Economics 690
Problem Set: Linear Regression

(1) My calculations imply

$$\beta_0|y \sim t(1.18, .0075, 1215).$$

(Since these make use of basic OLS-type statistics, nothing further is required here).

(2) After completing the square on β, the joint posterior $p(\beta, \sigma^2|y)$ can be written as follows (see lecture notes):

$$p(\beta,\sigma^2|y) \propto (\sigma^2)^{-([n+k]/2+a+1)} \exp\left[-\frac{1}{2\sigma^2}\left((\beta-\overline{\beta})'\overline{V}(\beta-\overline{\beta}) + (\mu-\hat{\beta})'[V_\beta + (X'X)^{-1}]^{-1}(\mu-\hat{\beta})\right)\right]$$
$$\times \exp\left[-\frac{1}{2\sigma^2}SSE\right]\exp\left(-\frac{1}{b\sigma^2}\right).$$

where $\overline{V} = V_\beta^{-1} + X'X$ and $\overline{\beta} = \overline{V}^{-1}(V_\beta^{-1}\mu + X'X\hat{\beta})$. Since we are interested in the marginal posterior for σ^2, let's group terms involving β together. Note that the only term involving β appears in the kernel of the exponential term.

Thus, we can write:

$$p(\beta,\sigma^2|y) \propto (\sigma^2)^{-k/2}\exp\left[-\frac{1}{2\sigma^2}(\beta-\overline{\beta})'\overline{V}(\beta-\overline{\beta})\right] \times \exp\left[-\frac{1}{2\sigma^2}SSE\right]\exp\left(-\frac{1}{b\sigma^2}\right)$$
$$\times(\sigma^2)^{-[(n/2)+a+1]}\exp\left[-\frac{1}{2\sigma^2}(\mu-\hat{\beta})'[V_\beta+(X'X)^{-1}]^{-1}(\mu-\hat{\beta})\right],$$

where we have "borrowed" a $(\sigma^2)^{-k/2}$ and grouped it with the exponential kernel for β.

The marginal posterior for σ^2 is obtained by integrating the above over β. Everything except the first part of this expression will move outside the integral. When "borrowing" a $(\sigma^2)^{-k/2}$ and grouping it with the part of the joint posterior involving β, note that this reduces to the integral of a normal density for β with mean $\overline{\beta}$ and covariance matrix $\overline{V}^{-1}$ (except the $(2\pi)^{-k/2}$ part). Thus, the integral over β will simply equal a constant, not involving σ^2. Therefore, we can write:

$$p(\sigma^2|y) \propto (\sigma^2)^{-[(n/2)+a+1]}\exp\left[-\frac{1}{2\sigma^2}(\mu-\hat{\beta})'[V_\beta+(X'X)^{-1}]^{-1}(\mu-\hat{\beta})\right]$$
$$\times\exp\left[-\frac{1}{2\sigma^2}SSE\right]\exp\left(-\frac{1}{b\sigma^2}\right)$$

Grouping terms together, we obtain

$$p(\sigma^2|y) \propto (\sigma^2)^{-[(n/2)+a+1]}\exp\left[-\frac{1}{\sigma^2}\left(b^{-1}+\frac{1}{2}SSE+\frac{1}{2}(\mu-\hat{\beta})'[V_\beta+(X'X)^{-1}]^{-1}(\mu-\hat{\beta})\right)\right].$$

Written in this form, it is seen that

$$\sigma^2|y \sim IG\left[\frac{n}{2}+a, \left(b^{-1}+\frac{1}{2}SSE+\frac{1}{2}(\mu-\hat{\beta})'[V_\beta+(X'X)^{-1}]^{-1}(\mu-\hat{\beta})\right)^{-1}\right].$$