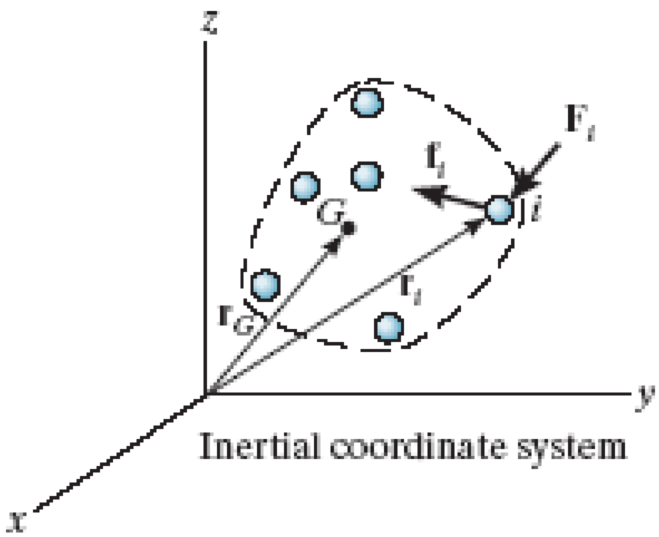
$$0+F_y\Delta t=mv_2\sin 40°$$

$$F_y(0.15)=\frac{4/16}{32.2}(120\cos 40°)$$

$$F_y=39.9\,\text{lb}$$

$$\vec{F}=(89\,\text{lb})\vec{i}+(39.9\,\text{lb})\vec{j},\quad F=97.5\,\text{lb}$$

Principle of Linear Impulse & Momentum: System of Particles



For a system of particles, the internal forces f_i between particles always occur in pairs with equal magnitude and opposite directions. *Thus the internal impulses sum to zero.*

The linear impulse and momentum equation for this system only includes the impulse of ***external*** forces.

$$\sum m_i(v_i)_1 + \sum \int_{t_1}^{t_2} F_i \, dt = \sum m_i(v_i)_2$$

Motion of the Center of Mass

For a system of particles, we can define a "fictitious" center of mass of an aggregate particle of mass m_{tot}, where m_{tot} is the sum ($\sum m_i$) of all the particles.

The position vector $r_G = (\sum m_i r_i)/m_{tot}$ describes the *motion of the center of mass.*

This system of particles then has an *aggregate velocity* of

$$v_G = (\sum m_i v_i)/m_{tot}.$$

The motion of this fictitious mass is based on motion of the center of mass for the system.

$$m_{tot}(v_G)_1 + \sum \int_{t_1}^{t_2} F_i \, dt = m_{tot}(v_G)_2$$

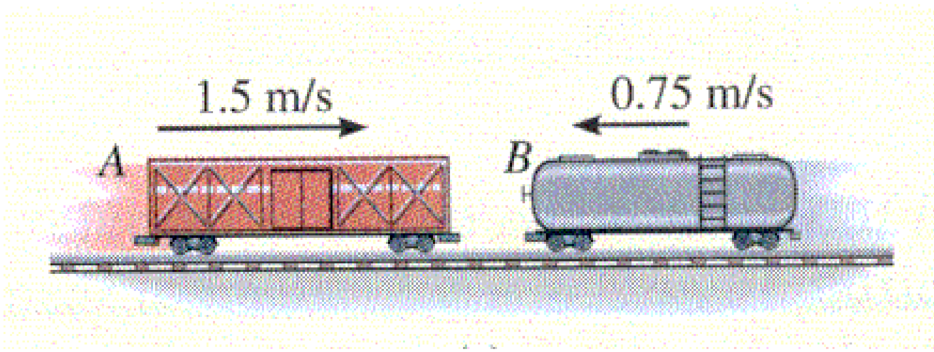
Conservation of Linear Momentum for a system of Particles

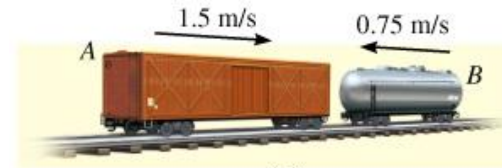
When the *sum of external impulses* acting on a system of objects is *zero*, the linear impulse-momentum equation simplifies to

$$\sum m_i (\mathrm{v}_i)_1 = \sum m_i (\mathrm{v}_i)_2$$

Conservation of linear momentum equation

Example





(a)

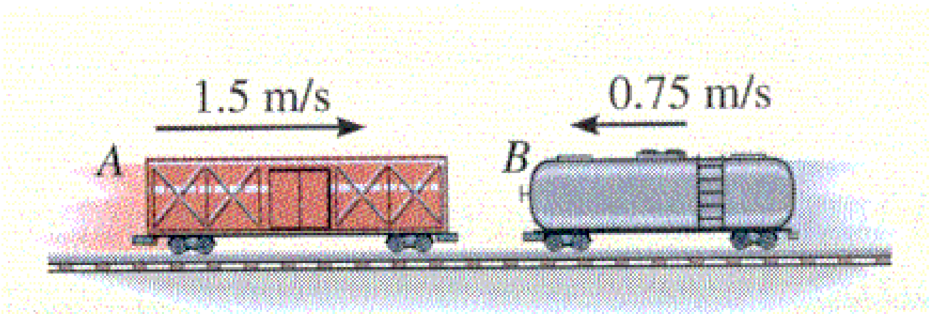
m_A=15 Mg
m_B=12 Mg
Couple together
V_2=? After coupling
F_{avg} = ? In 0.8 s

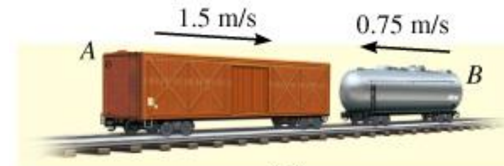
The 15-Mg boxcar *A* is coasting at 15 m/s on the horizontal track when it encounters a 12-Mg tank car *B* coasting at 0.75 m/s toward it. If the cars collide and couple together, determine

(a) the speed of both cars just after the coupling, and

(b) the average force between them if the coupling takes place in 0.8 s.

