(Near) Substitute Preferences and Equilibria with Indivisibilities*

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Abstract

An obstacle to using market mechanisms to allocate indivisible goods is the non-existence of competitive equilibria (CE). To surmount this, Arrow and Hahn proposed the notion of social-approximate equilibria: a price vector and corresponding excess demands that are 'small'. We identify a class of preferences called Δ-substitutes, and show that social approximate equilibria where the bound on excess demand, good-by-good, is $2(\Delta - 1)$ independent of the size of the economy. When $\Delta = 1$ existence of CE is guaranteed even with income effects. This sufficient condition strictly generalizes prior conditions such as single improvement, no complementarities, gross substitutes, and net substitutes. $\Delta > 1$ allows for preferences that accommodate a richer pattern of substitutes and complementarity relations. In this way, we expand the range of settings where market mechanisms can be used to allocate indivisible goods.

1 Introduction

Under certain conditions, competitive equilibrium (CE) allocations are Pareto Optimal and envy-free but not guaranteed to exist. This limits the deployment of market mechanisms

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to allocate scarce resources, particularly indivisible ones such as courses or worker shifts. Existence is often guaranteed by excluding complementarities in preferences. The gross substitutes condition of Kelso Jr and Crawford (1982) is an example. It requires that when the price of a good rises, an agent can drop it or substitute it with a new item while not reducing her demand for other goods. This is limiting as complementarities are an essential feature of many applications (e.g., spectrum, trucking routes).

This paper makes two contributions. First, it enlarges the domain of preferences to capture not just complementarities but financing constraints and other income effects. Second, following the first are new market mechanisms for allocating indivisible goods.

We introduce a new class of preferences we call Δ-*substitutes*, in which every non-singleton demand correspondence contains two bundles that can be obtained from each other by replacing a items in one bundle with b items from the other bundle, i.e., an a-for-b exchange where $\max\{a, b\} \leq \Delta$. An implication is that after a small price change, an agent can substitute away from a previously optimal bundle by exchanging up to Δ units of goods. Gross substitutes, in contrast, only allows an exchange of up to one unit of goods. We show that gross substitutes are an instance of 1-substitutes. $\Delta > 1$ allows for both substitutes *and* complements, essential in a setting with more than two goods. For instance, goods g_1 and g_2 by themselves may complement each other, but in the presence of a third good, they become substitutes for each other. Therefore, the larger Δ is, the richer the pattern of substitutes and complementary relations allowed. An obvious example of Δ-substitute preferences is those that are satiated outside of a finite region (Shapley and Shubik (1971), Quinzii (1984), Budish (2011). Thus, while the size of the bundles consumed is restricted, there is no limit on the substitutability and complementarity relations between goods in a bundle.

When all preferences satisfy Δ-substitutes, we exhibit what Arrow and Hahn (1971) called a social approximate equilibrium–prices and utility maximizing demands such that the excess demand for each good is at most $2(\Delta - 1)$. Δ-substitutes does not require quasi-linearity, so it accommodates financing constraints and other income effects. When $\Delta = 1$, the existence

of CE is guaranteed. 1-substitutes preferences *strictly* generalizes many earlier preference conditions, such as gross substitutes (Kelso Jr and Crawford, 1982), single-improvement, no complementarity property (Gul and Stacchetti, 1999) and net substitutes (Baldwin et al., 2020).

For general Δ, the magnitude of the excess demand, $2(\Delta - 1)$, is the 'shadow' cost of dividing the indivisible that depends on the degree of preference complementarity and not on the size of the economy. Knowing the excess demand a priori, a planner can withhold a quantity equal to the excess (or inject additional goods) to 'add back in' to ensure that each agent's demand is satisfied. In other words, 'burning' some of the supply (or subsidizing it) ensures feasibility. Injecting or burning resources, as documented by Young (1995), is a common way to 'divide' the indivisible.

While our results rely on preferences that depend on money, our second contribution shows how to adapt them to cover pseudo-markets where money is absent. We summarize two applications, one involving ordinal preferences and deterministic allocations and the other, von Neumann-Morgenstern utilities and probabilistic outcomes.

The first application is to the combinatorial assignment problem discussed in Budish (2011) in the context of course allocation. Goods are courses, and the number of seats in a course represents the supply of the corresponding good. Students have ordinal preferences over bundles of goods up to some size, Δ, say.

Budish (2011) proposed a modification of the Competitive Equilibrium with Equal Income (CEEI) mechanism of Hylland and Zeckhauser (1979) that circumvented the non-existence of a CE by randomly perturbing agents' budgets and allowing the market to clear approximately. The resulting allocation is approximately ex-post Pareto optimal and approximately ex-post envy-free.

Our results lead to an alternative market mechanism with similar fairness and efficiency properties to those in Budish (2011) but differs in the bound on the mismatch between supply and demand. In Budish (2011), the mismatch between supply and demand is quantified using

the Euclidean norm of the excess demand vector, resulting in a bound of $O(\sqrt{\Delta m})$, where m denotes the number of goods. This bound limits aggregate excess demand across all courses but allows for significant excess demand, potentially on the order of $O(\sqrt{m})$ for a specific course. In contrast, we offer a *constant* upper limit of $2(\Delta - 1)$ on the excess demand for *each* good. If we relax the assumption that each student desires only one unit of each good (referred to as single copy demand), then the bound in Budish (2011) on the ℓ_2-norm of excess demand becomes $O(\Delta\sqrt{m})$. Our mechanism satisfies this bound while limiting the excess demand for each good to be at most $2(\Delta - 1)$. If one allows under-allocation to compensate for over-allocation, our mechanism can be modified to yield an $O(\Delta)$ bound on aggregate excess demand.

The second application is to the implementability of lotteries in pseudo markets. It is natural to circumvent the problem of the non-existence of CE by interpreting fractional quantities of a good as a probability share in the good. While a competitive equilibrium in probability shares always exists, it is a fundamental challenge is to implement the equilibrium allocation of probability shares as a lottery over feasible allocations of indivisible goods. Hylland and Zeckhauser (1979) show this to be possible if agents have von Neumann-Morgenstern preferences and unit demand (each agent demands at most one unit of any good). Gul et al. (2019) extends this to multi-unit demand under the assumption of gross substitute preferences. Beyond gross substitute preferences, however, equilibrium probability share allocations cannot be implemented as a lottery over feasible allocations.

Our resolution of the implementability problem relies on strengthening our main result. Given Δ-substitute preferences, any equilibrium allocation of probability shares can be implemented as a lottery over social approximate equilibrium allocations. Therefore, a competitive equilibrium allocation of probability shares can be realized as a lottery over allocations, where each good is over-allocated by at most $2(\Delta - 1)$.

We generalize this approach to accommodate additional constraints on the resulting allocations. For example, in online advertising, advertisers may prefer to diversify the audiences

they reach, which can be encoded using bounds on the probability of receiving bundles from a particular category. In a school choice setting, these additional constraints encode proportionality requirements to ensure diversity.[1] We identify lotteries that satisfy, ex-ante, *any* set of linear constraints on allocations. Unlike Echenique et al. (2021) who also consider additional constraints, we do not rely on personalized prices to support a pseudo-equilibrium.[2]

1.1 Related Literature

Prior work deals with the non-existence of CE in three distinct ways. First, by restricting agent's preferences, for example, quasi-linearity and gross substitutes (Kelso Jr and Crawford (1982), Murota and Tamura (2003) and Gul and Stacchetti (1999)). Subsequent work by Danilov et al. (2001) and Baldwin et al. (2020) relaxed the quasi-linearity assumption. They introduce a condition that Baldwin et al. (2020) calls net substitutes, which guarantees the existence of a CE in the presence of income effects and contains gross substitutes as a particular case. Section 3.1 shows that all of these preferences satisfy 1-substitutes. The 1-substitutes condition overlaps with the notion of unimodular demand types in Baldwin and Klemperer (2019) but is not implied by it.

The literature has progressed to preference conditions that accommodate complementarities such as Sun and Yang (2006), Candogan et al. (2015) and Rostek and Yoder (2020). These conditions are restrictive in that they exclude substitutability entirely, or they permit complements only between pairs of goods. Our generalization of 1-substitutes to $\Delta-$substitutes allows both complements *and* substitutes.

The second way is to determine prices that produce small excess demand, what Arrow and Hahn (1971) called social approximate equilibria. Prior work does not restrict preferences but relies on the economy growing to infinity to ensure that the excess demand becomes negligible. See, for example, Starr (1969), Dierker (1971), Ali Khan and Rashid (1982), Mas-Colell (1977), and Azevedo et al. (2013). However, 'large enough' often exceeds the

[1] See He et al. (2018).

[2] Echenique et al. (2021) does not consider the problem of implementing a pseudo-equilibrium as a lottery.

scale of most applications.

This paper links the magnitude of excess demand in a social-approximate equilibrium to preferences via the parameter Δ. To the best of our knowledge, this is the first bound on excess demand in a social approximate CE independent of either the number of goods or the number of agents.[3]

The third is to focus on finding allocations that 'approximate' the welfare properties of a CE outcome. This usually requires adopting a cardinal measure of welfare and showing that it scales gracefully with the size of the economy. See, for example, Akbarpour and Nikzad (2020), Cole and Rastogi (2007), Milgrom and Watt (2021) and Feldman and Lucier (2014). This paper does not commit to a particular cardinal measure of welfare. Instead, the approximation quality is measured in terms of the mismatch between supply and demand.

2 Notation & Preliminaries

The economy has m indivisible goods and one divisible good, which we interpret as money. Let M denote the set of m indivisible goods. A bundle of goods is denoted by a vector $x \in \mathbb{Z}_+^m$ whose i^{th} component, denoted x_i, indicates the quantity of good $i \in M$ in the bundle.

Let N denote the set of agents. The utility of each agent $j \in N$ for a bundle x and the amount of money $w \in \mathbb{R}$ is denoted $U_j(x, w)$.[4] We assume $U_j(x, w)$ is continuous and strictly increasing in w, and $U_j(\vec{0}, 0) = 0$.[5] Each agent j is endowed with $b_j \geq 0$ units of money only.

Agent j's utility for a bundle x which costs t is $U_j(x, b_j - t)$. Associated with each $j \in N$ is a finite set of bundles $X_j \subset \mathbb{Z}_+^m$ that they can feasibly consume, their feasible bundles. The bundle $\vec{0}$ is always assumed to be feasible. If $x \notin X_j$, then, $U_j(x, b_j - t) = -\infty$ for all t.

Our results require a standard bounded willingness to pay condition, described next (see,

[3]The bound in Nguyen and Vohra (2018) is also independent of the market size, but it is valid for a strictly smaller class of preferences and stable solutions rather than CE.

[4]These are called expenditure augmented utilities in Deb et al. (2021).

