Spiky Strings and Giant Magnons on $S^5$

M. Kruczenski

Purdue University

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(Russo, Tseytlin, M.K.)
Summary

- **Introduction**

  String / gauge theory duality (AdS/CFT)

  Classical strings and field theory operators:
  - folded strings and spin waves in spin chains
  - folded strings and twist two operators

- **Spiky strings and higher twist operators** (M.K.)

  Classical strings moving in AdS and their field theory interpretation
• **Spiky strings on a sphere and giant magnon limit**
  (Ryang, Hofman-Maldacena)

• **Spin chain interpretation of the giant magnon**
  (Hofman-Maldacena)

• **More generic solutions:**
  Spiky strings and giant magnons on $S^5$
  (Russo, Tseytlin, M.K.)

• **Other solutions on $S^2$** (work in prog. w/ R.Ishizeki)

• **Conclusions**
String/gauge theory duality: Large N limit (‘t Hooft)

String picture $\rightarrow$ Fund. strings

( Susy, 10d, Q.G. )

Mesons

$\pi, \rho, \ldots$

Quark model

$q \bar{q}$

QCD [ SU(3) ]

Large N-limit [SU(N)]

Effective strings

Strong coupling

More precisely: $N \rightarrow \infty$, $\lambda = g_{YM}^2 N$ fixed (‘t Hooft coupl.)

Lowest order: sum of planar diagrams (infinite number)
**AdS/CFT correspondence** (Maldacena)

Gives a precise example of the relation between strings and gauge theory.

**Gauge theory**

\[ \mathcal{N} = 4 \text{ SYM } \text{SU}(N) \text{ on } \mathbb{R}^4 \]

\[ A_\mu, \Phi^i, \Psi^a \]

Operators w/ conf. dim. \( \Delta \)

\[ g_s = g_{YM}^2 ; \quad R / l_s = (g_{YM}^2 N)^{1/4} \]

\[ N \to \infty, \quad \lambda = g_{YM}^2 N \quad \text{fixed} \]

\[ \lambda \text{ large } \rightarrow \text{ string th.} \]

\[ \lambda \text{ small } \rightarrow \text{ field th.} \]
Can we make the map between string and gauge theory precise?

It can be done in particular cases.

Take two scalars $X = \Phi_1 + i \Phi_2$; $Y = \Phi_3 + i \Phi_4$

$O = \text{Tr}(XX...Y..Y...X)$, $J_1$ $X$’s, $J_2$ $Y$’s, $J_1 + J_2$ large

Compute 1-loop conformal dimension of $O$, or equiv. compute energy of a bound state of $J_1$ particles of type $X$ and $J_2$ of type $Y$ (but on a three sphere)
Large number of ops. (or states). All permutations of Xs and Ys mix so we have to diag. a huge matrix.

Nice idea (Minahan-Zarembo). Relate to a phys. system

\[ \text{Tr}(X X \ldots Y X X Y) \]

operator

mixing matrix

\[ | \uparrow \uparrow \uparrow \ldots \downarrow \uparrow \uparrow \downarrow \uparrow \downarrow > \]

conf. of spin chain

op. on spin chain

\[ H = \frac{\lambda}{4\pi^2} \sum_{j=1}^{J} \left( \frac{1}{4} - \vec{S}_j \cdot \vec{S}_{j+1} \right) \]

Ferromagnetic Heisenberg model!
Ground state \( (s) \)

\[
|\uparrow\uparrow\uparrow\ldots\uparrow\uparrow\uparrow\uparrow\uparrow\rangle \quad \longleftrightarrow \quad \text{Tr}(X X \ldots X X X X X)
\]

\[
|\downarrow\downarrow\downarrow\ldots\downarrow\downarrow\downarrow\downarrow\downarrow\rangle \quad \longleftrightarrow \quad \text{Tr}(Y Y \ldots Y Y Y Y Y)
\]

First excited states

\[
|k\rangle = \sum_{l} e^{ikl} |\uparrow\uparrow\ldots\downarrow\ldots\uparrow\uparrow\rangle, \quad k = \frac{2\pi n}{J}; (J = J_1 + J_2)
\]

\[
\varepsilon(k) = \frac{\lambda}{4\pi^2} \left(1 - \cos k\right) \xrightarrow{k \to 0} \frac{\lambda n^2}{2J^2} \quad \text{(BMN)}
\]

More generic (low energy) states: Spin waves

(FT, BFST, MK, \ldots)
Spin waves of long wave-length have low energy and are described by an effective action in terms of two angles $\theta, \varphi$: direction in which the spin points.

\[
S_{\text{eff.}} = J \left\{ -\frac{1}{2} \int d\sigma d\tau \left[ \cos \theta \partial_\tau \phi - \frac{\lambda}{32\pi J^2} \left[ (\partial_\sigma \theta)^2 + \sin^2 \theta (\partial_\sigma \phi)^2 \right] \right] \right\}
\]

Taking $J$ large with $\lambda/J^2$ fixed: classical solutions. Moreover, this action agrees with the action of a string moving fast on $S^5$. What about the case $k \sim 1$?
Since \((\theta, \varphi)\) is interpreted as the position of the string we get the shape of the string from \(\langle \vec{S} \rangle(\sigma)\).

