Problem 1
The Lagrangian (density) for a Dirac fermion
\[ \mathcal{L} = \bar{\psi}(i\partial - m)\psi \] (0.1)
is invariant under the transformation
\[ \psi(x) \to e^{i\eta\alpha}\psi(x) \] (0.2)

a) Compute the corresponding conserved current (electromagnetic current).

b) Write the current in terms of oscillators and check that electron and positron have opposite charges.

c) Consider the transformation \( \psi(x) \to e^{i\alpha\gamma^5}\psi(x) \). Check that this is a symmetry of the Lagrangian when \( m = 0 \) but not if \( m \neq 0 \). Write the corresponding conserved current and compute its divergence in the \( m \neq 0 \) case using the Dirac equation.

Problem 2
In the chiral representation\(^1\) a Dirac spinor can be written as
\[ \psi = \begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix} \] (0.3)

where \( \xi_{L,R} \) are two component spinors that do not mix under Lorentz transformations.

a) Show that the Dirac equation mixes both components.

b) Show that if \( m = 0 \) one can set \( e.g. \xi_R = 0 \) and still satisfy the Dirac equation (Weyl spinor).

\(^1\)This is the representation used in class where \( \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \).
c) Show that $\xi^*_L$ (conjugate) transforms (after an appropriate change of basis) as $\xi_R$ and therefore it can be identified with $\xi_R$ (Majorana fermion).

d) Using the result of c) write a massive Dirac equation for just $\xi_L$ (Majorana fermion). Show that this equation is not invariant under the charge symmetry $\xi_L \rightarrow e^{iq\alpha} \xi_L$ of problem 1) and therefore this fermion has no charge ($q = 0$).

Problem 3

Consider solutions to the Dirac equation of the form $\psi(x) = u(p)e^{-ipx}$ with $p^2 = m^2$ where $u(p)$ solves the algebraic equation

$$(\gamma^\mu p_\mu - m)u(p) = 0 \quad (0.4)$$

If $\hat{p}$ is a unit vector along $\vec{p}$, define the helicity operator as

$$h = \hat{p}.\vec{S} = \frac{1}{2} \begin{pmatrix} \hat{p}.\vec{\sigma} & 0 \\ 0 & \hat{p}.\vec{\sigma} \end{pmatrix} \quad (0.5)$$

where $\hat{p}.\vec{\sigma} = \hat{p}_i\sigma^i$.

a) Find the two linearly independent solutions $u^{1,2}$ that are also eigenvectors of the helicity operator and normalize them such that

$$\bar{u}^r(p)u^s(p) = 2m\delta^{rs}, \quad \bar{u} = u^\dagger\gamma^0 \quad \text{(0.6)}$$

b) Repeat the same for solutions of the form $\psi(x) = v(p)e^{ipx}$ but normalize them now as

$$\bar{v}^r(p)v^s(p) = -2m\delta^{rs}, \quad \text{(0.7)}$$

c) Check the identities

$$\bar{u}^r(p)v^s(p) = 0 \quad \text{(0.8)}$$

$$\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \gamma^\mu p_\mu + m \quad \text{(0.9)}$$

$$\sum_{s=1,2} v^s(p)\bar{v}^s(p) = \gamma^\mu p_\mu - m \quad \text{(0.10)}$$