[5]Our key result, Theorem 5.2, extends to the case when utilities are not strictly monotone in money.

for example, Fleiner et al. (2019)). Specifically, there is a monetary amount B, which can depend on all other parameters of the economy, such that if the price of a bundle is at least B, then an agent is better off not consuming anything, that is

$$U_j(x, b_j - B) < U_j(\vec{0}, b_j) \text{ for all } j \text{ and } x \in X_j.$$

Quasi-linear preferences, where $U_j(x, b_j - t) = v_j(x) + b_j - t$ for some valuation function $v_j(\cdot)$ satisfy these conditions. Hard budget constraints can be approximated by allowing $U_j(x, b_j - t)$ to approach $-\infty$ as t approaches the budget. While we state our results in terms of utility functions that depend on money, we show, in Section 4.1, how to accommodate token money.

Let $p \in \mathbb{R}^m_+$ be a price vector where p_i is the unit price of good $i \in M$. The utility of agent j for bundle x at price p will be $U_j(x, b_j - p \cdot x)$. Given a price vector p, agent j's **choice correspondence**, denoted $Ch_j(p)$, is defined as follows:

$$Ch_j(p) = \arg\max\{U_j(x, b_j - p \cdot x) : x \in X_j\}.$$

Denote the convex hull of $Ch_j(p)$ by $conv(Ch_j(p))$. Let $(x-y)^+$ denote the vector whose i^{th} component is $\max\{x_i - y_i, 0\}$. The ℓ_1 norm of these vectors will play an important role, particularly,

$$||(x-y)^+||_1 = \vec{1} \cdot (x-y)^+ \text{ and } ||x-y||_1 = ||(x-y)^+||_1 + ||(y-x)^+||_1.$$

Let $n = |N|$, $s_i \in \mathbb{Z}_+$ denote the supply of good $i \in M$, $\vec{s} \in \mathbb{Z}^m_+$ the supply vector and $\vec{b}$ the vector of cash endowments. An economy is the collection $\{\{U_j\}_{j \in N}, \vec{b}, \vec{s}\}$.

Definition 2.1 *A* <u>*competitive equilibrium*</u> *for the economy* $\{\{U_j\}_{j \in N}, \vec{b}, \vec{s}\}$ *is a price vector* p *and demands* $x^j \in Ch_j(p)$ *for all* $j \in N$ *such that* $\sum_{j \in N} x^j \leq s$ *with equality for each* $i \in M$ *for which* $p_i \neq 0$.

Definition 2.2 *A* <u>*α-approximate competitive equilibrium*</u> *for the economy* $\{\{U_j\}_{j\in N}, \vec{b}, \vec{s}\}$ *is a competitive equilibrium for the economy* $\{\{U_j\}_{j\in N}, \vec{b}, \vec{s'}\}$*, where* $|s_i - s'_i| \leq \alpha$ *for every good* $i \in M$*.*

Our results rely on the concept of a pseudo-equilibrium (see Milgrom and Strulovici (2009)) where agent preferences are 'convexified' by replacing each agent $j \in N$'s choice correspondence by its convex hull, $conv(Ch_j(p))$. Each $x \in conv(Ch_j(p))$, because it is a convex combination of the bundles in $Ch_j(p)$, can be interpreted as a lottery over those bundles, with x itself being the 'expected' bundle. A pseudo-equilibrium allocates to each agent a lottery over bundles.

Definition 2.3 *A price vector* p *and* $x^j \in conv(Ch_j(p))$ *for all* $j \in N$ *is a* <u>*pseudo-equilibrium*</u> *if* $\sum_{j\in N} x^j \leq s$ *with equality for every good* $i \in M$ *with* $p_i \neq 0$*.*

A pseudo-equilibrium exists in our setting. See Appendix A for a proof.

3 Δ-substitutes

In keeping with the literature, we assume that each agent is interested in consuming at most one copy of each good, i.e., single copy demand.[6] We accommodate multi-copy demand by reduction to single-copy demand.

Definition 3.1 *A utility function* $U(x, b-p\cdot x)$ *satisfies the (single copy demand)* <u>*Δ-substitutes*</u> *property if for any price vector* p *and endowment* b*, either* $|Ch(p)| = 1$ *or there exist two different bundles* $x, y \in Ch(p)$ *such that* $||(x-y)^+||_1 \leq \Delta$ *and* $||(y-x)^+||_1 \leq \Delta$*.*

$\Delta-$ substitutes is a condition on demand, not preferences. Demand is the primitive rather than preferences, consistent with many textbook treatments (e.g. Varian (2010)). To understand the intuition behind the condition, suppose $|Ch(p)| > 1$ and a perturbation in

[6]Sometimes called unit demand, but this term is also used when agents demand at most one unit of any good, so we avoid it.

price to p'. For a sufficiently small perturbation, we expect (and this is proved below) that $Ch(p) \cap Ch(p') \neq \emptyset$. If at price p the agent was consuming $x \in Ch(p)$, they can switch to $y \in Ch(p) \cap Ch(p')$ after the change. $\Delta-$ substitutes restricts the number of goods that can be swapped in and out of x to obtain y. When $\Delta = 1$, only one-for-one substitution after a price change is allowed. For example, a consumer can substitute a quantity of chocolate delivered today for the same quantity delivered tomorrow, or he may substitute one kind of cocoa bean for another. $\Delta > 1$ allows for a 1-for-2 exchange involving substituting one unit of sugar with two units of an artificial sweetener. A 2-for-1 exchange might involve dropping two complementary items, coffee and cream, for a caffeinated power drink. In this way, we sidestep the conceptual problem of extending the notion of complementarity between pairs of goods to larger sets of them.[7] As Samuelson (1974) observed, a pair of goods by themselves may complement each other, but in the presence of a third good, they become substitutes for each other.

To accommodate multi-copy demand, we employ the standard trick (see, for example, Baldwin et al. (2020)) of treating each copy of a good as a separate good. For example, three oranges are represented as three distinct objects: orange copy #1, orange copy #2, and orange copy #3. In this way, any bundle containing multiple copies of any good can be represented as a 0-1 vector, which we call its binary representation.

Formally, let $C \in \mathbb{Z}_+$ be a constant at least as large as the maximum number of copies of a good that an agent consumes. Make C copies of each good. Let $y \in \{0,1\}^{C \cdot m}$ be a binary representation of a bundle. The total number of copies of good $i \in M$ contained in y is

$$\mathcal{T}_i(y) = \sum_{k=C\cdot(i-1)+1}^{C\cdot i} y_k. \tag{1}$$

Thus, $y \in \{0,1\}^{C \cdot m}$ represents the bundle $\mathcal{T}(y) = (\mathcal{T}_1(y), .., \mathcal{T}_m(y))$.

In addition, we allow each copy of each good to have its price. Thus, a price vector

[7] By way of contrast, Candogan et al. (2018) builds a limited notion of pairwise complementarity directly into valuations

$p \in \mathbb{R}^{Cm}$ will have one component for each copy of each good.[8] Call a price vector $p \in \mathbb{R}^{Cm}$ **non-discriminatory** if all copies of the same good have the same price.

The following is the definition of Δ-substitutes for the multi-copy demand case.

Definition 3.2 *Multi-copy preferences satisfy the Δ -substitutes property if it is Δ -substitutes in its binary representation when each copy of each good has its own price.*

While a CE need not exist under Δ-substitutes preferences, one can perturb the supply of each good to guarantee the existence of CE. Our main result is the following.

Theorem 3.1 *If all agent's preferences possess the Δ-substitutes property, then for every supply vector s the economy $\{\{U_j\}_{j\in N}, \vec{b}, \vec{s}\}$ has a $2(\Delta - 1)$-approximate CE.*

Therefore, there exists a price vector where the excess demand for *each* good is at most $2(\Delta - 1)$, a quantity independent of both the number of agents, goods and the number of copies each good an agent might consume. The condition on preferences that guarantees this does not require quasi-linear preferences.

Theorem 3.1 is a consequence of a more general result regarding the approximate implementation of a pseudo-equilibrium. Specifically, let $(p, \{x^j\}_{j=1}^n)$ be a pseudo-equilibrium of an economy $\{\{U_j\}_{j\in N}, \vec{b}, \vec{s}\}$. Each $x^j \in conv(Ch_j(p))$ can be interpreted as the average of a lottery over bundles in $Ch_j(p)$. However, the joint allocation $\{x^j\}_{j=1}^n$ need not be implementable because there is no lottery over deterministic feasible allocations that gives, on average, to each agent j the bundle x^j. This is the implementability problem.[9] The following theorem shows that we can solve the implementability problem approximately.

Theorem 3.2 *Let $(p, \{x^j\}_{j=1}^n)$ be a pseudo-equilibrium of an economy $\{\{U_j\}_{j\in N}, \vec{b}, \vec{s}\}$ in which each U_j satisfies the Δ-substitutes property. Then, the joint allocation $(x^1, ..x^n)$ can*

[8] Even though we allow each copy of a good to have its own price, we show that our main results Theorems 3.1 and 3.2 hold with non-discriminatory prices. Under quasi-linearity, this follows from Bikhchandani and Mamer (1997), but we do not assume that here.

[9] See Budish et al. (2013) for more on this issue.

be implemented as the average of a lottery over $Ch_1(p) \times .. \times Ch_n(p)$, such that each of the element in this lottery is an allocation of $2(\Delta - 1)$-approximate competitive equilibria.

Recall that preferences satisfy non-satiation because $U(x, w)$ is strictly increasing in w. Thus, CE allocations will be efficient. If we remove the assumption that $U(x, w)$ is strictly increasing in w, preferences violate non-satiation, and CE allocations need not be Pareto optimal. This is because some agents do not purchase their least expensive optimal bundle. To ensure efficiency, attention has focused on CE with slack (eg. McLennan (2017)) or 'paper money' (eg. Kajii (1996)). The first allows for reallocating unspent wealth (slack). The second interprets the slacks (or the dividends) as paper money allocated to the consumers before the market takes place. If the conditions for the existence of these efficient CE hold, they can be used in our implementation results (Theorem 3.2). The resulting lottery will be over allocations that are efficient CE.

Next, we show how Δ-substitutes is related to other classes of preferences. Throughout, we restrict attention to single copy demand, making direct comparison easier. We begin with the case $\Delta = 1$ and show that several well known preference classes are a special case of 1-substitutes. Then, we show that natural extensions of these preferences to capture complementary are special cases of Δ-substitutes.

3.1 1-substitutes verus gross substitutes

When $\Delta = 1$, Theorem 3.1 guarantees the existence of a CE. We show that 1-substitutes preferences strictly subsume many known preference conditions that guarantee the existence of a CE.

Recall the definition of gross substitutes:

Definition 3.3 *A utility function satisfies <u>gross substitutes</u> if for all budgets b and for every pair of price vectors p and q such that $p \leq q$ and for all $x \in Ch(p)$, there exists $y \in Ch(q)$ such that $y_i \geq x_i$ for all $i \in M$ such that $q_i = p_i$.*

We display an example of a utility function satisfying 1-substitutes but not gross substitutes.