Example

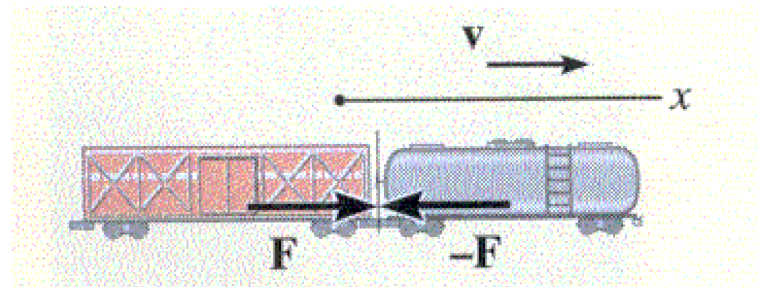




(a)

m_A=15 Mg
m_B=12 Mg
Couple together
V_2=? After coupling
F_{avg} = ? In 0.8 s

Consider conservation of momentum for the system of A and B (the coupling force is internal to the system and thus cancels out)

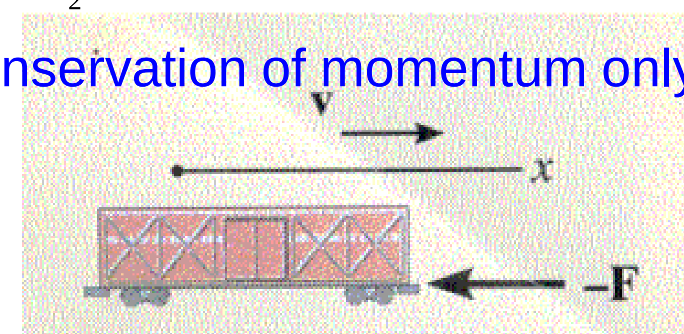


$$\left(\overset{+}{\rightarrow}\right) \quad m_A(\upsilon_A)_1 + m_B(-\upsilon_B)_1 = (m_A + m_B)\upsilon_2$$

$$(15000)(1.5) - (12000)(0.75) = (27000)\,\upsilon_2$$

$$\upsilon_2 = 0.5 \;\; m/s \quad \rightarrow$$

conservation of momentum only for A

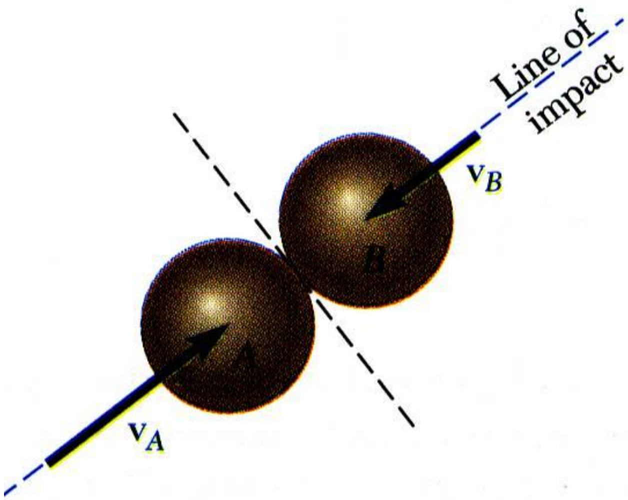


$$(\overset{+}{\rightarrow}) \quad m(\upsilon_A)_1 + F_{avg}(t) = m(\upsilon_A)_2$$

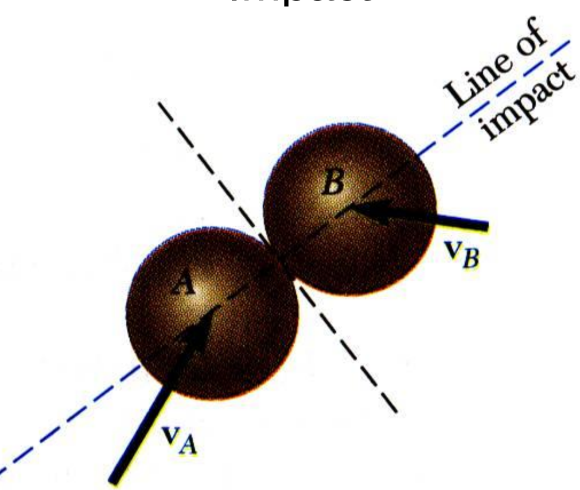
$$(15000)(1.5) - F_{avg}(0.8) = (15000)(0.5)$$