**Examples**

| \[ \uparrow\uparrow\uparrow \ldots \uparrow\uparrow\uparrow \uparrow \uparrow \rangle | \text{point-like} |
|-----------------------------------------------|

\( (\theta, \varphi) \)

Folded string
Rotation in $\text{AdS}_5$? (Gubser, Klebanov, Polyakov)

\[ Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 - Y_5^2 - Y_6^2 = -R^2 \]

\[ \sinh^2 \rho; \Omega_{[3]} \quad \cosh^2 \rho; t \]

\[ ds^2 = -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_{[3]}^2 \]

\[ E \equiv S + \frac{\sqrt{\lambda}}{2\pi} \ln S, \quad (S \to \infty) \]

\[ O = Tr(\Phi \nabla^S_+ \Phi), \quad x_+ = z + t \]

\[ \theta = \omega t \]
Verification using Wilson loops (MK, Makeenko)

The anomalous dimensions of twist two operators can also be computed by using the cusp anomaly of light-like Wilson loops (Korchemsky and Marchesini).

In AdS/CFT Wilson loops can be computed using surfaces of minimal area in AdS$_5$ (Maldacena, Rey, Yee)

The result agrees with the rotating string calculation.
Generalization to higher twist operators \((\text{MK})\)

\[
O_{[2]} = Tr \left( \Phi \nabla^S \Phi \right) \quad \rightarrow \quad O_{[n]} = Tr \left( \nabla^S \Phi \nabla^S \Phi \nabla^S \Phi \cdots \nabla^S \Phi \right)
\]

In flat space such solutions are easily found in conf. gaug

\[
x = A \cos[(n-1)\sigma_+] + A(n-1) \cos[\sigma_-] \\
y = A \sin[(n-1)\sigma_+] + A(n-1) \sin[\sigma_-]
\]
**Spiky strings in AdS:**

\[ E \equiv S + \left( \frac{n}{2} \right) \frac{\sqrt{\lambda}}{2\pi} \ln S, \quad (S \to \infty) \]

\[ O = Tr \left( \nabla_{+}^{S/n} \Phi \nabla_{+}^{S/n} \Phi \nabla_{+}^{S/n} \Phi \ldots \nabla_{+}^{S/n} \Phi \right) \]

\[ S = \frac{\sqrt{\lambda}}{2\pi} \int dt \sum_{j} (\cosh 2\rho_{j} - 1) \dot{\theta}_{j} - \frac{\sqrt{\lambda}}{8\pi} \int dt \sum_{j} \left\{ 4\rho_{j} + \ln \left( \sin^{2} \left( \frac{\theta_{j+1} - \theta_{j}}{2} \right) \right) \right\} \]
Spiky strings on a sphere: (Ryang)

Similar solutions exist for strings rotating on a sphere:

The metric is:

\[ ds^2 = -dt^2 + d\theta^2 + \sin^2 \theta \, d\phi^2 \]

We use the ansatz: 

\[ t = \kappa \tau, \quad \varphi = \omega \tau + \sigma, \quad \theta = \theta(\sigma) \]

And solve for \( \theta(\sigma) \). Field theory interpretation?
**Special limit:** (Hofman-Maldacena)

\[
\frac{d\theta}{d\sigma} = \frac{\kappa \sin \theta}{A} \sqrt{\frac{\kappa^2 \sin^2 \theta - A^2}{\kappa^2 - \omega^2 \sin^2 \theta}}
\]

\[\omega = \kappa\]

\[
\frac{d\theta}{d\sigma} = \frac{\sin \theta}{A \cos \theta} \sqrt{\kappa^2 \sin^2 \theta - A^2}
\]

\[
\sin \theta = \frac{A}{\kappa \sin \sigma}
\]

(top view)  

**giant magnon**
The energy and angular momentum of the giant magnon solution diverge. However their difference is finite:

\[ E - J = \frac{\sqrt{\lambda}}{2\pi} \frac{A}{\kappa} \int \frac{d\sigma}{\sin^2 \sigma} = \frac{\sqrt{\lambda}}{\pi} \sin \frac{\Delta \phi}{2} \]

\[ \cos \frac{\Delta \phi}{2} = \frac{A}{\kappa}, \quad \Delta \phi = \text{Angular distance between spikes} \]