Example 3.1 *Suppose two goods and single copy demand. Let*

$$U(x_1, x_2, t) = v(x_1, x_2) + f_\epsilon(t)$$

where , $v(0,0) = 0, v(1,0) = 1, v(0,1) = 2$ *and* $v(1,1) = 3$, *and* $\epsilon = 0.01$.

Where $f_\epsilon(.)$ *captures the soft budget of agents. In particular,*

- $f_\epsilon(t) = -\infty$ *for* $t < -\epsilon$
- $f_\epsilon(.)$ *is strictly increasing on* $(-\epsilon, \infty)$,
- $f_\epsilon(0) = 0$, $f_\epsilon(-\frac{\epsilon}{2}) = -\epsilon$,
- $\lim_{t\to\infty} f_\epsilon(t) = \epsilon$ *and* $\lim_{t\to-\epsilon} f_\epsilon(t) = -\infty$.

Figure 1 provides a graphical illustration of $f(\cdot)$.

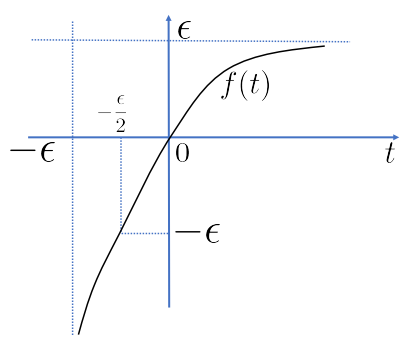

Figure 1: $f_\epsilon(.)$

Proposition 3.1 *The utility function defined in Example 3.1 is* 1*-substitutes but not gross substitutes.*

Proof. To show that the utility function in Example 3.1 is 1-substitutes, we argue that the agent's choice correspondence at any price vector $p = (p_1, p_2)$ can never be $\{(0,0),(1,1)\}$.

Assume, for a contradiction, that there is a price vector p such that $Ch(p) = \{(0,0),(1,1)\}$. For any money endowment $b \geq 0$, the utility of bundle $(0,0)$ is $v(x_1,x_2) + f(b - p_1x_1 - p_2x_2)$. Therefore, $U(0,0,b) = f(b) < \epsilon$. If $b - p_1 \geq -\frac{\epsilon}{2}$, then, $U(1,0,b-p_1) = v(1,0) - \epsilon > f(b) = U(0,0,b)$, contradicting the assumption that $(0,0)$ is in the choice correspondence. Hence, $b - p_1 < -\frac{\epsilon}{2}$. Similarly $b - p_2 < -\frac{\epsilon}{2}$. Thus,

$$b - p_1 - p_2 \leq b - p_1 + b - p_2 < -\epsilon.$$

Therefore, the utility of bundle $(1,1)$ is $U(1,1,b-p_1-p_2) = -\infty$, contradicting the fact that $(1,1)$ is in the choice correspondence.

Now we show that the utility function in Example 3.1 violates gross substitutes. Gross substitutes requires that if the price of a single good increases, the demand for the other goods does not decline. Suppose $b = 1, p_1 = p_2 = 0.4$. The unique utility-maximizing bundle is $(1,1)$. Now increase p_2 to 0.7. The unique utility maximizing bundle is $(0,1)$. Thus, the utility function in Example 3.1 is not gross substitutes. ■

Proposition 3.2 *Gross substitutes is a strict subset of* 1*-substitutes.*

Proof. Example 3.1 shows that it suffices to prove that gross substitutes implies 1-substitutes. Let $|Ch(p) > 1$. Choose $x, y \in Ch(p)$ so that x and y form an edge (1-dimensional face) of $conv(Ch(p))$.

First, we show that we can perturb p and the budget b so that the new choice correspondence contains x and y only.

Because x, y is an edge of $conv(Ch(p))$, there exists a weight vector $v \in \mathbb{R}^m$ such that

$$v \cdot x = v \cdot y < v \cdot z \text{ for all } z \in Ch(p) \setminus \{x, y\}. \tag{2}$$

Set $p' = p + \epsilon \cdot v$, for small $\epsilon > 0$, and change the agent's budget to $b' = b + \epsilon \cdot v \cdot x$. Choose ϵ so that $Ch(p') = \{x, y\}$. This corresponds to determining an ϵ that satisfies:

$$U(x, b + \epsilon \cdot v \cdot x - p' \cdot x) = U(y, b + \epsilon \cdot v \cdot x - p' \cdot y) \tag{3}$$

$$U(x, b + \epsilon \cdot v \cdot x - p' \cdot x) > U(x, b + \epsilon \cdot v \cdot x - p' \cdot z) \; \forall z \in X \setminus \{x, y\} \tag{4}$$

Equation (3) follows from (2). Existence of a suitable ϵ satisfying (4) follows from $U(.,.)$ being strictly decreasing and continuous in transfers.

Now, assume, for a contradiction, that $||(x - y)^+||_1 \geq 2$, and without loss of generality that $x_1 = x_2 = 1$, $y_1 = y_2 = 0$. We now invoke the gross substitute property for the given choice correspondence at p'. If we increase the price of good 1 by $\delta > 0$ to get a new price q, then the cost of bundle x increases, while the cost of bundle y is unchanged. Hence, at q,, the agent will only choose y, but $y_2 < x_2$ even though $q_2 = p'_2$, which contradicts the definition of gross substitutes. Thus, $||(x - y)^+||_1 \leq 1$. By symmetry, we also obtain $||(y - x)^+||_1 \leq 1$. This shows that the utility function is 1-substitutes. ■

Quasi-linear gross substitutes with single copy demand is equivalent to the single improvement property as well as no complementarities (see Gul and Stacchetti (1999)). This equivalence fails in the non-quasilinear setting.[10] We show these properties are a special case of 1-substitutes. Together with Theorem 3.1, this shows that single improvement/ no complementarities implies the existence of CE in a non-quasi-linear setting as well. To the best of our knowledge, this extension of Gul and Stacchetti (1999) to the non-quasi-linear setting is new.[11]

We use the following definition of single improvement and no complementarities that are modified to account for the cash endowment b.

[10] Schlegel (2018) shows that the law of aggregate demand is needed to guarantee equivalence.
[11] It is not implied by either Echenique (2012) or Baldwin et al. (2020).

Definition 3.4 *We say $U(x, b-t)$ satisfies the* <u>*single improvement*</u> *property if for all b and all price vectors p if $x \notin Ch(p)$, there exists a superior bundle y, (that is $U(x, b-p\cdot x) < U(y, b-p\cdot y)$) such that $||(x-y)^+||_1 \leq 1$ and $||(y-x)^+||_1 \leq 1$.*

Definition 3.5 *We say $U(x, b-t)$ satisfies the* <u>*no complementarities*</u> *property if for all b and all price vectors p if $x, y \in Ch(p)$ and any bundle $x' \leq x$, then there exists a bundle $y' \leq y$ such that $x - x' + y' \in Ch(p)$.*

Proposition 3.3 *If $U(.,.)$ satisfies either the single improvement property or no complementarities, then it satisfies 1-substitutes.*

Proof. Let $x, y \in Ch(p)$ determine an edge of $conv(Ch(p))$. As in the proof of Proposition 3.2, we perturb the price vector to p' and the cash endowment of the agent so that $Ch(p') = \{x, y\}$. We will show that $||(x-y)^+||_1 \leq 1$ and $||(y-x)^+||_1 \leq 1$. Assume not, without loss we may assume $||(x-y)^+||_1 \geq 2$. and $x_1 = x_2 = 1; y_1 = y_2 = 0$.

If $U(.,.)$ satisfies the no complementarity property, we can remove good 1 from x and replace it with a subset of goods from y to obtain a new bundle in the choice correspondence. However, this contradicts the fact that $Ch(p') = \{x, y\}$.

Suppose that $U(.,.)$ satisfies single improvement. Then, we can increase the price of good 1 by a small ϵ to get a new price vector p'' such that $x \notin Ch(p'')$ and the utility at x is greater than all other bundles except bundle y, which is the only bundle in $Ch(p'')$. In other words, y is the only bundle utility superior to x. This contradicts single improvement because $||(x-y)^+||_1 \geq 2$. ■

Baldwin et al. (2020) accommodate non-quasi-linear preferences using Hicksian demand. Given a price vector p and a target level of utility u, the Hicksian demand at (p, u) is

$$D_H(p, u) = \arg\min\{px : x \in X, U(x, b-p\cdot x) \geq u\}.$$

They introduce an analog of gross substitutes for Hicksian demand called (strong) net substitutes.[12]

DEFINITION 3.6 *$U(x, b - p \cdot x)$ satisfies* <u>*net substitutes*</u> *if for all utility levels u, budget b and price vectors p and $\lambda > 0$ whenever $D_H(p, u) = x$ and $D_H(p + \lambda e^i, u) = x'$ where e^i is the ith unit vector, we have that $x'_k \geq x_k$ for all $k \neq i$.*

Baldwin et al. (2020) show that if all preferences satisfy net substitutes, a CE exists.

The net substitutes property must hold for all prices *and* budgets *and* all utility levels. 1-substitutes, however, is a property that needs to hold for all prices and budgets but not utility levels. We require it to hold at the optimal utility level only.

Proposition 3.4 *Suppose $U(x, b - t)$ satisfies net substitutes, then, $U(.,.)$ is 1-substitutes.*

Proof. In the proof, we invoke the net-substitutes condition only at the maximum utility level. Given price vector p, let $u^* = \max_{x \in X} U(x, b - p \cdot x)$. At the optimal utility level u^*, Hicksian demand coincides with the choice correspondence. That is, $D_H(p, u^*) = Ch(p)$.

Baldwin et al. (2020) show that one can associate with $D_H(p, u^*)$ an auxilliary quasi-linear utility function such that $D_H(p, u^*)$ is a choice correspondence of that auxilliary function. Under net-substitutes, the auxilliary function satisfies the gross substitute property. Hence, by Proposition 3.2, $U(.,.)$ is 1-substitutes. ■

We conclude this subsection by noting, without proof, that gross substitutes, net substitutes, and 1-substitutes are equivalent under quasi-linearity.

3.2 Special classes of Δ-substitutes

We start with an obvious example of Δ-substitutes.

[12] A similar approach is taken in Danilov et al. (2001), but this condition is not explicitly articulated in that paper.

Example 3.2 *If an agent is satiated outside a finite region, as is true in our case, an agent's preferences satisfy the Δ-substitutes property for a suitable chosen Δ. Examples are Shapley and Shubik (1971), Dierker (1971)), Quinzii (1984), Budish (2011) and Milgrom and Watt (2021). In Budish (2011), goods correspond to courses in an academic term, and there is usually an upper limit on the number of courses a student can take in the term. If that limit is Δ, i.e., no bundle can contain more than Δ items, the preferences satisfy Δ substitutes. In this example, no limit is imposed on the pattern of substitutes and complements between goods within a bundle of size Δ.*

Agents can have preferences for bundles whose size exceeds some Δ, but nevertheless, satisfy Δ-substitutes. Next, we describe natural properties of utility functions that are special cases of Δ-substitutes.