$$F_{avg} = 18.8 \; kN$$

Impact



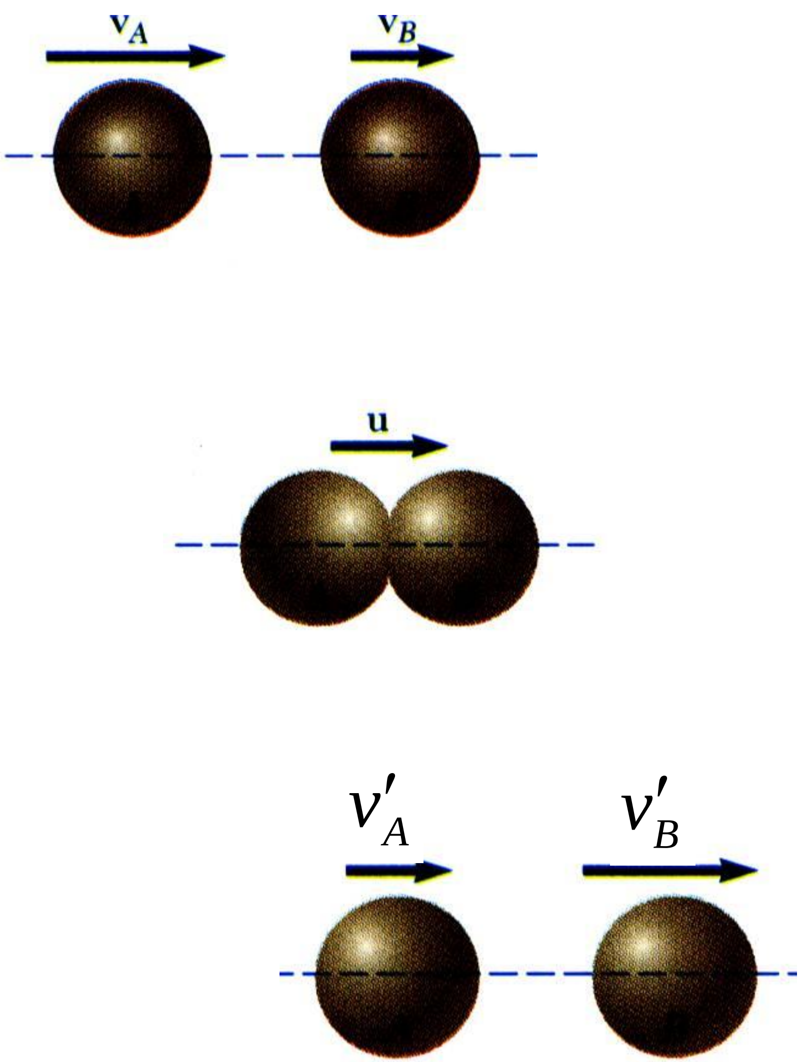
Direct Central Impact



Oblique Central Impact

- *Impact:* Collision between two bodies which occurs during a small time interval and during which the bodies exert large forces on each other.
- *Line of Impact:* Common normal to the surfaces in contact during impact.
- *Central Impact:* Impact for which the mass centers of the two bodies lie on the line of impact; otherwise, it is an *eccentric impact.*
- *Direct Impact:* Impact for which the velocities of the two bodies are directed along the line of impact.
- *Oblique Impact:* Impact for which one or both of the bodies move along a line other than the line of impact.

Direct Central Impact

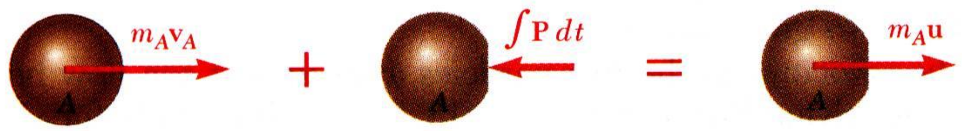


- Bodies moving in the same straight line, $v_A > v_B$.
- Upon impact the bodies undergo a *period of deformation*, at the end of which, they are in contact and moving at a common velocity.
- A *period of restitution* follows during which the bodies either regain their original shape or remain permanently deformed.
- Wish to determine the final velocities of the two bodies. The total momentum of the two body system is preserved,

$$m_A v_A + m_B v_B = m_A v'_A + m_B v'_B$$

- A second relation between the final velocities is required.

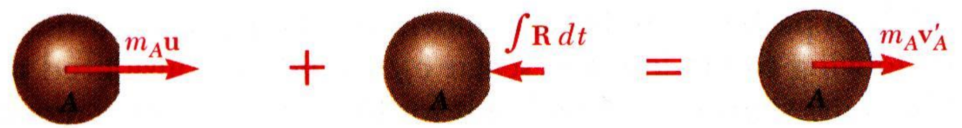
Direct Central Impact



- Period of deformation: $m_A v_A - \int P dt = m_A u$

$e = \text{coefficient of restitution}$

$$= \frac{\int R dt}{\int P dt} = \frac{u - v'_A}{v_A - u}$$



- Period of restitution: $m_A u - \int R dt = m_A v'_A$

$0 \le e \le 1$

- A similar analysis of particle *B* yields

$$e = \frac{v'_B - u}{u - v_B}$$

- Combining these relations leads to the desired second relation between the final velocities.

$$v'_B - v'_A = e(v_A - v_B)$$

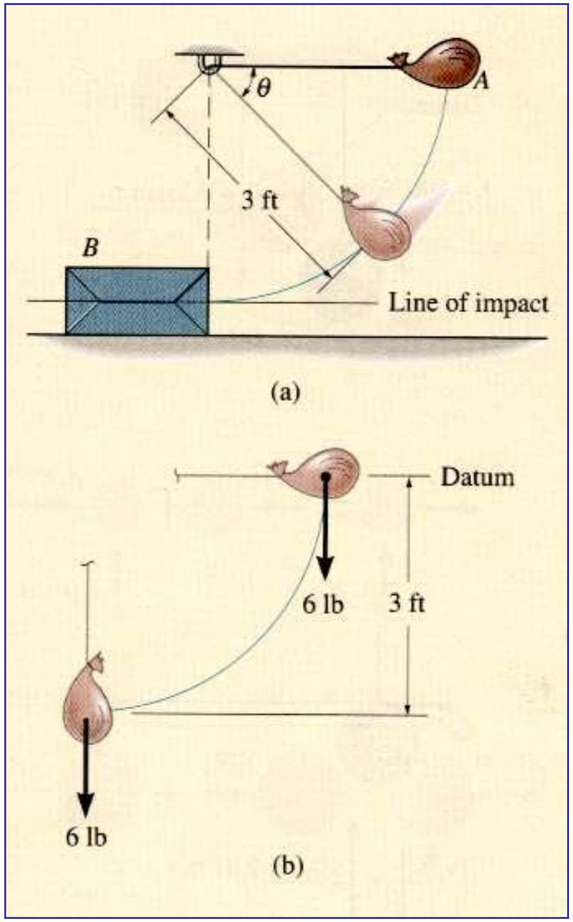
- *Perfectly plastic impact, e = 0:* $v'_B = v'_A = v'$ $\quad m_A v_A + m_B v_B = (m_A + m_B) v'$

- *Perfectly elastic impact, e = 1:*
  Can show that the kinetic energy of the particles and total momentum are conserved.

$$v'_B - v'_A = v_A - v_B$$

$$\frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2 = \frac{1}{2} m_A \left(v'_A\right)^2 + \frac{1}{2} m_B \left(v'_B\right)^2$$

Example



$W_A = 6\,Ib$

from rest to $\theta = 90^o$

$W_B = 18\,Ib$

$e = 0.5$

$(v_A)_2 = ?\quad (v_B)_2 = ?$

loss of energy = ?