Interpolating expression:

\[ E - J = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{\Delta \phi}{2}} \approx \begin{cases} 
\frac{\sqrt{\lambda}}{\pi} \sin \frac{\Delta \phi}{2}, & \lambda \gg 1 \\
1 + \frac{\lambda}{2\pi^2} \sin^2 \frac{\Delta \phi}{2}, & \lambda \ll 1 
\end{cases} \]
Field theory interpretation: (Hofman-Maldacena)

\[ H = \frac{\lambda}{4\pi^2} \sum_{j=1}^{J} \left( \frac{1}{4} - \vec{S}_j \cdot \vec{S}_{j+1} \right) \]

\[ |k\rangle = \sum e^{ikl} |\uparrow \uparrow \downarrow \ldots \uparrow \uparrow\rangle, \quad k = \frac{2\pi n}{J}; (J = J_1 + J_2) \]

\[ \mathcal{E}(k) = \frac{\lambda}{4\pi^2} \left(1 - \cos k\right) = \frac{\lambda}{2\pi^2} \sin^2 \frac{k}{2} \]

States with one spin flip and \( k \sim 1 \) are giant magnons
More spin flips: (Dorey, Chen-Dorey-Okamura)

In the string side there are solutions with another angular momentum $J_2$. The energy is given by:

$$E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}} \sin^2 \frac{\Delta \varphi}{2} \approx J_2 + \frac{\lambda}{2J_2 \pi^2} \sin^2 \frac{\Delta \varphi}{2}, \, \lambda \ll 1$$

Justifies interpolating formula for $J_2=1$

In the spin chain, if we flip a number $J_2$ of spins there is a bound state with energy:

$$\varepsilon(k) = \frac{\lambda}{2J_2 \pi^2} \sin^2 \frac{k}{2}$$

$k \rightarrow \Delta \varphi$

($J_2$ is absorbed in $J$)
More general solutions: (Russo, Tseytlin, MK)

**Strategy:** We generalize the spiky string solution and then take the giant magnon limit.

In flat space:

\[ x = A \cos[(n-1)\sigma_+] + A(n-1) \cos[\sigma_-] \]
\[ y = A \sin[(n-1)\sigma_+] + A(n-1) \sin[\sigma_-] \]

namely:

\[ x + iy = X = x(\xi)e^{i\omega \tau}, \quad \xi = \alpha \sigma + \beta \tau \]

Consider \( txS^5 \):

\[ ds^2 = -dt^2 + \sum_{a=1}^{3} dX_a d\bar{X}_a, \quad \sum_{a=1}^{3} X_a \bar{X}_a = 1 \]

Use similar ansatz:

\[ X_a = x_a(\xi)e^{i\omega_a \tau} = r_a(\xi)e^{i\mu_a(\xi) + i\omega_a \tau} \]
The reduced e.o.m. follow from the lagrangian:

\[ L = (\alpha^2 - \beta^2) x'_a \bar{x}'_a + i \beta \omega_a (x'_a \bar{x}_a - \bar{x}'_a x_a) - \omega_a^2 x_a \bar{x}_a + \Lambda (x_a \bar{x}_a - 1) \]

If we interpret \( \xi \) as time this is particle in a sphere subject to a quadratic potential and a magnetic field. The trajectory is the shape of the string. The particle is attracted to the axis but the magnetic field curves the trajectory.
Using the polar parameterization we get:

\[
L = (\alpha^2 - \beta^2) r_a^2 - \frac{1}{(\alpha^2 - \beta^2)} \frac{C_a^2}{r_a^2} - \frac{\alpha^2}{(\alpha^2 - \beta^2)} \omega_a^2 r_a^2 + \Lambda (r_a^2 - 1)
\]

\[
\mu_a' = \frac{1}{(\alpha^2 - \beta^2)} \left[ \frac{C_a^2}{r_a^2} + \beta \omega_a \right], \quad x_a = r_a e^{i\mu_a}
\]

Constraints: \( \omega_a C_a + \beta \kappa^2 = 0 \), \( H = \frac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2} \kappa^2 \)

Three ang. momenta: \( J_a = \int d\xi \left( \frac{\beta C_a}{\alpha (\alpha^2 - \beta^2)} + \frac{\alpha}{(\alpha^2 - \beta^2)} \omega_a r_a^2 \right) \)

Corresponding to phase rotations of \( \chi_{1,2,3} \)
Solutions:

- One angular momentum:
  \[ x_3 = 0, \quad x_2 \text{ real (} \mu_2 = 0\text{)}, \quad r_1^2 + r_2^2 = 1, \quad \text{one variable.} \]

- Two angular momenta:
  \[ x_3 = 0, \quad r_1^2 + r_2^2 = 1, \quad \text{one variable} \]

Since only one variable we solve them using conservation of H. Reproduced Ryang, Hofman-Maldacena and Chen-Dorey-Okamura

- Three angular momenta: \( r_1^2 + r_2^2 + r_3^2 = 1, \quad r_1, r_2 \)
Therefore the three angular momenta case is the first “non-trivial” and requires more effort. It turns that this system is integrable as shown long ago by Neumann, Rosochatius and more recently by Moser.