To motivate the next definition, consider what may happen when the price of a good, i, say, is increased. The demand for good i as well as goods whose prices are unchanged, may decline. These goods might be complements of good i. Gross substitutes forbids this. The next definition relaxes this.

Definition 3.7 *A utility function satisfies the* <u>*Λ-bounded substitutes property*</u> *if, for every two price vectors (p, q) differing in one coordinate and $p \leq q$, then for all $x \in Ch(p)$, there exists $y \in Ch(q)$ such that $|\{i \in M : y_i < x_i, q_i = p_i\}| \leq \Lambda$.*

Gross substitutes corresponds to $\Lambda = 0$.

Proposition 3.5 *A utility function that satisfies the Λ-bounded substitutes property is $(\Lambda + 1)-$substitutes.*

Proof. Suppose for some price vector p we have $|Ch(p)| > 1$. Choose $x, y \in Ch(p)$ so that x and y form an edge (1-dimensional face) of $conv(Ch(p))$. As in the proof of Proposition 3.2, we perturb the price vector to p' and the cash endowment of the agent so that $Ch(p') = \{x, y\}$.

We now show that $||(x - y)^+||_1 \leq \Lambda + 1$. If not, alter p' by choosing an item in x but not y and increasing its price by ϵ. The new choice correspondence contains only y. However,

compared with x, there are more than Λ items whose prices are unchanged but are no longer demanded at this new price.

By symmetry, we also have $||(y-x)^+||_1 \leq \Lambda + 1$. Hence, the utility function is $(\Lambda+1)$-substitutes. ■

We illustrate Definition 3.7 with an example of a preference that we call *bundled gross substitutes.*

Example 3.3 *Each agent is interested in at most one copy of each good. Associated with each agent is a partition $P_1, P_2 \ldots, P_k$ of M such that $|P_r| \leq \Delta$ for all $r = 1, \ldots, k$. The partitions can vary among agents. If x is a bundle, let $x|_{P_r}$ denote the sub-bundle consisting only of goods in P_r. Suppose the price of one item in P_r increases, then, the demand for goods outside P_r does not decrease. Thus, within P_r, there can be complementarities among the goods. Fox and Bajari (2013) suggests that spectrum preferences resemble bundled substitutes. Spectrum licenses within the same geographical cluster complement each other but are substitutes across different clusters. It is easy to see that bundled gross substitutes satisfies $\Delta-$substitutes property.*

We generalize the single improvement property of Gul and Stacchetti (1999) as follows.

Definition 3.8 *A utility function satisfies the <u>Δ-improvement property</u> if, whenever $x \notin Ch(p)$ there is a bundle y with higher utility such that $||(y-x)^+||_1$ and $||(x-y)^+||_1 \leq \Delta$.*

Definition 3.8 extends single improvement in two ways. It relaxes quasi-linearity and allows for the swap of up to Δ goods to increase utility, as opposed to the 1-for-1 swap in Gul and Stacchetti (1999).

Proposition 3.6 *A utility function that satisfies Δ-improvement is $\Delta-$substitutes.*

Proof. Let $x, z \in Ch(p)$ with $x \neq z$. Without loss, we may assume that $x_1 = 1$ and $z_1 = 0$. Consider price vector q obtained from p by increasing p_1 by a small $\epsilon > 0$ such that

$$U(x, b - q \cdot x) > U(x', b - q \cdot x'), \text{ for all } x' \notin Ch(p). \tag{5}$$

We can choose such an $\epsilon > 0$ because $U(.,.)$ is strictly decreasing and continuous in transfers. Let S be the bundles with higher utility than x at price q. Because of (5), $S \subset Ch(p)$. Futhermore, $S \neq \emptyset$ because $z \in S$. Now, the Δ-improvement property implies that there exists $y \in S$ with $||(y - x)^+||_1 \leq \Delta$ and $||(x - y)^+||_1 \leq \Delta$. But $S \subset Ch(p)$, hence, $y \in Ch(p)$ and therefore, the utility function satisfies Δ-substitutes. ■

We postpone a discussion of the relationship between $\Delta-$ substitutes and the notion of demand types introduced in Baldwin and Klemperer (2019) to Section 5.1.

4 Applications to Market Design

Theorems 3.1 and Theorem 3.2 are relevant for the design of (approximate) market-clearing mechanisms where money is present. They are also relevant, as we show, in settings without money and ordinal preferences, in particular, the design of pseudo-market mechanisms that use artificial currency because of fairness concerns or a desire to limit certain trades.[13] We illustrate with two examples.

4.1 Combinatorial Assignment

Our first application is to the combinatorial assignment problem (Budish, 2011), which aims to allocate bundles of goods to agents fairly and efficiently, solely based on their ordinal

[13] In course allocation, for example, all students are considered to have an equal claim on available classes. In this case, the goal of a pseudo-market mechanism is to allocate the available classes fairly (see Budish (2011)). In the context of food banks (Prendergast (2017)), the goal is to reallocate food to reduce waste. Artificial currency ensures that donated food remains within the food bank network rather than sold to the 'outside.'

preferences, without resorting to monetary exchanges. Budish (2011) proposed a modification of the competitive equilibrium with equal incomes (CEEI) mechanism due to Hylland and Zeckhauser (1979), called A-CEEI (approximate CEEI). Unlike the CEEI mechanism, the A-CEEI mechanism returns a deterministic allocation. To circumvent the non-existence of a CE, agents receive randomly perturbed budgets, and the market only approximately clears. The outcome is a deterministic allocation that is approximately Pareto optimal and approximately envy-free.

We introduce an alternative modification of the CEEI mechanism to achieve results similar to Budish (2011). Our approach involves creating auxiliary utility functions that mimic ordinal preferences; then, we leverage Theorem 3.1. Unlike Budish (2011) we measure the deviation from market clearing using the ℓ- infinity norm instead of the ℓ_2.

Under single copy demand Budish (2011) shows that the ℓ_2 norm of the excess demand vector is at most $\frac{\sqrt{\min\{2\Delta,m\}m}}{2}$, where Δ is the size of a maximum bundle that an agent is interested in consuming and m is the number of goods. Our mechanism guarantees a good-by-good bound on excess demand of $2(\Delta - 1)$.[14]

The bound based on the ℓ_2 norm has two disadvantages. First, to extend it to multi-copy demand, one needs to make each individual copy a separate good yielding a bound of $\frac{\sqrt{C\min\{2\Delta,m\}m}}{2}$ where C is the maximum number of copies of each good that an agent can consume. Second, the ℓ_2-norm aggregates excess demand across courses, so the excess demands may be disproportionally distributed across goods, which can be impractical in some applications. In particular, the A-CEEI mechanism has been implemented to assign students to courses at the Wharton School (see Budish et al. (2017)). Agents are students, and objects are courses; no student wants more than one copy of any course, and the number of seats in a course is the supply of that course. Seat supply is upper bounded by Fire Safety

[14] Nguyen et al. (2016) propose an alternative mechanism based on the probabilistic serial mechanism (see Bogomolnaia and Moulin (2001)). It returns an allocation in which the excess demand for each course is at most $\Delta - 1$. However, it enjoys different efficiency and fairness properties than the A-CEEI mechanism. An argument in Nguyen et al. (2016), specific to the case where no agent demands a bundle whose size exceeds Δ, yields a bound of $\Delta - 1$ on excess demand.

regulations. Each semester, students take about five courses, thus $\Delta = 5$. Because the bound on excess demand is in terms of the ℓ_2-norm, it is possible that the number of students assigned to a course will exceed the regulated limit. When this happens, the capacity of that course is adjusted downwards, and the mechanism is rerun. This heuristic has no theoretical guarantees and is time-consuming as each iteration requires the computation of a fixed point. Our mechanism, on the other hand, is independent of both m and C, holds for a broader class of preferences, and guarantees that the excess demand of each course is bounded by a constant. Thus, one knows a-priori how many seats in each class to 'withhold' to ensure feasibility.

Even though the two bounds involve different metrics, they can be compared. In the worst case, our bound in the ℓ_∞-norm implies a bound of $O(\Delta\sqrt{m})$ in the ℓ_2-norm, which is slightly worse than the bound of $O(\sqrt{\Delta m})$ due to Budish (2011) under single copy demand. Under multi-copy demand, assuming $C = \Delta$, both bounds become $O(\Delta\sqrt{m})$ in the ℓ_2-norm.

Another contrast can be drawn based on the aggregate excess demand, which is the sum over all goods of the number of units of each over-allocated good. Suppose the Euclidean distance between supply and demand is $O(\sqrt{\Delta m})$. In that case, the aggregate excess demand can be of order $O(\sqrt{\Delta} \cdot m)$.[15] In our case, each good is over-allocated by at most $O(\Delta)$ units. However, if one allows under-allocation to compensate for over-allocation, our mechanism can be modified to yield $O(\Delta)$ aggregate excess demand. Namely, introduce a dummy good g_0 representing the aggregate resource. A bundle x in the original economy corresponds to a new bundle consisting of x and $||x||_1$ units of good g_0. The maximum bundle size is now 2Δ, and our result bounds the excess demand on all the goods, including the good g_0, corresponding to the aggregate demand by at most $2(2\Delta - 1)$.

We next describe how Theorem 3.1 can be adapted to accommodate ordinal preferences. Suppose each student j has a feasible set of bundles (course schedules) X_j and a budget of

[15] For example, when each good has an excess demand of $\sqrt{\Delta}$, the ℓ_2-norm is $\sqrt{\Delta m}$ and the total excess demand is $m\sqrt{\Delta}$.

one unit of artificial currency.[16] Let $\succeq_j$ denote the ordinal preference of agent j and let $v_j(x)$ be a utility function that represents these preferences. We assume $0 \in X_j$ and $v_j(0) = 0$. Without loss, we can assume that for each $x \in X_j$, $v_j(x) \geq 0$ is a rank score of the bundles.

Recall, Theorem 3.1 on $2(\Delta - 1)$ approximate CE relies on expenditure augmented utilities. In course allocation, agents' preferences depend on goods only. To apply Theorem 3.1, we must mimic each student's preferences using an appropriately chosen expenditure augmented utility function. Define the following auxiliary utility function for each student. Let

$$U_j^\epsilon(x, t) := v_j(x) + f_\epsilon(t) \tag{6}$$

where $f_\epsilon(t)$ is defined as in example 3.1. Thus, given a money endowment (budget) of \$1, and price p, the auxiliary utility of agent j for bundle x is $v_j(x) + f_\epsilon(1 - p \cdot x)$. Intuitively, for ϵ sufficiently small, the preference ordering over budget feasible bundles induced by the auxiliary utility function should closely approximate that of the actual utility function. We now formalize this intuition.

