Spin $\frac{1}{2}$ chain

\[ H = -J \sum_{\langle i, j \rangle} \hat{S}_i \cdot \hat{S}_{i+1} \]

\[ \hat{S}_x = \frac{\hbar}{2} \sigma_x, \quad \hat{S}_y = \frac{\hbar}{2} \sigma_y, \quad \hat{S}_z = \frac{\hbar}{2} \sigma_z \]

\[ \sigma_x \uparrow \uparrow = \uparrow \uparrow \quad \sigma_y \uparrow \uparrow = \uparrow \downarrow \quad \sigma_z \uparrow \uparrow = \uparrow \uparrow \]

\[ \sigma_x \downarrow \downarrow = \downarrow \downarrow \quad \sigma_y \downarrow \downarrow = \downarrow \uparrow \quad \sigma_z \downarrow \downarrow = \downarrow \downarrow \]

\[ (\sigma_x^{(i)} \sigma_x^{(j)} + \sigma_y^{(i)} \sigma_y^{(j)} + \sigma_z^{(i)} \sigma_z^{(j)}) \mid \uparrow \uparrow \rangle = \mid \uparrow \uparrow \rangle - \mid \downarrow \downarrow \rangle + \mid \uparrow \downarrow \rangle + \mid \downarrow \uparrow \rangle = \mid \uparrow \uparrow \rangle \]

\[ \mid \uparrow \downarrow \rangle = \mid \uparrow \downarrow \rangle + \mid \downarrow \uparrow \rangle - \mid \uparrow \uparrow \rangle = 2 \mid \uparrow \downarrow \rangle - \mid \downarrow \uparrow \rangle \]

\[ \mid \downarrow \uparrow \rangle = \mid \downarrow \uparrow \rangle + \mid \downarrow \uparrow \rangle - \mid \downarrow \downarrow \rangle = 2 \mid \downarrow \uparrow \rangle - \mid \downarrow \downarrow \rangle \]

\[ \mid \downarrow \downarrow \rangle = \mid \downarrow \downarrow \rangle - (\mid \uparrow \downarrow \rangle + \mid \downarrow \uparrow \rangle) = \mid \downarrow \downarrow \rangle \]

\[ H = -2 \sum_{i} H_{c_{i,i+1}} \]

\[ H_{c_{i,i+1}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ H_{c_{i,i+1}} = 1 + 2 (-1 + P_{c_{i,i+1}}) \]

\[ P_{c_{i,i+1}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow H_{c_{i,i+1}} = 1 + 2 (-1 + P_{c_{i,i+1}}) \]

\[ H = -\lambda \sum_{j} (1 - P_{c_{i,i+1}}) \]

\[ \text{Hence,} \quad H = 2 \lambda \sum_{j} (1 - P_{c_{i,i+1}}) \]
Many grandfathers all spun in the same stall.

\[ | \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \rangle \]

\[ \mathbf{H} \mathbf{| 1 \rangle} = \mathbf{0} \]

\[ \mathbf{H} \mathbf{| 1 \rangle} \text{ preserves the # of spins up.} \]

\[ \mathbf{| 1 \rangle} = | \uparrow \downarrow \downarrow \downarrow \rangle \]

Spin up in position \( j \)

\[ \mathbf{H} | \downarrow \rangle = (2 \lambda | \downarrow \rangle - 2 \lambda | \downarrow \rangle_{j+1}) + (2 \lambda | \downarrow \rangle - 2 \lambda | \downarrow \rangle_{j-1}) \]

\[ + \mathbf{0} = \mathbf{0} \]

\[ = 2 \lambda (2 | \downarrow \rangle_{j} - | \downarrow \rangle_{j+1} - | \downarrow \rangle_{j-1}) \]

\[ | \Psi \rangle = \sum_{j} a_{j} | \downarrow \rangle \]

\[ \mathbf{H} | \Psi \rangle = 2 \lambda \sum_{j} a_{j} (2 | \downarrow \rangle_{j} - | \downarrow \rangle_{j+1} - | \downarrow \rangle_{j-1}) \]

