Economics 690
Problem Set: Hypothesis Testing

(1) Under equal prior odds, the Bayes factor reduces to:

$$K_{12} = \frac{p(y|\mathcal{M}_1)}{p(y|\mathcal{M}_2)}.$$

Noting

$$p(y|\mathcal{M}_j) = \int_\Theta p(y|\theta, \mathcal{M}_j) p(\theta|\mathcal{M}_j) d\theta,$$

it follows that when the prior is *dogmatic*, the above integral reduces to

$$p(y|\mathcal{M}_j) = p(y|\theta = c_j), \quad j = 1, 2.$$

Therefore, the Bayes factor reduces to the likelihood ratio, as claimed.

(2) Using the result above, we have

$$p(y|\mathcal{M}_1) = (2\pi)^{-1/2} \exp\left(-\frac{1}{2}(y+1)^2\right).$$

Similarly,

$$p(y|\mathcal{M}_2) = (2\pi)^{-1/2} \exp\left(-\frac{1}{2}(y-1)^2\right).$$

Therefore,

$$K_{12} = \exp\left(-\frac{1}{2}[(y+1)^2 - (y-1)^2]\right) = \exp(-2y).$$

The result is sensible. If $y = 0$ the odds equal 1, which makes sense since the realization is half-way in between our two hypotheses. When $y > 0$, the odds are less than one, suggesting a preference for $\mathcal{M}_2$. Similarly, when $y < 0$ the odds exceed one, suggesting preference for $\mathcal{M}_1$.

To address the final question, note that

$$\Pr(\mathcal{M}_1|y) = \frac{p(y|\mathcal{M}_1)\Pr(\mathcal{M}_1)}{p(y)} = \frac{p(y|\mathcal{M}_1)\Pr(\mathcal{M}_1)}{p(y|\mathcal{M}_1)\Pr(\mathcal{M}_1) + p(y|\mathcal{M}_2)\Pr(\mathcal{M}_2)} = \frac{p(y|\mathcal{M}_1)}{p(y|\mathcal{M}_1) + p(y|\mathcal{M}_2)},$$

where the last line follows under the assumption of equal prior odds.

If $y = 1$,

$$p(y|\mathcal{M}_1) = .34, \quad p(y|\mathcal{M}_2) = 2.51$$

so that

$$\Pr(\mathcal{M}_1|y) = \frac{.34}{.34 + 2.51} = .12$$

and

$$\Pr(\mathcal{M}_2|y) = .88.$$

(3) Matlab code for performing these calculations is provided on the website. When choosing $V_\beta = 1.0 \times 10^{100} I_2$, I obtained a (log) marginal likelihood for the model containing education equal to -1,164.4 and a (log) marginal likelihood for the model containing only an intercept equal to $-1,134.5$. Thus, under this prior, our testing procedure now suggests a preference for the *restricted model* that excludes education! This is an illustration of Bartlett's paradox - as our prior becomes increasingly flat, we tend to favor the restricted model, regardless of the evidence provided by the data.