Let $Ch_j^\epsilon(p)$ denote the choice correspondence of the auxiliary utility function. We show that for any bundle in the choice correspondence of the auxiliary utility function, there is a way to perturb the budget so that under the original ordinal preference, the bundle continues to be the optimal choice. Formally, we have the following.

Claim 4.1 *Let $x \in Ch_j^\epsilon(p)$, then, there exists a budget b such that $1 \leq b < 1 + \epsilon$ and*

$$x = max_{(\succeq_j)}\{x' \in X_j \text{ and } p \cdot x' \leq b\}.$$

Proof. If $x \in Ch_j^\epsilon(p)$, then, the utility at bundle x is at least the utility of bundle $\vec{0}$:

$$U_j^\epsilon(x, 1 - p \cdot x) = v_j(x) + f_\epsilon(1 - p \cdot x) \geq U_j^\epsilon(\vec{0}, 1 - p \cdot \vec{0}) > 0.$$

[16] The approach trivially extends to cases where the agents have differing budgets.

Therefore, $1 - p \cdot x > -\epsilon$, which implies $p \cdot x < 1 + \epsilon$.

Define $b := \max\{p \cdot x, 1\}$. Clearly, $1 \leq b < 1 + \epsilon$. Suppose first that $p \cdot x \geq 1$. Then, $b = p \cdot x$. Since $x \in Ch_j^\epsilon(p)$, for any bundle x' such that $x' \succ_j x$, we have

$$U_j^\epsilon(x', 1 - p \cdot x') \leq U_j^\epsilon(x, 1 - p \cdot x)$$

$$\Rightarrow v_j(x') + f_\epsilon(1 - p \cdot x') \leq v_j(x) + f_\epsilon(1 - p \cdot x).$$

As $v_j(x') > v_j(x)$ it follows that $1 - p \cdot x' < 1 - p \cdot x$ and therefore, $p \cdot x' > p \cdot x = b$. Hence, x' is not feasible.

Now, suppose $p \cdot x \leq 1$. Then, $b = 1$. Furthermore, for any bundle x' such that $x' \succ_j x$ we have $U_j^\epsilon(x, 1 - p \cdot x) < v_j(x) + \epsilon < v_j(x')$. The first inequality holds because $f(.) < \epsilon$. Thus, for any bundle x' such that $x' \succ_j x$, we have $p \cdot x' > 1$, otherwise $U_j^\epsilon(x', 1 - p \cdot x') \geq v_j(x') > U_j^\epsilon(x, 1 - p \cdot x)$, contradicting $x \in Ch_j^\epsilon(p)$. ■

To formally state our result, we modify the notion of approximate competitive equilibrium with equal incomes to account for the ℓ_∞-norm instead of the ℓ_2 norm as in Budish (2011).

Definition 4.1 *The allocation $(x^1, .., x^n)$, budgets $(b_1, .., b_n)$ and item prices $(p_1, .., p_m)$ are a (Δ, ϵ)-approximate competitive equilibrium with almost equal incomes (ACEEI) if the following holds:*

- $x^j = max_{(\succeq_j)}\{x \in X_j$ *and* $p \cdot x \leq b_j\}$ *for all* $j \in N$
- $1 \leq b_j < 1 + \epsilon$ *for all* $j \in N$
- $\max_{1 \leq i \leq m} |z_i| \leq \Delta$, *where*
  - *a)* $z_i = \sum_j x_i^j - s_i$ *if* $p_i > 0$, *and*
  - *b)* $z_i = \max\{0, \sum_j x_i^j - s_i\}$ *if* $p_i = 0$

Thus, we obtain the following result as a direct consequence of Theorem 3.1 and Theorem 3.2.

Proposition 4.1 *For every $\epsilon > 0$, if U_j^ϵ is defined as in (6) and is Δ-substitutes, there exists a $(2(\Delta-1), \epsilon)$-ACEEI. Moreover, all pseudo equilibria of the economy with the utility profile $\{U_j^\epsilon\}_{j \in N}$ can be implemented as a lottery over a set of $(2(\Delta-1), \epsilon)$-AEEI.*

4.2 Implementability of probability shares

The second application generalizes the CEEI mechanism of Hylland and Zeckhauser (1979), who proposed the use of lotteries that were determined via CE allocations with *equal* artificial currency endowments. Goods are treated as divisible with fractional quantities interpreted as probability shares. Assuming agents have unit demand and von Neumann-Morgenstern preferences, the CEEI mechanism returns a lottery over feasible allocations that is both ex-ante Pareto optimal and ex-ante envy-free.[17]

The implementability problem described in Section 3 is the fundamental obstacle to extending this mechanism to multi-copy demand. In this section, we show that Theorem 3.2 can be used to extend the CEEI mechanism while satisfying, ex-ante, *any* finite collection of linear constraints on the probability shares an agent can consume. These constraints can be endowment limitations, bounds on the probability of receiving bundles of a specific category as well as budget constraints. Such requirements could be baked into the utility function but might destroy any nice properties the utility function enjoyed. We show that under Δ-substitutes, a CE in probability shares can be implemented as a lottery over approximately feasible allocations.

The critical step is to show that a CE allocation in probability shares with Δ-substitute preferences corresponds to a pseudo-equilibrium allocation of a related economy with quasi-linear preferences that are also Δ-substitutes. Therefore, we can apply Theorem 3.2 directly to obtain an implementation of the CE in probability shares. This differs from a similar step in Gul et al. (2019). Under substitute preferences, they show there is an equilibrium lottery over feasible allocations that is the limit of an infinite sequence of equilibrium lotteries of

[17] For a general discussion of efficiency in these settings see Miralles and Pycia (2021).

quasi-linear economies. In our case, starting from *any* equilibrium price of probability shares, we can construct a lottery over (approximate) feasible allocations that satisfy the equilibrium condition. Additionally, we extend this result to Δ-substitute preferences and incorporate other side constraints.

Let $v_j(x)$ be the value of agent $j \in N$ for bundle $x \in X_j$, which is the set of feasible bundles for j. Each agent j also has a budget of artificial currency of b_j. Assume $\vec{0} \in X_j$, $v_j(0) = 0$ and let $\mathcal{L}(X_j)$ denote the set of lotteries over bundles in X_j. For each lottery $\tilde{y} \in \mathcal{L}(X_j)$, denote by $\bar{v}_j(\tilde{y})$ its expected utility, i.e.,

$$\bar{v}_j(\tilde{y}) := \sum_{x \in X_j} v_j(x) \cdot Pr(\tilde{y} = x).$$

Let $\overline{y}$ be the average consumption of the lottery $\tilde{y}$, i.e.

$$\overline{y} := \sum_{x \in X_j} x \cdot Pr(\tilde{y} = x).$$

Thus, $p \cdot \overline{y}$ is the expected price of the lottery y.

Linear constraints on probability shares that an agent can consume can be represented in matrix form. Associated with each agent $j \in N$, is a $k \times m$ matrix $A_j(p)$ and vector $b^j(p) \in \mathbb{R}^k_+$. We assume $A_j(p)$ and $b^j(p)$ are continuous functions of p. Agent j is restricted to choosing lotteries $\tilde{y} \in \mathcal{L}(X_j)$ such that $A_j(p) \cdot \overline{y} \leq b^j(p)$. The budget constraint, $p \cdot \overline{y} \leq b$, would corresponds to one row of matrix $A_j(p)$. Examples of additional constraints are furnished below.

1. Quota constraints that arise from 'controlled-choice' in school assignment such as no more than K goods from a particular subset, G, of goods can also be accommodated. If e_G is a 0-1 characteristic vector of the set G, the constraint will be $e_G \cdot \overline{y} \leq K$. In this case, the corresponding row of $A_j(p)$ is independent of p.

2. The entries of $A_j(p)$ need not be positive. Hence, a requirement that the quantity of goods consumed from group G be at least 25% of the quantity consumed from group G' can also be encoded:

$$e_G \cdot \overline{y} - 0.25 e_{G'} \cdot \overline{y} \geq 0.$$

3. In the same spirit, the amount spent on goods in G must be at least as much as what is spent on goods in G'. If p_G is the price vector obtained from p by setting the prices of goods not in G to zero, we can express this constraint as follows:

$$-p_G \cdot \overline{y} + p_{G'} \cdot \overline{y} \leq 0.$$

Let

$$Ch_j^*(p) := \arg\max\{\bar{v}_j(\tilde{y}) : \tilde{y} \in \mathcal{L}(X_j), A_j(p) \cdot \overline{y} \leq b^j(p)\}.$$

If $v_j(\cdot)$ is agent j's von Neumann-Morgenstern utility function, then $Ch_j^*(p)$ is agent j's set of optimal lotteries all of whose average consumption satisfy the constraints $A_j(p) \cdot \overline{y} \leq b^j(p)$.

As before, let N be the set of agents, M the set of goods and s the supply vector. The economy described above is denoted by $(\{v_j(\cdot), X_j, A_j(p), b^j(p)\}_{j \in N}, \{s_i\}_{i \in M})$.

As in Hylland and Zeckhauser (1979), we consider a probability share equilibrium, in which each agent j directly consumes the lottery $\tilde{y}^j$, and the market clearing condition holds in expectation.

Definition 4.2 *A* <u>*probability share equilibrium*</u> *of an economy* $(\{v_j(\cdot), X_j, A_j(p), b^j(p)\}_{j \in N}, \{s_i\}_{i \in M})$ *is a price vector* p^* *and a profile of lotteries* $(\tilde{y}^{*1}, \ldots, \tilde{y}^{*n})$ *such that* $\tilde{y}^{*j} \in Ch_j^*(p^*)$ *and* $\sum_{j=1}^n \overline{y}^{*j} \leq s$*, with equality for any good* i *if* $p_i^* > 0$.

The existence of a probability share equilibrium follows a standard fixed-point argument. Refer to Appendix A for details. Even though, in a probability share equilibrium, the random assignment induced by $(\tilde{y}^{*1}, \ldots, \tilde{y}^{*n})$ satisfies the supply constraints in expectation, it

does not necessarily imply that it can be jointly implemented by a lottery over deterministic allocations, each of which satisfies the supply constraints. Our following definition of α-approximate randomized competitive equilibrium relaxes the requirement of exact implementability.

DEFINITION 4.3 *Given the economy* $(\{v_j(\cdot), X_j, A_j(p), b^j(p)\}_{j\in N}, \{s_i\}_{i\in M})$*, a vector price* p^* *and a lottery over the set* $X_1 \times \ldots \times X_n$ *is an* <u>*α-approximate randomized*</u> <u>*competitive equilibrium*</u> *if* p^* *and the corresponding lottery profile assigned to the agents form a probability share equilibrium and any realization of the lottery over allocates the supply of each good by at most* α*.*

The main result of this section is the following.