\[ = 2 \lambda \sum_{j} (2 a_{j} - a_{j-1} - a_{j+1}) | \downarrow \rangle_{j} = \varepsilon \sum_{j} a_{j} | \downarrow \rangle_{j} \]

\[ 2 a_{j} - a_{j-1} - a_{j+1} = \frac{\varepsilon}{2 \lambda} a_{j} \]

\[ a_{j}^{2} = \frac{q^{j}}{q^{j}} \quad 2 q^{j} - q^{j-1} - q^{j+1} = \frac{\varepsilon}{2 \lambda} q^{j} \]

\[ 2 - \frac{1}{q} - q = \frac{\varepsilon}{2 \lambda} \]

but \( q \) should be a phase, otherwise the probability diverges for \( j \to \infty \).
\[ q = e^{i k} \]

\[ \xi = 2 - e^{-ik} - e^{ik} = 2 - 2 \cos k = 2 \sin^2 \frac{k}{2} \]

\[ \xi = 8 \lambda \sin^2 \frac{k}{2} \]

\[ \left< u_k \right> = \sum_{j=-\infty}^{\infty} e^{ikj} |j> \]

\[ k \rightarrow k + 2\pi \]

\[ \left< u_{k+2\pi} \right> = \sum_{j=-\infty}^{\infty} e^{i(k+2\pi)j} |j> = \sum_{j=-\infty}^{\infty} e^{ikj} |j> = \left< u_k \right> \]

\[-\pi < k < \pi \]
\[ \langle j_1, j_2 \rangle = \langle -\downarrow \uparrow \rangle \]

\[ \langle \psi \rangle = \sum_{j_1 = -\infty}^{\infty} \sum_{j_2 = j_1 + 1}^{\infty} a_{j_1 j_2} \langle j_1, j_2 \rangle \]

\[ H \langle j_1, j_2 \rangle = 2 \lambda \sum_{\ell = -\infty}^{\infty} (\mathbb{1}_{\ell, e_{\ell + 1}} - P_{\ell, e_{\ell + 1}}) \langle -\downarrow \uparrow \rangle \]

\[ \langle j_1, j_2 \rangle > \]

\[ H \langle j_1, j_2 \rangle > = 2 \lambda \left( -\langle j_1, j_2 \rangle > + \langle j_1 + 1, j_2 \rangle > - \langle j_1 - 1, j_2 \rangle > - \langle j_1, j_2 + 1 \rangle > + 4 \langle j_1, j_2 \rangle > \right) \]

\[ j_2 = j_1 + 1 \]

\[ H \langle j_1, j_2 \rangle > = 2 \lambda \left( -\langle j_1, j_2 \rangle > - \langle j_1 - 1, j_2 + 1 \rangle > + 2 \langle j_1, j_2 \rangle > \right) \]

\[ H \langle \psi \rangle = \mathcal{E} \langle \psi \rangle \]

\[ H \langle \psi \rangle = 2 \lambda \sum_{j_1 = -\infty}^{\infty} \sum_{j_2 = j_1 + 2}^{\infty} a_{j_1 j_2} \left( \langle j_1, j_2 \rangle \right)_{58}^{57} \]

\[ + 2 \lambda \sum_{j_1 = -\infty}^{\infty} a_{j_1 j_1 + 1} \left( 2 \langle j_1, j_1 + 1 \rangle > - \langle j_1 - 1, j_1 + 1 \rangle > - \langle j_1, j_1 + 2 \rangle > \right)_{58}^{57} \]

\[ = \mathcal{E} \sum_{j_1 = -\infty}^{\infty} \sum_{j_2 = j_1 + 1}^{\infty} a_{j_1 j_2} \langle j_1, j_2 \rangle > \]
Match coefficients:

\[ \mathbf{j}_2 > \mathbf{j}_1 + 1 \]