Can be solved by doing a change of variables to $\zeta_+, \zeta_-$

$$r_a^2 = \frac{(\zeta_+ - \omega_a^2)(\zeta_- - \omega_a^2)}{\prod_{a \neq b} (\omega_a^2 - \omega_b^2)}$$
In the new variables, the system separates if we use the Hamilton-Jacobi method:

Compute the Hamiltonian: \( H(p_\pm, \zeta_\pm) \)

Find \( W(\zeta_\pm) \) such that \( H\left(p_\pm = \frac{\partial W}{\partial \zeta_\pm}, \zeta_\pm\right) = E = \text{const.} \)

In this case we try the ansatz: \( W = W(\zeta_+) + W(\zeta_-) \) and it works! Variables separate!

A lengthy calculation gives a solution for \( \zeta_+, \zeta_- \) which can then be translated into a solution for \( r_a \)
The resulting equations are still complicated but simplify in the giant magnon limit in which \( J_1 \to \infty \)

We get for \( r_a \):

\[
r_a^2 = \frac{(\omega_2^2 - \omega_3^2)}{(\omega_1^2 - \omega_2^2)} s^2 \frac{1 - A_2^2}{(s_3 A_3 - s_2 A_2)^2}
\]

\[
r_3^2 = \frac{(\omega_2^2 - \omega_3^2)}{(\omega_1^2 - \omega_3^2)} s^2 \frac{A_3^2 - 1}{(s_3 A_3 - s_2 A_2)^2}
\]

with \( A_2 = \tanh \left( -\frac{s_2 \xi}{1 - \beta^2} + B_2 \right), \quad A_3 = \coth \left( -\frac{s_3 \xi}{1 - \beta^2} + B_3 \right) \)

We have \( r_a \) explicitly in terms of \( \xi \) and integration const.
We can compute the energy and angular extension of the giant graviton obtaining:

\[
E - J_1 = E_2 + E_3 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2} \sin^2 \frac{\varphi_2}{2}} + \sqrt{J_3^2 + \frac{\lambda}{\pi^2} \sin^2 \frac{\varphi_3}{2}}
\]

\[
\Delta \varphi = \varphi_2 + \varphi_3
\]

We get two superposed magnons. However there is a relation: same group velocity.

\[
v_2 = v_3, \quad v_j = \frac{\partial E_j}{\partial \varphi_j}
\]
In the spin chain side we need to use more fields to have more angular momenta. We consider therefore operators of the form:

$$\text{Tr}(\ldots XXXYYYZYYYZZZYYXXYZZZXXXXXXXXXXX\ldots)$$

Where $X = \Phi_1 + i \Phi_2$, $Y = \Phi_3 + i \Phi_4$, $Z = \Phi_5 + i \Phi_6$

The $J_2$ Y’s form a bound state and the $J_3$ Z’s another, both superposed to a “background” of $J_1$ X’s ($J_1 \to \infty$)

$$E - J_1 \approx J_2 + J_3 + \frac{\lambda}{2J_2\pi^2} \sin^2 \frac{\varphi_2}{2} + \frac{\lambda}{2J_3\pi^2} \sin^2 \frac{\varphi_3}{2}, \quad \lambda \ll 1$$

The condition of equal velocity appears because in the string side we use a rigid ansatz which does not allow relative motion of the two lumps.
Some examples of solutions.

\[ r_{1,2,3}(\xi) \]
Other solutions on $S^2$: (Work in progr. w/ R. Ishizeki)

It turns out that looking at rigid strings rotating on a two-sphere one can find other class of solutions and in particular another limiting solution:

Antiferromagnetic magnon? (see Roiban, Tirziu, Tseytlin)

(Goes around infinite times)
**Conclusions:**
Classical string solutions are a powerful tool to study the duality between string and gauge theory.

**We saw several examples:**

- folded strings rotating on $S^5$
- spiky strings rotating in AdS$_5$ and $S^5$
- giant magnons on $S^2$ and $S^3$
- giant magnons with three angular momenta
- work in progress on other sol. on $S^2$