THEOREM 4.1 *Fix the economy* $(\{v_j(\cdot), X_j, A_j(p), b^j(p)\}_{j\in N}, \{s_i\}_{i\in M})$*, where the auxiliary function* $v_j(x) - p \cdot x$ *is* Δ*-substitutes for all* p *and for all* j*. Let* p^* *be the price of a probability share equilibrium; then there exists a lottery over the joint allocations such that together* p^* *it is a* $2(\Delta - 1)$*-approximate randomized competitive equilibrium.*

Theorem 1 in Gul et al. (2019) corresponds to a particular case of Theorem 4.1 when $A_j(p)$ represents budget constraints and $\Delta = 1$. Gul et al. (2019) accommodate additional constraints provided they can be folded into the agent's utility functions while preserving gross substitutes. Furthermore, the proof of Theorem 4.1 gives a polynomial time algorithm to construct an (approximate) randomized competitive equilibrium from any probability share equilibrium price, while Gul et al. (2019) relies on a limit sequence of fixed-point computations.

In contrast to Echenique et al. (2021), which also considers lottery equilibrium in a pseudo market, we avoid using personalized prices. Section 6 of Akbarpour and Nikzad (2020) contains an approximate competitive equilibrium result that accommodates side constraints on agents' choices, akin to what we have presented here. However, the quality of the approximation is measured in terms of agent utility and relies on a large market assumption.

Proof of Theorem 4.1

We start with a probability share equilibrium $(p^*, \tilde{y}^{*1}, .., \tilde{y}^{*n})$ which is guaranteed to exist by Lemma A.2. Now $\tilde{y}^{*j} \in \mathcal{L}(Y_j)$ means that $\overline{y}^{*j} \in conv(Y_j)$. This suggests interpreting $(p^*, \overline{y})$ as a pseudo-equilibrium which will allow us to invoke Theorem 3.2 to obtain a lottery implementation. In order to make the invocation, the following lemma is essential. The proof is given in Appendix B.

LEMMA 4.1 *If $v_j(x) - p \cdot x$ is $\Delta-$substitutes for all p, then, there exists u_j such that $u_j(x) - p \cdot x$ is also $\Delta-$ substitutes and*

$$Ch_j^*(p^*) = \mathcal{L}(Y_j(p^*)), \text{ where } Y_j(p^*) := \arg\max_{x \in X_j} \{u_j(x) - p^* \cdot x\}.$$

Lemma 4.1 characterizes the set of optimal lotteries at price p^* as the set of optimal lotteries over p^* of an auxiliary quasi-linear function. In the invocation of Theorem 3.2 we use $u_j(x) - p \cdot x$ in place of $v_j(x)$. The corresponding pseudo-equilibrium, can be expressed as a lottery over allocations such that each realized allocation is a $2(\Delta - 1)$ approximate competitive equilibrium at price p^*. Therefore, each agent j receives a lottery over $Y_j(p^*)$, i.e. $\mathcal{L}(Y_j(p^*)) = Ch_j^*(p^*)$. This show that the pseudo-equilibrium is a $2(\Delta - 1)$-approximate randomized competitive equilibrium for the economy $(\{v_j(\cdot), X_j, A_j(p), b^j(p)\}_{j \in N}, \{s_i\}_{i \in M})$.

5 Outline of Proof of Theorem 3.2

This section outlines the proof of our main result, which is based on two key ideas. First is the geometric characterization of Δ-substitute preferences, which shows that the convex hull of a choice correspondence has a special property that we call Δ-uniform. Δ-uniform polytopes have edges with at most Δ positive and Δ positive coordinates. This directly corresponds to the fact that one can obtain a nearby bundle in the choice correspondence

by swapping at most Δ items. Second is a novel extension of the Shapley-Folkman-Starr lemma, which states that *any* vector in the Minkowski sum of Δ-uniform polytopes is 'close' to a vector that can be expressed as a sum of the *vertices* of these same polytopes. In our setting, the 'any' vector is the aggregate demand of a pseudo-equilibrium. The nearby vector is a social approximate competitive equilibrium allocation.

5.1 Geometry of Δ-substitutes

First, we characterize $\Delta-$ substitutes in terms of the edges of the convex hull of the choice correspondence. This will also allow us to describe the connection between $\Delta-$substitutes and the notion of demand types introduced in Baldwin and Klemperer (2019).

Some terminology from convex geometry will be helpful. A **polyhedron** is defined by the intersection of a finite number of half-spaces. A bounded polyhedron is called a **polytope**, it is also the convex hull of a finite set of points.

A subset F of a polytope $Q \subseteq \mathbb{R}^m$ is called a **face** of Q if there is a hyperplane $\{x : h \cdot x = \gamma\}$ such that $F = Q \cap \{x : h \cdot x = \gamma\}$ and $Q \subseteq \{x : h \cdot x \leq \gamma\}$. The face F of the polytope Q is itself a polytope. A face of F is also a face of Q. The polytope Q is a (trivial) face of itself, where $h = \vec{0}, \gamma = 0$.

A zero-dimensional face, which is a single point, is called an **extreme point**, and a face of dimension 1 is called an **edge** of the polytope.

Definition 5.1 *Call a polytope P* **binary** *if it is the convex hull of 0-1 vectors. Denote its extreme points by $ext(P)$. A binary polytope is* **Δ-uniform** *if each of its edges, which is a $\{-1, 0, 1\}$ vector, has at most Δ positive and at most Δ negative coordinates.*

The following is a characterization of Δ-substitutes. See Appendix C for the proof.

Theorem 5.1 *A multi-copy preference satisfies Δ-substitutes, if and only if $conv(Ch(p))$ in its binary presentation, for all price vectors $p \in \mathbb{R}^{Cm}$, is Δ-uniform.*

Δ-uniform polytopes with $\Delta = 1$ have been studied extensively in combinatorial optimization and found applications in economics. Under single copy demand, 1-uniform polytopes are a special case of unimodular polytopes Danilov et al. (2001), that was recently interpreted as demand types in Baldwin and Klemperer (2019), that we explain next. To our knowledge, our paper is the first to study the case $\Delta > 1$ and its economic implications.

Baldwin and Klemperer (2019) proposed that an agent's preferences over bundles of indivisible goods be characterized in terms of demand changes in response to a small generic price change.[18] The set of vectors that summarize the possible demand changes is called the **demand type** of an agent. They give a variety of definitions that, under quasi-linearity are equivalent. One involves the edges of $conv(Ch(p))$. Scale the edges of $conv(Ch(p))$ so that their greatest common divisor is one and call them **primitive edge directions**. The demand type of an agent is the set of primitive edge directions of $Ch(p)$ for all price vectors p. Baldwin and Klemperer (2019) show that if utilities are quasi-linear, concave, and all agents' demand types form a unimodular vector system, then a CE exists.

The column vectors of a network matrix, which is a $0, \pm 1$ matrix with at most two non-zero entries in each column and these being of opposite sign, is a unimodular vector system. When the matrix of vectors of the demand type is a network matrix, the underlying quasi-linear preferences are gross substitutes. Hence, Baldwin and Klemperer (2019) extends Kelso Jr and Crawford (1982) but maintains quasi-linearity.

Under single copy demand, $\Delta-$ substitutes can be interpreted as requiring the demand type to consist of vectors with at most Δ positive and at most Δ negative coordinates. Hence, 1-substitutes is a subset of unimodular demand types under single-copy demand. However, the single copy version of Theorem 3.1 is not a consequence of Baldwin and Klemperer (2019) because we do not assume quasi-linearity. In addition, the multi-copy version of 3.1 does not assume the concavity of the utility function.

In the multi-copy case, even if the demand types come from a network matrix, it does not

[18] This idea is implicit in Danilov et al. (2001).

follow that the convex hull of choice correspondence in its binary presentation is 1-uniform (see the example below).

Example 5.1 *Let P be the convex hull of $(2,0)$ and $(0,2)$. P consists of a single edge whose vector representation is $(2,-2)$. The corresponding primitive edge direction is $(1,-1)$. Now, P in its binary presentation is the convex hull of the following points: $\{(1,1,0,0),(0,0,1,1)\}$. This binary polytope is not 1-uniform but 2-uniform.*

Unimodular demand types and 1-substitutes do not exhaust the space of preference conditions that guarantee the existence of a CE. The class of quasi-linear tree valuations introduced in Candogan et al. (2015) admits the existence of a CE but does not correspond to either a unimodular demand type or 1-substitutes.

5.2 Proof of Theorem 3.2

As stated earlier, we first express the aggregate demand of a pseudo-equilibrium as the Minkowski sum of vectors in the convex hull of the choice correspondences and use Theorem 5.2 below to derive Theorem 3.2.

Suppose a pseudo-equilibrium with price vector $p \in \mathbb{R}^m$ and demands $x^j \in conv(Ch_j(p))$ for all $j \in N$ such that $\sum_{j\in N} x^j \leq s$, and equality for each good i whenever $p_i > 0$. For economy of exposition only, suppose $\sum_{j\in N} x^j = s$, i.e., demand meets supply. This can be guaranteed by adding dummy agents consuming goods that are under demanded.

Let $P_j \subset [0,1]^{Cm}$ be the convex hull of agent j's choice correspondence in its binary presentation at the *non-discriminatory* price p. Let $y^j \in P_j$ be a binary representation of x^j. Then, by definition (see (1)), $\mathcal{T}(y^j) = x^j$. Thus, $\sum_{j\in N} x^j = s$ implies that $\sum_{j\in N} \mathcal{T}(y^j) = s$. Under a pseudo-equilibrium price p, s is in the Minkowski sum of the sets $\{conv(Ch_j(p))\}_{j\in N}$. For a pseudo-equilibrium to be an actual equilibrium, we need s to be in the Minkowski sum of $\{Ch_j(p)\}_{j\in N}$ instead. If each $y^j \in P_j$ was, in fact, an extreme point of P_j, then the pseudo-equilibrium is an actual equilibrium. Under appropriate conditions, this is 'approximately'

true.[19]

THEOREM 5.2 *Let $P_1, \ldots, P_n$ be binary Δ-uniform polytopes in $\mathbb{R}^{Cm}$. Then, each $y = (y^1, \ldots, y^n) \in P_1 \times \ldots \times P_n$ can be expressed as a convex combination of points in $ext(P_1) \times \ldots \times ext(P_n)$. Furthermore, for each $(z^1, \ldots, z^n) \in ext(P_1) \times \ldots \times ext(P_n)$ in the support of the convex combination, $||\sum_{j=1}^n \mathcal{T}(z^j) - \sum_{j=1}^n \mathcal{T}(y^j)||_\infty < 2\Delta - 1$.*

The proof of Theorem 5.2 is constructive and can be found in Appendix D. Its relationship to the Shapley-Folkman-Starr Lemma is discussed below.