\[ \mathbf{E} a_{\mathbf{j}_2} = 2\lambda \begin{pmatrix} 4a_{\mathbf{j}_2} \ 
\frac{5}{8} \times \frac{5}{7} - \frac{3}{5} \times \frac{2}{7} \ 
\frac{6}{8} \times \frac{6}{7} - \frac{6}{7} \ 
\frac{6}{7} \times \frac{6}{7} - \frac{6}{7} \ 
\frac{6}{7} \times \frac{6}{7} - \frac{6}{7} \ 
\frac{6}{7} \times \frac{6}{7} - \frac{6}{7} \end{pmatrix} \]

\[ \mathbf{j}_2 = \mathbf{j}_1 + 1 \]

\[ \mathbf{E} a_{\mathbf{j}_1} = 2\lambda \begin{pmatrix} 2a_{\mathbf{j}_1+1} - a_{\mathbf{j}_1-1} \ 
\frac{5}{8} \times \frac{5}{7} - \frac{3}{5} \times \frac{2}{7} \ 
\frac{6}{8} \times \frac{6}{7} - \frac{6}{7} \ 
\frac{6}{7} \times \frac{6}{7} - \frac{6}{7} \ 
\frac{6}{7} \times \frac{6}{7} - \frac{6}{7} \ 
\frac{6}{7} \times \frac{6}{7} - \frac{6}{7} \end{pmatrix} \]

\[ a_{\mathbf{j}_1} = e^{i\theta_2} e^{-ik_1j_1} + e^{-i\theta_2} e^{ik_1j_1+i k_2j_1} \]

\[ \frac{\mathbf{E}}{2\lambda} e^{-i\theta_2} e^{-ik_1j_1+i k_2j_1} + \frac{\mathbf{E}}{2\lambda} e^{i\theta_2} e^{ik_1j_1+i k_2j_1} = e^{i\theta} e^{ik_1j_1+i k_2j_1} \]

\[ \left(4 - e^{ik_1} - e^{-ik_1} - e^{i k_2} - e^{-i k_2}\right) + e^{i\theta} e^{ik_1j_1+i k_2j_1} \]

\[ \frac{\mathbf{E}}{2\lambda} = 4 - e^{ik_1} - e^{-ik_1} - e^{ik_2} - e^{-ik_2} = 4 - 2\cos k_1 - 2\cos k_2 = 4\sin^2\frac{k_1}{2} + 4\sin^2\frac{k_2}{2} \]

\[ \mathbf{E} = 8\lambda \sin^2\frac{k_1}{2} + 8\lambda \sin^2\frac{k_2}{2} = \mathbf{E}(k_1) + \mathbf{E}(k_2) \]

But also:

\[ \frac{\mathbf{E}}{2\lambda} \left( e^{i\theta} e^{ik_1j_1+i k_2j_1} e^{ik_1j_1+i k_2j_1} e^{ik_1j_1+i k_2j_1} \right) = e^{i\theta} e^{ik_1j_1+i k_2j_1} \]

\[ \left(2e^{i\theta/2} e^{ik_2} - e^{i\theta/2} e^{-ik_2} e^{i\theta/2} e^{-ik_2} \right) + \mathbf{E} \left(2e^{i\theta} e^{ik_2} e^{i\theta} e^{-ik_2} \right) \]
\[
\begin{align*}
(4 - e^{ik_1} - e^{-ik_1} - e^{ik_2} - e^{-ik_2}) (e^{i\Theta/2} e^{ik_2} + e^{-i\Theta/2} e^{-ik_1}) &= \\
= 4 e^{i\Theta/2} e^{ik_2} + 4 e^{-i\Theta/2} e^{-ik_1} - e^{i\Theta} e^{-i\Theta} - e^{-i\Theta} e^{i\Theta} \\
- e^{i\Theta} e^{ik_2} - e^{-i\Theta} e^{-ik_1} = 2 e^{i\Theta} e^{ik_2} - e^{-i\Theta} e^{-ik_1} \\
= 2 e^{i\Theta} e^{ik_2} - 2 e^{-i\Theta} e^{-ik_1} - 2 e^{i\Theta} e^{ik_2} + 2 e^{-i\Theta} e^{-ik_1} = 0
\end{align*}
\]