The bag *A*, having a weight of 6lb, is released from rest at the position θ= 0°. After falling to θ =90°, it strikes an 18-lb box *B.* If the coefficient of restitution between the bag and box is *e* = 0.5, determine the velocities of the bag and box just after impact. What is the loss of energy during collision?

Conservation of Energy (bag A)

$$T_0 + V_0 = T_1 + V_1$$

$$0 + 0 = \frac{1}{2}\frac{6}{32.2}(v_A)_1^{\,2} - 6*3$$

$$\therefore (v_A)_1 = 13.9\ ft/s$$

Compute the velocity of the bag before it hits the box

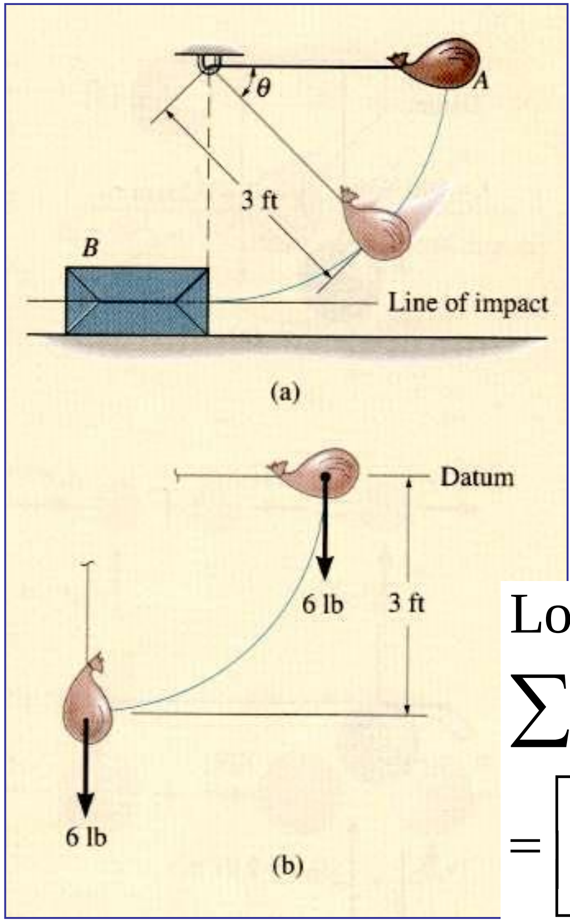
Conservation of Momentum (A and B)

$$\leftarrow m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2$$

$$\frac{6}{32.2}13.9 + 0 = \frac{6}{32.2}(v_A)_2 + \frac{18}{32.2}(v_B)_2$$

$$\therefore (v_A)_2 = 13.9 - 3(v_B)_2 \quad (1)$$

Example



Coefficient of Restitution

$$\leftarrow e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1} = \frac{(v_B)_2 - (v_A)_2}{13.9 - 0} = 0.5$$

$$(v_A)_2 = (v_B)_2 - 6.95 \qquad (2)$$

$$(v_A)_2 = 13.9 - 3(v_B)_2 \qquad (1)$$

$$\therefore (v_A)_2 = -1.74\ ft/s \rightarrow \ \ and \ \ (v_B)_2 = 5.21\ ft/s \leftarrow$$

Loss of Energy

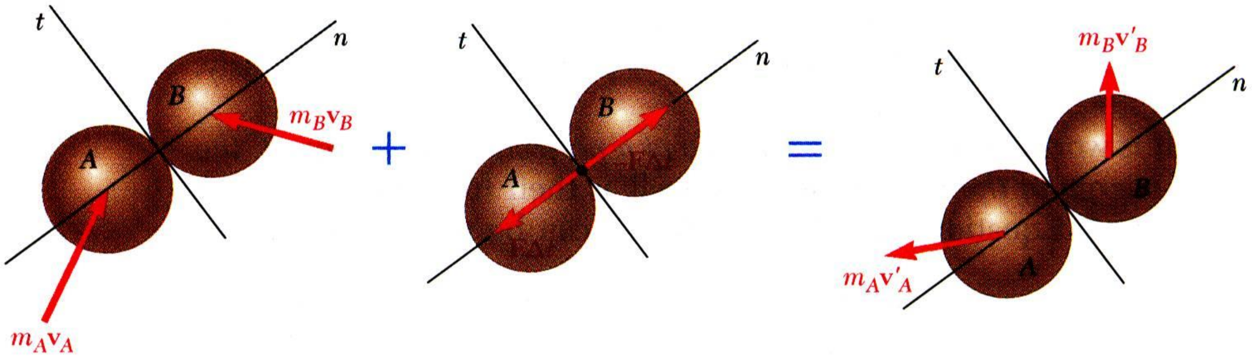
$$\sum U_{12} = T_2 - T_1$$

$$= \left[\frac{1}{2}\frac{18}{32.2}5.2^2 + \frac{1}{2}\frac{6}{32.2}1.74^2\right] - \frac{1}{2}\frac{6}{32.2}13.9^2 = -10.1\ ft \cdot lb$$

Note: The ***energy loss*** during the collision is calculated on the basis of the difference in the particles' kinetic energy.

$$\sum U_{1\text{-}2} = \sum T_2 - \sum T_1 \quad \text{where } T_i = 0.5 m_i (v_i)^2$$

Oblique Central Impact



- Final velocities are unknown in magnitude and direction. Four equations are required.