Because of Theorem 5.1, when utilities are Δ-substitutes, the convex hull of the choice correspondences in their binary presentation, P_j-s, are Δ-uniform. Invoking Theorem 5.2, we can express the pseudo-equilibrium $(y^1, .., y^n)$ as a lottery over vectors in $ext(P_1) \times \ldots \times ext(P_n)$. Each such vector $(z^1, \ldots, z^n)$ is a binary representation of the allocation $(\mathcal{T}(z^1), \ldots, \mathcal{T}(z^n))$ in $Ch_1(p) \times \ldots \times Ch_n(p)$, which is a CE of the economy if we change the supply vector to be $\sum_{j=1}^n \mathcal{T}(z^j)$.

Notice, the supply vector $s = \sum_{j=1}^n \mathcal{T}(y^j)$ is integral, and because each z^j is an integral vector and $||\sum_{j=1}^n \mathcal{T}(z^j) - \sum_{j=1}^n \mathcal{T}(y^j)||_\infty < 2\Delta - 1$ it follows that $||\sum_{j=1}^n \mathcal{T}(z^j) - \sum_{j=1}^n \mathcal{T}(y^j)||_\infty \leq 2(\Delta - 1)$. This yields Theorem 3.2. ■

Because Theorem 3.2 relies on Theorem 5.2, we can relax the assumptions Theorem 3.2 and preserve its conclusions. Continuity in money and the convex hull of the choice correspondence being Δ-uniform suffice.

Theorem 5.2 has applications beyond obtaining social approximate equilibria. One can interpret any mechanism as giving to each agent $j \in N$, depending on what they report, a 'budget' or 'option' set, denoted B_j, from which they may choose. Let $Q_j \subseteq B_j$ be the set of agent j's most preferred elements of B_j. Now, 'convexify' the sets B_j and Q_j by allowing lotteries. Suppose these convexified sets have been chosen so that 'average' demand equals

[19] Recall that a polytope P is **binary** if it is the convex hull of 0-1 vectors and denote its set of extreme points by $ext(P)$. In our case, P will correspond to the convex hull of a choice correspondence and $ext(P)$ to the choice correspondence itself.

supply. If the sets Q_j are uniform polytopes, one may invoke Theorem 5.2 to implement these allocations as lotteries over approximately feasible allocations of the goods.

Theorem 5.2 is related to the Shapley-Folkman-Starr Lemma, which states, roughly, that the aggregate of many sets is approximately convex. Let $Q_1, \ldots, Q_n$ be a collection of sets. Suppose a vector y is contained in the Minkowski sum of the $conv(Q_i)$s, i.e., $y \in conv(Q_1) + \ldots + conv(Q_n)$. Is it the case that $y \in Q_1 + \ldots + Q_n$? In general, no, but according to the Shapley-Folkman-Starr lemma, for large n, the answer is 'close' to yes. We recall a version due to Cassels (1975) which allows for a direct comparison.[20]

THEOREM 5.3 *Let $Q_1, \ldots, Q_n$ be a collection of compact sets in $[0,1]^m$ with $n > m$. For any $y \in conv(Q_1) + \ldots + conv(Q_n)$ there exist vectors $\{z^j\}_{j=1}^n$ such that*

1. *$z^j \in conv(Q_j)$ for all j,*
2. *$|\{j : z^j \in ext(Q_j)\}| \geq n - m$, and*
3. *$||y - \sum_{j=1}^n z^j||_\infty \leq m$.*

Theorem 5.2 does not require that $n > m$ and the nearby vector is *actually* in the Minkowski sum of the Q_js. The stronger conclusion obtains because we restrict the Q_j's to be $\Delta-$uniform.

6 Conclusion

This paper makes two contributions. The first identifies a new sufficient condition for the existence of a competitive equilibrium which generalizes gross substitute preferences to non-quasi-linear settings and subsumes other known sufficient conditions. Second, a relaxation of the sufficient condition yields a social approximate equilibrium where the mismatch between supply and demand depends on preferences rather than the size of the economy. The usefulness of this approximation is illustrated in the context of pseudo-markets.

[20] See Budish and Reny (2020) for an example of another.

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Appendix

A Existence of pseudo-equilibrium and probability share equilibrium

A.1 Existence of pseudo-equilibrium

The next result establishes the existence of a pseudo-equilibrium and recapitulates the conditions under which it does so.

LEMMA A.1 *Let X_j denote the finite set of bundles that agent $j \in N$ can feasibly consume. Each agent j's utility function $U_j(x, b - p \cdot x)$ satisfies*

1. $U_j(0, 0) = 0$,
2. $U_j(x, b - p \cdot x) = -\infty$ *for* $x \notin X_j$
3. $U_j(x, b - p \cdot x)$ *is continuous in* $p \in \mathbb{R}^m$ *for each* $x \in X_j$ *and*
4. *there exists* $B > 0$ *such that if* $p \cdot x \geq B$, *then,* $U_j(x, b - p \cdot x) < U_j(\vec{0}, b)$.

Then, there exists a pseudo-equilibrium.

Proof. Denote by X_j the set of feasible bundles available to agent $j \in N$. Let $\mathcal{L}(X_j)$ be the

set of lotteries over X_j. We construct a correspondence

$$f : [0, B]^m \times \mathcal{L}(X_1) \times \ldots \times \mathcal{L}(X_n) \rightrightarrows [0, B]^m \times \mathcal{L}(X_1) \times \ldots \times \mathcal{L}(X_n)$$

and use Kakutani's fixed point theorem to show that it has a fixed point, $(p, x^1, .., x^n)$. This fixed point will correspond to a pseudo-equilibrium.

Given $\{x^j \in \mathcal{L}(X_j)\}_{j=1}^n$, let $\overline{x} = \sum_{j=1}^n x^j$ be the aggregate consumption. The excess demand vector is $\overline{x} - s$. Let $g(p, x^1, .., x^n) := (p + \overline{x} - s)^+$. Notice, for $p \in \mathbb{R}_+^m$, $g(p, x^1, .., x^n) = p$ implies that for all $i \in M$, if $p_i > 0$, then $\overline{x}_i = s_i$ and if $p_i = 0$, then $\overline{x}_i \leq s_i$. This is the condition that price and excess demand must meet to be considered a pseudo equilibrium.

We need the function g to be bounded to apply Kakutani's Theorem, so we modify it. As B is the bound on the willingness to pay of each agent, let

$$z_i(p, x^1, .., x^n) := \min\{(p_i + \overline{x}_i - s_i)^+, B\} \text{ for all } i \in M.$$

Each z_i is bounded. Define the following correspondence.

$$f(p, x^1, .., x^n) := (z, conv(Ch_1(p)), .., conv(Ch_n(p)).$$

It is easy to see that f satisfies all the conditions of Kakutani's Theorem. Let $(p, x^1, .., x^n)$ be a fixed point of the correspondence f.

At this fixed point $x^j \in conv(Ch_j(p))$. Because the correspondence is defined on the set of prices between 0 and B, we have $p_i \leq B$. We will show that strict inequality holds, that is, $p_i < B$. This implies that $p_i = (p_i + \overline{x}_i - s_i)^+$. Assume the contradiction that $p_i = B$, then because of the bounded willingness to pay assumption, no agent will consume good j. Thus, for all $x^j \in conv(Ch_j(p))$, $x_i^j = 0$. Therefore,

$$\min\{(p_i + \overline{x}_i - s_i)^+, B\} = (B - s_i)^+ < B = p_i.$$

This contradicts the fixed point condition. Hence, at the fixed point $p_i = (p_i + \overline{x}_i - s_i)^+$, implying that $(p, x^1, .., x^n)$ is a pseudo-equilibrium. ■

Note that the proof of Lemma A.1 does not use the fact that $U(,.)$ is strictly monotone in transfers.

A pseudo-equilibrium is an equilibrium with respect to the 'convexified' choice correspondences. As these may violate non-satiation, some competitive equilibria may be Pareto inefficient (with respect to the convexified choice correspondences). To avoid this, one can, when possible, select an **efficient disposal** equilibrium as defined in McLennan (2017).

A.2 Existence of probability share equilibrium

LEMMA A.2 *Every economy* $(\{v_j(\cdot), X_j, A_j(p), b^j(p)\}_{j\in N}, \{s_i\}_{i\in M})$ *has a probability share equilibrium.*

Proof. Lemma A.2 is not a direct corollary of Lemma A.1 because condition 4 in Lemma A.1 is not assumed. Nevertheless, we can employ a proof analogous to that of Lemma A.1 if we can show that there exists $B > 0$ such that if price $p_i = B$ for some good i, then the aggregate consumption for good i is smaller than its supply. In other words, $\overline{x}_i < s_i$. This is true because each agent j has a budget constraint b_j, thus if $B > \max_j n \cdot s_i b_j$, then when $p_i \geq B$, each agent consumes less than $\frac{s_i}{n}$ units of goof i. Hence, the total consumption is less than s_i.

B Proof of Lemma 4.1

First, observe that $Ch_j^*(p^*)$ can be characterized by optimal solutions to a linear program. To describe this linear program, for convenience, we omit the dependence of A_j and b^j on p^*. For each $x \in X_j$ let $w_x^j \in [0, 1]$ denote the fraction of bundle x selected. The average bundle $\bar{y}^j := \sum_{x\in X_j} w_x^j \cdot x$.

The constraint $A_j \cdot \bar{y}^j \leq \vec{b}^j$ can be reformulated in terms of w_x^j as $A_j \cdot (\sum_{x \in X_j} w_x^j \cdot x) \leq \vec{b}^j$. Thus, the problem of selecting a utility-maximizing lottery over X_j can be represented as follows:

$$\begin{aligned} \max \quad & \sum_{x \in X_j} w_x^j \cdot v_j(x) \\ \text{s.t} \quad & \sum_{x \in X_j} w_x^j = 1 \\ & A_j \cdot (\sum_{x \in X_j} w_x^j \cdot x) \leq \vec{b}^j \\ & w_x^j \geq 0 \ \forall x \in X_j. \end{aligned}$$

Let α^j be the dual variable associated with the first constraint and $\vec{\beta}^j \geq 0$ be the dual variable associated with the second constraint. For every good $i \in M$, denote

$$\gamma_i^j := (\vec{\beta}^j)^T \cdot A_j^i, \text{ where } A_j^i \text{ is the column } i \text{ of matrix } A_j.$$

Dual feasibility and complementary slackness imply:

$$v_j(x) - \sum_{i \in M} \gamma_i^j \cdot x_i \leq \alpha^j \text{ for all } x \in X$$

and

$$\text{if } w_x^j > 0, \text{ then } v_j(x) - \sum_{i \in M} \gamma_i^j \cdot x_i = \alpha^j.$$

Therefore,

$$\alpha^j = \max_{x \in X_j} \{v_j(x) - \sum_{i \in M} \gamma_i^j \cdot x_i\}$$

Thus any lottery in $Ch_j^*(p^*)$ will be over the bundles in $\arg\max\{v_j(x) - \sum_{i \in M} \gamma_i^j \cdot x_i\}$. Moreover, because of the complementarity and slackness, the reverse is also true if a lottery $\tilde{z}$ over $\arg\max\{v_j(x) - \sum_{i \in M} \gamma_i^j \cdot x_i\}$ with the average $\overline{z}$ satisfying $A_j \overline{z} \leq b^j$, then $\tilde{z} \in Ch_j^*(p^*)$.