\[
e^{i\Theta} (2 e^{ik_2} - e^{-i(k_1+k_2)} - 1) = e^{i\Theta} (-2 e^{ik_1} + e^{i(k_1+k_2)} + 1)
\]

\[
e^{i\Theta} = \frac{e^{i(k_1+k_2)} - 2 e^{ik_1} + 1}{e^{i(k_1+k_2)} - 2 e^{ik_2} + 1}
\]

\[
\Theta(k_1,k_2) = -\Theta(k_2,k_1)
\]

\[
K = k_1 + k_2, \quad q = \frac{k_2 - k_1}{2}, \quad k + 2q = 2k_2, \quad K - 2q = 2k_1
\]

\[
e^{i\Omega} = -\frac{e^{iK/2} - 2 e^{iK/2} e^{-iq} + 1}{e^{iK/2} - 2 e^{iK/2} e^{-iq} + 1} = -\frac{e^{iK/2} - 2 e^{-iq} + e^{-iK/2}}{e^{iK/2} - 2 e^{-iq} + e^{-iK/2}}
\]

\[
e = \frac{-2 \cos K/2 - 2 e^{-iq}}{2 \sin K/2 - 2 e^{-iq}} = -\frac{\cos K/2 - e^{-iq}}{\cos K/2 - e^{-iq}}
\]

\[
\frac{\Theta + \pi}{2} = \arg \left( c \frac{K}{2} - e^{-iK} \right) = \arg \left( c \frac{K}{2} - \cos q + i\sin q \right)
\]
\[
\tan(\frac{\Theta \pm \pi}{2}) = \frac{\sin(\frac{h - \varphi}{2})}{\cos(\frac{h - \varphi}{2}) - \cos(\frac{k - \psi}{2})} = \frac{s_2 c_1 - s_1 c_2}{c_1 c_2 - s_1 s_2} = \frac{s_2 c_1 - s_1 c_2}{-2 s_1 s_2}
\]

\[\cot \frac{\Theta}{2} = \frac{1}{2} \cot \frac{k_1}{2} - \frac{1}{2} \cot k_2
\]

If \( k_1 \) and \( k_2 \) are real,

\[
\alpha_{n_1 n_2}^{(k_1, k_2)} = e^{i \varphi_{k_1}} e^{i \frac{h - \varphi}{2}} e^{i \frac{k_1 + k_2}{2}} e^{-i \varphi_{k_2}} e^{i \frac{h - \varphi}{2}} e^{i \frac{k_1 + k_2}{2}} e^{-i \varphi_{k_2}}
\]

\[
\alpha_{n_1 n_2}^{(k_1, k_2)} = e^{i \frac{k_1 + k_2}{2}} \left( e^{i \varphi_{k_1} + i \varphi_{k_2}} e^{i (h - \varphi)} + e^{-i \varphi_{k_1} - i \varphi_{k_2}} e^{-i (h - \varphi)} \right)
\]

\[
 q_{n_1 n_2} = e^{i \frac{k_1 + k_2}{2}} 2 \cos \left( \frac{\Theta}{2} + \varphi (k_2 - k_1) \right)
\]

As \( k \to \text{real} \), then \( \alpha_{n_1 n_2} \to \infty \), \( c_{k_1 + k_2} \to e^{\infty} \).
\[
\cos \left( \frac{\Theta}{2} + q \cdot j \right) = \frac{e^{i\eta} - e^{-i\eta}}{2} = \frac{e^{i\eta} - e^{-i\eta}}{-2i} = \frac{1}{i} \sin(q \cdot j)
\]

\[
q = q_r + i q_i \quad \text{for } q > 0
\]

\[
\cos \left( \frac{\Theta}{2} + q_r \cdot j \right) + i \eta \cdot j = \cos \left( \frac{\Theta}{2} + q_r \cdot j \right) \cos(i \eta \cdot j) - \sin \left( \frac{\Theta}{2} + q_r \cdot j \right) \sin(i \eta \cdot j)
\]