- No tangential impulse component; tangential component of momentum for each particle is conserved.

$$(v_A)_t = (v'_A)_t \qquad (v_B)_t = (v'_B)_t$$

- Normal component of total momentum of the two particles is conserved.

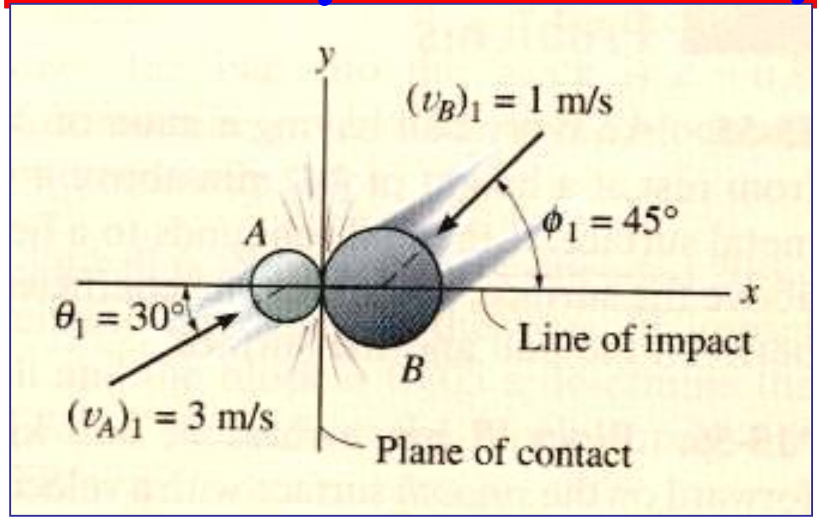
$$m_A (v_A)_n + m_B (v_B)_n = m_A (v'_A)_n + m_B (v'_B)_n$$

- Normal components of relative velocities before and after impact are related by the coefficient of restitution.

$$(v'_B)_n - (v'_A)_n = e\left[(v_A)_n - (v_B)_n\right]$$

Conservation of momentum and the coefficient of restitution equation are applied along the line of impact

Example: Oblique Central Impact



$$m_A = 1\ kg \qquad m_B = 2\ kg$$

$$e = 0.75$$

$$(v_{Ax})_1 = 3\cos 30° = 2.60\, m/s$$

$$(v_{Ay})_1 = 3\sin 30° = 1.50\, m/s$$

$$(v_{Bx})_1 = -1\cos 45° = -0.707\, m/s$$

$$(v_{By})_1 = -1\sin 45° = -0.707\, m/s$$

- Momentum of the system is conserved along the line of impact

$$\rightarrow m_A(v_{Ax})_1 + m_B(-v_{Bx})_1 = m_A(v_{Ax})_2 + m_B(v_{Bx})_2$$

$$2.6 + 2\cdot(-0.707) = (v_{Ax})_2 + 2(v_{Bx})_2$$

$$(v_{Ax})_2 + 2(v_{Bx})_2 = 1.18 \qquad \Rightarrow (1)$$

- Coefficient of Restitution

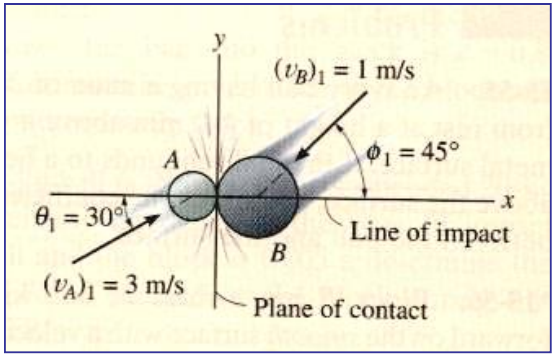
$$\rightarrow e = \frac{(v_{Bx})_2 - (v_{Ax})_2}{(v_{Ax})_1 - (v_{Bx})_1} = 0.75 = \frac{(v_{Bx})_2 - (v_{Ax})_2}{2.60 - (-0.707)}$$

$$\therefore\ (v_{Bx})_2 - (v_{Ax})_2 = 2.48 \qquad \Rightarrow (2)$$

$$\therefore\ (v_{Ax})_2 = -1.26\, m/s \qquad (v_{Bx})_2 = 1.22\, m/s$$

Example: Oblique Central Impact

- *Momentum of particle A,B is conserved along the y axis, since no impulse acts on particle A,B*



$$\uparrow m_A(v_{Ay})_1 = m_A(v_{Ay})_2 \;\Rightarrow\; (v_{Ay})_2 = 1.5\, m/s$$

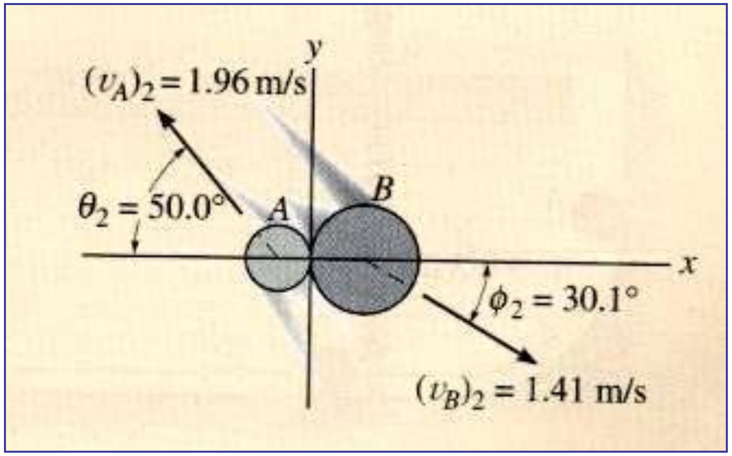
$$\uparrow m_B(v_{By})_1 = m_B(v_{By})_2 \;\Rightarrow\; (v_{By})_2 = -0.707\, m/s$$

