Problem 1
The Lagrangian (density) for a Dirac fermion
\[ \mathcal{L} = \bar{\psi}(i\partial - m)\psi \] (0.1)
is invariant under the transformation
\[ \psi(x) \rightarrow e^{i\alpha^\alpha} \psi(x) \] (0.2)
a) Compute the corresponding conserved current (electromagnetic current).
b) Write the current in terms of oscillators and check that electron and positron have opposite charges.
c) Consider the transformation \( \psi(x) \rightarrow e^{i\alpha^5} \psi(x) \). Check that this is a symmetry of the Lagrangian when \( m = 0 \) but not if \( m \neq 0 \). Write the corresponding conserved current and compute its divergence in the \( m \neq 0 \) case using the Dirac equation.

Problem 2
A Dirac field transforms in the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) representation of the Lorentz group. In the chiral representation\(^1\) the Dirac spinor can be written as
\[ \psi = \begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix} \] (0.3)
where \( \xi_{L,R} \) are two component spinors such that \( \xi_L \) is in the \((\frac{1}{2}, 0)\) and \( \xi_R \) in the \((0, \frac{1}{2})\).
a) Show that the Dirac equation mixes both components.
b) Show that of \( m = 0 \) one can set e.g. \( \xi_R = 0 \) and still satisfy the Dirac equation (Weyl spinor).

\(^1\)This is the representation used in class where \( \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \).
c) Show that $\xi_L^*$ (conjugate) transforms (after an appropriate change of basis) in the $(0, \frac{1}{2})$ representation and therefore can be identified with $\xi_R$.

d) Using the result of c) write a massive Dirac equation for just $\xi_L$ (Majorana fermion). Show that this equation is not invariant under the charge symmetry $\xi_L \rightarrow e^{iqs} \xi_L$ of problem 1) and therefore this fermion has no charge ($q = 0$).