Problem 1

Prove that
\[ [AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB \]  
\[ (0.1) \]

Problem 2

Consider a Hermitian operator \( A \), \( i.e. A = A^\dagger \). Let \( \{|a_i\rangle, i = 1 \ldots N\} \) be a basis of eigenstates \( |a_i\rangle \) of \( A \), with eigenvalues \( a_i \). Assume for simplicity that there is no degeneracy, namely all the \( a_i \) are different.

a) Prove that
\[ \prod_{i=1}^{N} (A - a_i) = 0 \]  
\[ (0.2) \]

b) For a given value of \( i \) consider the operator
\[ P_i = \prod_{j=1; j \neq i}^{N} \left( \frac{A - a_j}{a_i - a_j} \right) \]  
\[ (0.3) \]

What does \( P_i \) do when applied to an arbitrary state?

c) Illustrate points a) and b) by using the operator \( S_z \) of a spin 1/2 system.

d) Discuss how to modify the formulas if there is a degeneracy in the spectrum of \( A \).

Problem 3

Consider the following Hamiltonian of a two-state system
\[ H = E(|1\rangle\langle 1| - |2\rangle\langle 2|) + \Delta(|1\rangle\langle 2| + |2\rangle\langle 1|) \]  
\[ (0.4) \]

where \( E, \Delta \) have dimension of energy. Find the energy eigenvalues and the corresponding eigenstates as linear combinations of \( |1\rangle, |2\rangle \).
Problem 4
Consider the following Hamiltonian of a three-state system

\[
H = \frac{\epsilon}{\sqrt{2}} \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]  

(0.5)

where \( \epsilon \) has dimension of energy. Find the energy eigenvalues and the corresponding eigenstates.

Problem 5
Consider the following observables in a three-state system:

\[
A = \begin{pmatrix}
a & 0 & 0 \\
0 & -a & 0 \\
0 & 0 & -a
\end{pmatrix}, \quad B = \begin{pmatrix}
b & 0 & 0 \\
0 & 0 & -ib \\
0 & ib & 0
\end{pmatrix}
\]  

(0.6)

where \( a, b \) are real numbers.

a) The spectrum of \( A \) is degenerate. How about the spectrum of \( B \)?

b) Show that \( A \), and \( B \) commute.

c) Find a new orthonormal basis where both \( A \) and \( B \) are diagonal. Do \( A \) and \( B \) form a complete set of observables for this system?