$$(v_A)_2 = \sqrt{1.26^2 + 1.5^2} = 1.96\, m/s$$

$$(v_B)_2 = \sqrt{1.22^2 + (0.707)^2} = 1.41\, m/s$$

$$\theta_2 = \tan^{-1}\frac{1.5}{-1.26} = -50°$$

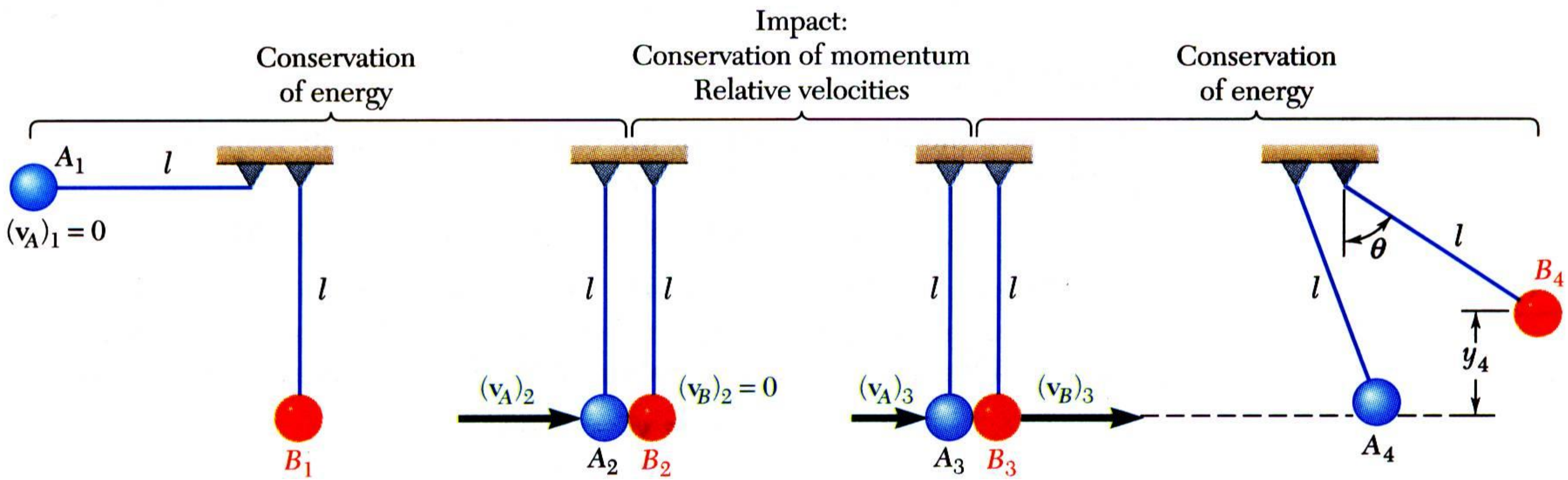
$$\varphi_2 = \tan^{-1}\frac{-0.707}{1.22} = -30.1°$$



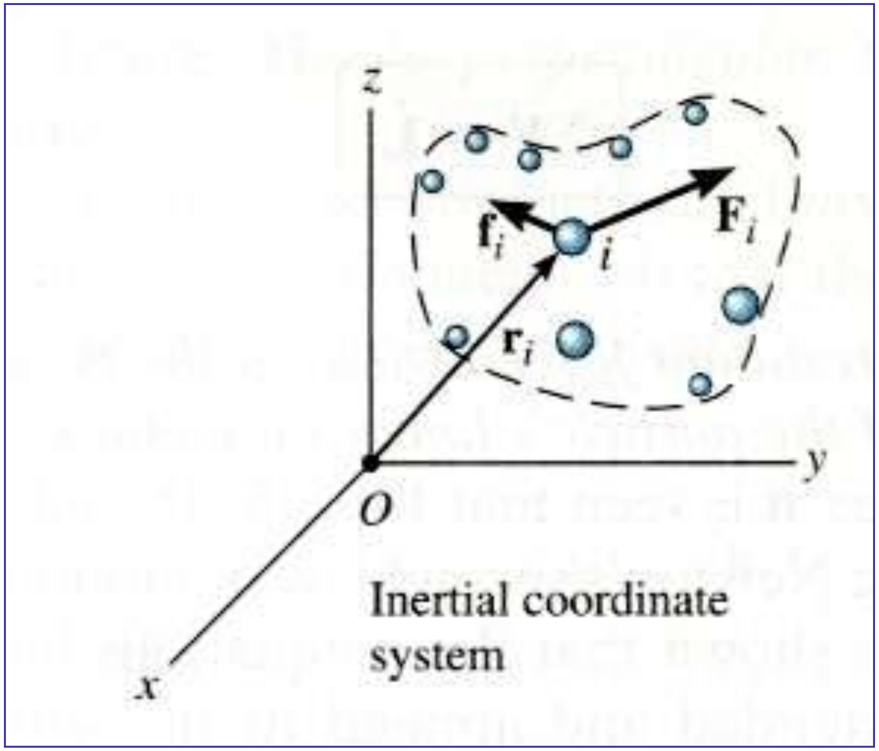
Problems Involving Energy and Momentum

- In summary, we have seen three methods for the analysis of kinetics problems:
  - Direct application of Newton's second law
  - Method of work and energy
  - Method of impulse and momentum

- Select the method best suited for the problem or part of a problem under consideration.