\[
= \cos \left( \frac{\Theta}{2} + q_r \cdot j \right) \cosh(i \eta \cdot j) - \sin \left( \frac{\Theta}{2} + q_r \cdot j \right) \sinh(i \eta \cdot j)
\]

\[
\eta \to \infty \quad \frac{\cos \left( \frac{\Theta}{2} + q_r \cdot j \right) e^{i \eta \cdot j} - i \sin \left( \frac{\Theta}{2} + q_r \cdot j \right) \sinh(i \eta \cdot j)}{2}
\]

\[
q_r \to 0 \quad \Theta_{\text{half}} - i \Theta_{\text{half}} = 0 \implies \Theta \text{ is imaginary also.}
\]

\[
e^{i \Theta} = -\frac{ck \eta - e^{-q_r} - e^{\eta}}{ck^{-2} - e^{iq_r} - e^{\eta}}
\]

\[
e^{i \Theta} = -\frac{ck \eta - e^{\eta}}{ck \eta - e^{-\eta}} \quad \text{real \( \Theta \) purely \( \eta \).}
\]

\[
\Theta = i \alpha
\]

\[
e^{-\alpha k_r - \eta (l_2 - l_1)} + e^{a k_r + \eta (l_2 - l_1)}
\]

\[
\eta = -\infty \quad (l_2 - l_1) \to \infty
\]

\[
e^{-\alpha k} = -\frac{ck \eta - e^{\eta}}{ck \eta - e^{-\eta}}
\]

\[
e^{\alpha} = -\frac{ck \eta - e^{-\eta}}{ck \eta - e^{\eta}}
\]

\[
c k \eta = e^{-\eta}
\]
\[ a_{11} = e^{i \frac{\lambda}{2} (1 + 1')} \left( e^{-\eta (1' - 1)} + e^{\alpha + \eta (1 - 1')} \right) \]

\[ \frac{c}{\lambda} = e^{-\eta} \]

\[ \frac{\epsilon}{\lambda} = 4 - e^{i \frac{\lambda}{2} - i \eta} - e^{-i \frac{\lambda}{2} + i \eta} - e^{i \frac{\lambda}{2} + i \eta} - e^{i (\frac{\lambda}{2} - i \eta)} \]

\[ = 4 - 2 \cos \frac{\lambda}{2} e^{i \eta} - 2 \cos \frac{\lambda}{2} e^{-i \eta} = 4 - 4 \cos \frac{\lambda}{2} \cos \eta \]

\[ \frac{\epsilon}{\lambda} = 4 - 2 \cos \frac{\lambda}{2} \left( e^i + e^{-i} \right) \]

\[ = 4 - 2 \cos \frac{\lambda}{2} \left( \cosh \frac{\lambda}{2} + \frac{1}{\sinh \frac{\lambda}{2}} \right) = 4 - 2 \left( 1 + \cosh \frac{\lambda}{2} \right) \]

\[ = 2 - 2 \cosh \frac{\lambda}{2} = 2 \cosh^2 \frac{\lambda}{2} \]

\[ \boxed{\epsilon = 4 \lambda \cosh^2 \frac{\lambda}{2}} \]

**bound**

\[ \epsilon_{\text{bound}} = 8 \lambda \cosh^2 \frac{\lambda}{2} + 8 \lambda \cosh^2 \frac{\lambda}{2} = 16 \lambda \cosh^2 \frac{\lambda}{2} \]

\[ \epsilon_0 = 16 \lambda \cosh^2 \frac{\lambda}{2} - 4 \lambda \cosh^2 \frac{\lambda}{2} \]

\[ = 16 \lambda \cosh^2 \frac{\lambda}{2} \]

\[ = 16 \lambda \cosh^2 \frac{\lambda}{2} \cosh \frac{\lambda}{2} = 16 \lambda \cosh \frac{\lambda}{2} \cosh^2 \frac{\lambda}{2} \]

\[ \boxed{\epsilon_0 = 16 \lambda \cosh^2 \frac{\lambda}{2}} \]