Angular Momentum for a System of Particles



$$\sum (\dot{H}_i)_0 = \sum (\mathbf{r}_i \times \mathbf{F}_i) + \sum (\mathbf{r}_i \times \mathbf{f}_i)$$

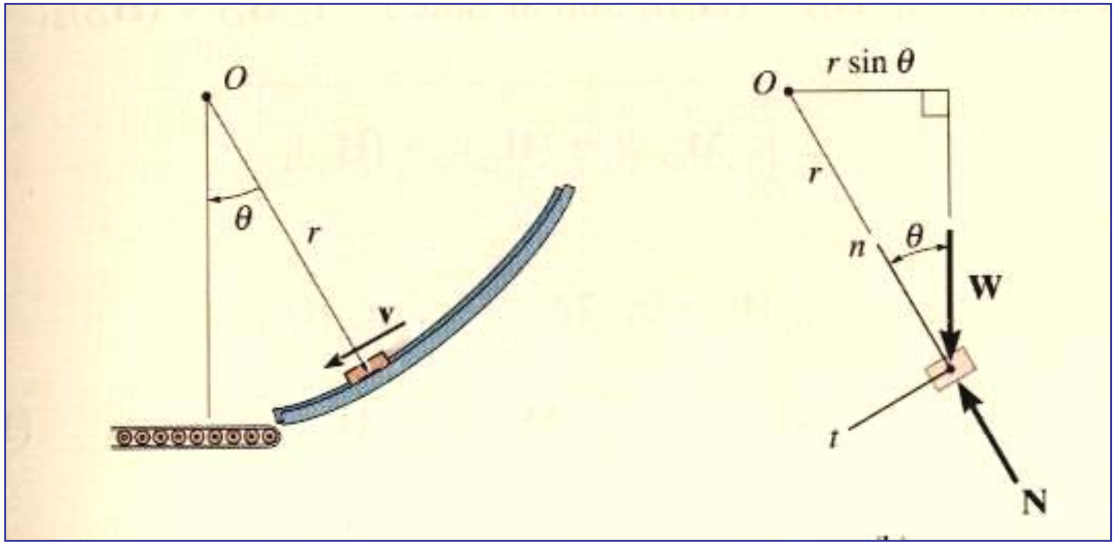
i th particle

Internal forces cancel each other

$$\sum \mathbf{M}_0 = \dot{H}_0$$

Sum of the moments about O of all the external forces acting on a system of particles = rate of change of the total angular momentum of the system of particles about point O

Example



The box shown has a mass *m* and travels down the smooth circular ramp such that when it is at the angle θ it has a speed v. Determine its angular momentum about point O at this instant and the rate of increase in its speed, *dv/dt.*

Angular momentum about O

$$H_0 = ?$$

$$H_0 = rmv$$

rate of increase in its speed (a_t)

$$\sum M_0 = \dot{H}_0 = r\,m\,\dot{v}$$

$$M_0 = (m\,g\sin\theta\,)\,r = \frac{d}{dt}(r\,m\,v)$$

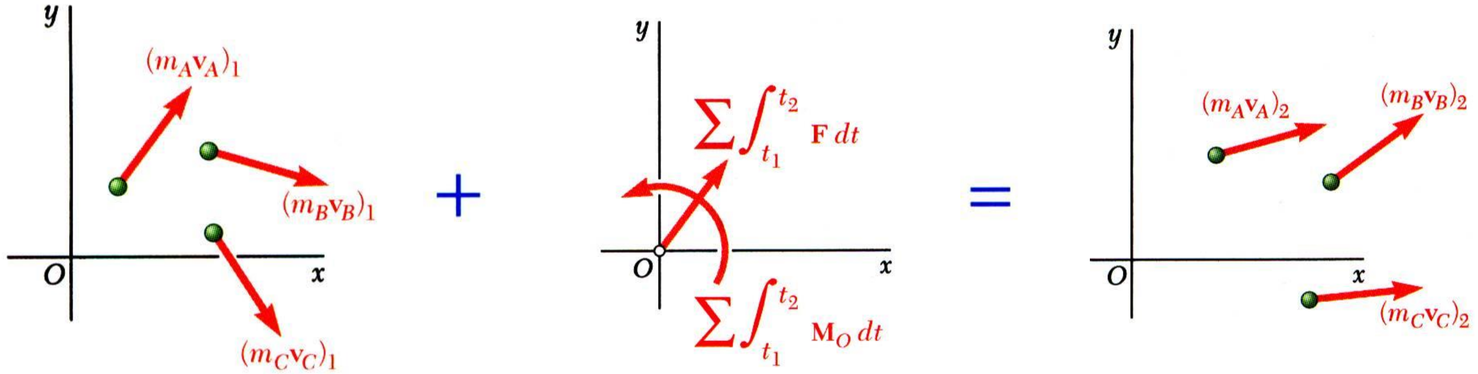
$$\therefore\ \frac{dv}{dt} = g\sin\theta$$

Similar results are obtained with Newton's law

$$\sum F_t = ma_t \qquad a_t = \dot{v}$$

$$m\,g\,\sin\theta = m\frac{dv}{dt} \qquad \therefore\ \frac{dv}{dt} = g\,\sin\theta$$

Principle of Impulse and Momentum



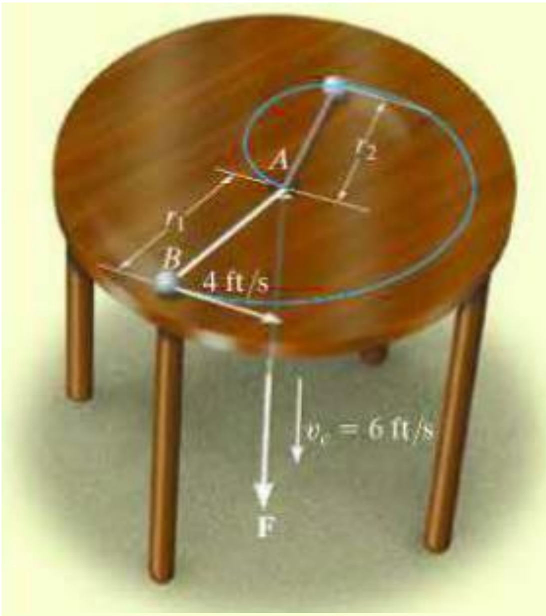
$$\sum \vec{F} = \dot{\vec{L}} \qquad \sum \vec{M}_O = \dot{\vec{H}}_O$$

$$\sum \int_{t_1}^{t_2} \vec{F} dt = \vec{L}_2 - \vec{L}_1 \qquad \sum \int_{t_1}^{t_2} \vec{M}_O dt = \vec{H}_2 - \vec{H}_1$$

$$\vec{L}_1 + \sum \int_{t_1}^{t_2} \vec{F} dt = \vec{L}_2 \qquad \vec{H}_1 + \sum \int_{t_1}^{t_2} \vec{M}_O dt = \vec{H}_2$$

- The momenta of the particles at time t_1 and the impulse of the forces from t_1 to t_2 form a system of vectors *equipollent* to the system of momenta of the particles at time t_2 .

Example: Angular Momentum + Work Equation



W = 0.8 Ib
Smooth table
r_1=1.7 ft
v_1= 4 ft/s
v_c=6ft/s (constant)
v_2= ? at (r_2)= 0.6ft
Work done = ?

$$H_1 = H_2$$

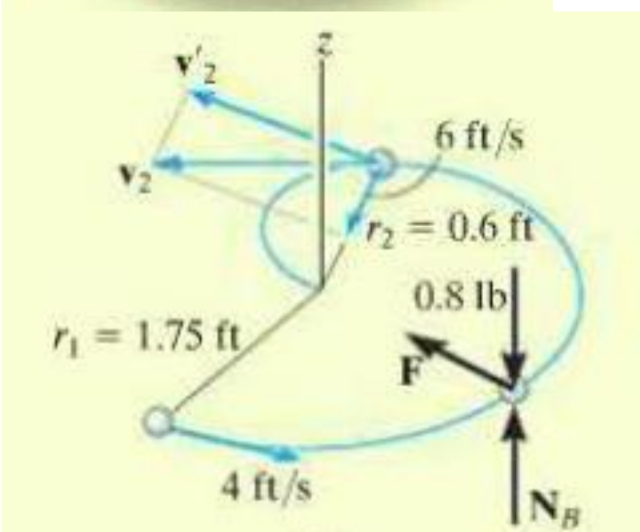
$$r_1 m_B \upsilon_1 = r_2 m_B \upsilon'_2$$

The cord force F (not constant) on the ball passes through the z axis, and the weight and N_B are parallel to it. Hence the moments, or angular impulses created by these forces, are all zero about this axis. Therefore, angular momentum is conserved about the z axis.

$$1.75\,(\frac{0.8}{32.2})\,4 = 0.6\,(\frac{0.8}{32.2})\,\upsilon'_2$$

$$\upsilon'_2 = 11.67 \;\; ft/s$$

$$\upsilon_2 = \sqrt{(11.67)^2 + (6)^2} = 13.1 \;\; ft/s$$



Work Done (energy balance for the ball)

$$T_1 + \sum U_{1-2} = T_2$$

$$\frac{1}{2}(\frac{0.8}{32.2})(4)^2 + U_F = \frac{1}{2}(\frac{0.8}{32.2})(13.1)^2$$

$$U_F = 1.94 \text{ ft.Ib}$$

Motion of the Center of Mass

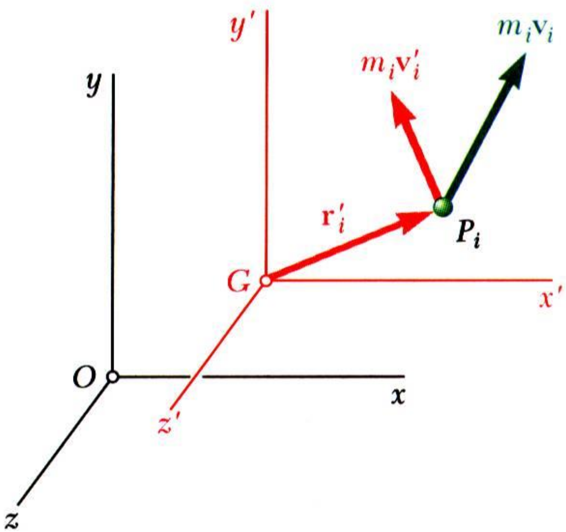
- Mass center *G* of system of particles is defined by position vector $\vec{r}_G$ which satisfies

$$m\vec{r}_G = \sum_{i=1}^{n} m_i \vec{r}_i$$

- Differentiating twice, $m\dot{\vec{r}}_G = \sum_{i=1}^{n} m_i \dot{\vec{r}}_i,\ m\vec{v}_G = \sum_{i=1}^{n} m_i \vec{v}_i = \vec{L},\ m\vec{a}_G = \dot{\vec{L}} = \sum \vec{F}$

- The mass center moves as if the entire mass and all of the external forces were concentrated at that point.

- Angular momentum about *G* of the particle momenta can be calculated with respect to either the Newtonian or centroidal frames of reference.



$$\vec{H}_G = \sum_{i=1}^{n} \left(\vec{r}_i' \times m_i \vec{v}_i \right)$$

$$= \sum_{i=1}^{n} \left(\vec{r}_i' \times m_i \left(\vec{v}_G + \vec{v}_i' \right) \right)$$

$$= \left(\sum_{i=1}^{n} m_i \vec{r}_i' \right) \times \vec{v}_G + \sum_{i=1}^{n} \left(\vec{r}_i' \times m_i \vec{v}_i' \right) = \sum_{i=1}^{n} \left(\vec{r}_i' \times m_i \vec{v}_i' \right)$$

$$\dot{\vec{H}}_G = \dot{\vec{H}}'_G = \sum \vec{M}_G = \sum_{i=1}^{n} \left(\vec{r}_i' \times \vec{F}_i \right)$$

The moment resultant about G of the external forces is equal to the rate of change of angular momentum about G of the system of particles

Thank you for your attention!