Now, consider the following valuation function

$$u_j(x) := v_j(x) + \sum_{i \in M} (p_i^* - \gamma_i^j) \cdot x_i.$$

Because $u_j(.), v_j(.)$ are shifted by a linear function, under u_j the demand at price p is the same as the demand under v at the corresponding shifted price. Because Δ-substitution is a property on the choice correspondence at all prices, thus if the auxiliary utility $v_j(x) - p \cdot x$ is Δ-substitutes, then so is $u(x) - p \cdot x$. Furthermore,

$$\arg\max\{v_j(x) - \sum_{i \in M} \gamma_i^j \cdot x_i\} = \arg\max\{u_j(x) - p^* \cdot x\}.$$

Hence, $Ch^*(p^*)$ is exactly the set of lotteries over $\arg\max\{u_j(x) - p^* \cdot x\}$. ■

C Proof of Theorem 5.1

First, we show the 'only if' direction, i.e. $conv(Ch(p))$ is Δ-uniform. By the hypothesis of the Theorem, for all prices p, $Ch(p)$ in its binary presentation is either a singleton or there exist $x, y \in Ch(p)$ such that $\max\{||(y-x)^+||_1, ||(x-y)^+||_1\} \leq \Delta$. Suppose, for a contradiction, that $conv(Ch(p))$ is not $\Delta-$uniform. Then, there exists $u, v \in Ch(p)$ that form an edge such that $\max\{||(u-v)^+||_1, ||(v-u)^+||_1\} > \Delta$.

Similar to the proof of Proposition 3.5, we can perturb prices p to p' such that $Ch(p') = \{u, v\}$. Because of the Δ-substitutes property $\max\{||(u-v)^+||_1, ||(v-u)^+||_1\} \leq \Delta$, a contradiction. Hence, $conv(Ch(p))$ is a Δ uniform polytope.

The 'if' direction is straightforward. If $conv(Ch(p))$ is Δ-uniform, then, when $|Ch(p)| \geq 2$, two bundles $x, y \in Ch(p)$ forming an edge of $conv(Ch(p))$ will satisfy $\max\{||(x-y)^+||_1, ||(y-x)^+||_1\} \leq \Delta$. ■

D Proof of Theorem 5.2

D.1 Lemmas

We start with some lemmas. The following basic result is needed for our lottery implementation result.

LEMMA D.1 *Given an $y \in \mathbb{R}^d$, and some property $(*)$, y can be expressed as a lottery over a set of vectors satisfying $(*)$ if and only if for every weight vector $w \in \mathbb{R}^d$, there exists $z \in \mathbb{R}^d$ satisfying $(*)$ and $w \cdot z \geq w \cdot y$.*

Proof. Let Z be the set of all vectors satisfying property $(*)$. Now, $y \notin conv(Z)$ if and only if a hyperplane separating y from $conv(Z)$ exists. This means that there is a $w \in \mathbb{R}^m$ such that $w \cdot y > w \cdot z$ for all $z \in Z$. ■

Any algorithm that finds a vector z satisfying $(*)$ and $w \cdot z \geq w \cdot y$ can be used to express y as a convex combination of vectors in Z (see Nguyen et al. (2016)).

We also need the following, which is standard, but we provide its proof for completeness.

LEMMA D.2 *Let $Q \subset \mathbb{R}^d$ be a polytope, B a $k \times d$ matrix and $b \in \mathbb{R}^k$. Let x^* be an extreme point of $Q \cap \{x \in \mathbb{R}^d | Bx = b\}$. Let Q' be the minimal face of Q containing x^*, then the dimension of Q is at most k. Equality occurs when all the k row vectors of B are independent and Q' is affine independent with $\{x \in \mathbb{R}^d | Bx = b\}$.*

Proof. As Q is a polytope, it may be expressed as the intersection of finitely many hyperplanes, i.e. $Q = \{x \in \mathbb{R}^d | Ax \leq a\}$, for some matrix A and vector a. Because x^* is an extreme point of $\{x \in \mathbb{R}^d | Ax \leq a; Bx = b\}$, in this inequality system, the number of *independent* binding constraints at x^* is equal to d. At most k of these can be credited to the system $\{Bx = b\}$. Thus, at least $d-k$ independent binding constraints belong to the system $\{Ax \leq a\}$. The minimal face containing x^*, Q', is in the intersection of these constraints. Thus, the dimension of Q' is at most k. Equality occurs only when all the k row vectors of B are independent and are independent of the binding constraints in Q.

For the subsequent lemmas, we need some terminology. Given a polytope $Q \in \mathbb{R}^d$, a coordinate $i \in \{1, \ldots, d\}$ is called **fixed with respect to** Q if $x_i = y_i$ for all $x, y \in Q$. Otherwise, it is called **free**. In other words, if a coordinate i is fixed with respect to Q, there is a constant θ such that $x_i = \theta$ for all $x \in Q$.

LEMMA D.3 *If Q is a binary Δ-uniform polytope, the number of free coordinates with respect to Q is at most $2\Delta \cdot dim(Q)$.*

Proof. Fix an extreme point v of Q and let E_v be a maximal linearly independent set of edges of Q that are incident to v. Recall that the dimension of the space spanned by the edges incident to any extreme point is equal to the dimension of the polytope. Thus, $|E_v| = dim(Q)$.

As Q is binary and Δ-uniform, the components of each of the vectors in E_v belongs to $\{-1, 0, 1\}$ and the number of non-zero components in each of them is at most 2Δ (with at most Δ positive and at most Δ negative coordinates). Hence the total number of non-zero coordinates among the vectors in E_v is at most $2\Delta \cdot |E_v| = 2\Delta \cdot dim(Q)$.

If j is a free coordinate with respect to Q, there exists $v' \in Q$ such that the jth coordinate of $v' - v$ is not 0. But $v' - v$ is in the span of E_v, thus, there must be a vector in E_v, whose j^{th} coordinate is not 0. This implies that all free coordinates with respect to Q are the non-zero coordinates among the vectors in E_v. Therefore, the number of free coordinates of Q is at most $2\Delta \cdot dim(Q)$. ∎

LEMMA D.4 *Let $Q \subset \mathbb{R}^d$ be a binary polytope and $x^* \in Q$. let Q' be the minimal face of Q containing x^*, then the set of free coordinates with respect to Q' is $\{l \in \{1, .., d\} | 0 < x^*_l < 1\}$.*

Proof. If $x^*_l = 1$, consider $Q'' := Q' \cap \{x \in \mathbb{R}^d | x_l = 1\}$. Q'' is a face of Q', which is also a face of Q, that contains x^*. However, Q' is the minimal face of Q containing x^*, therefore $Q' = Q''$. Thus the coordinate l is fixed with respect to Q'.

Similarly if $x_l^* = 0$, consider $Q'' := Q' \cap \{x \in \mathbb{R}^d | x_l = 0\}$. $Q'' = Q'$ and, again, the coordinate l is fixed with respect to Q'.

If $0 < x_l^* < 1$, then because Q' is a binary polytope, there exists $x \in Q'$ such that $x_l = \lfloor x_l^* \rfloor$=0, thus coordinate l is free with respect to Q'.

D.2 Proof of Theorem 5.2

The proof is based on the algorithm described in Figure 2.

At each iteration of the algorithm, we solve a linear program (7) with the objective given by the weight vector w and one less supply constraint than the iteration before. Thus, the linear program (7) is feasible at each iteration, and the optimal objective function value is non-decreasing. Therefore, the terminal solution z will satisfy $w \cdot z \geq w \cdot y$.

By Lemma D.1, if the algorithm is correct and terminates in the desired solution z for any weight vector w, then y can be expressed as a lottery over approximately feasible allocations as in Theorem 5.2.

We now show that the algorithm is correct and terminates in the desired solution z.

First, at each iteration of the algorithm, we delete (Step 2(b)) at least one supply constraint. By Lemma D.2, the minimal face $F \subset Q$ containing the extreme point solution of (7) satisfies $dim(F) \leq |S|$, the outcome of the algorithm belongs to a polytope that shrinks in dimension at every Step of the algorithm. Hence, the algorithm terminates in at most m iterations, and its output will be a zero-dimensional face of Q, which is a vertex.

Second, observe that if each P_i is $\Delta-$uniform, then $Q = P_1 \times .. \times P_n$ is also $\Delta-$uniform. In the subsequent iterations, Q is replaced by one of its faces, F. The edges of a face are a subset of the edges of the corresponding polytope. Thus, Q and F remain $\Delta-$uniform throughout the algorithm.

By Lemma D.3 there are at most $2\Delta \cdot dim(F)$ free coordinates with respect to F. By Lemma D.2, $dim(F) \leq |S|$. Note that the supply constraints do not share a common variable. In other words, the coordinates of the non-zero components of distinct supply constraints

Figure 2: ALGORITHM

Input: $\{P_j\}_{j=1}^n$ Δ-uniform binary polytopes, $y = (y^1, \ldots, y^n) \in P_1 \times \ldots \times P_n$, $\sum_{j=1}^n \mathcal{T}_i(y^j) = s_i \in \mathbb{R}$, and weight vector $w \in \mathbb{R}^{Cmn}$.

Output: $z \in ext(P_1) \times \ldots \times ext(P_n)$ such that $w \cdot z \geq w \cdot y$ and for all $i \in M$, $|\sum_j \mathcal{T}_i(z^j) - s_i| < 2\Delta - 1$.

Step 0: Initiate $Q = P_1 \times \ldots \times P_n$, and let $S := M$ to be the set of active supply constraints.

Step 1: Let

$$R := \{x = (x^1, \ldots, x^n) | \sum_{j=1}^n \mathcal{T}_i(x^j) = s_i \text{ for all } i \in S\}.$$

Solve

$$\{\max w \cdot x | x \in Q \cap R\}. \tag{7}$$

Let $x^* = (x^{*1}, \ldots, x^{*n})$ be an optimal extreme point solution and F be the minimal face of Q containing x^*.

Step 2a: If $dim(F) = 0$, that is $F = \{x^*\}$, call it z and STOP.

Step 2b: Else, each constraint i such that $\sum_{j=1}^n \mathcal{T}_i(x^j) = s_i$, can be written as $\alpha(i) \cdot x = s_i$, where $\alpha(i)$ is a $\{0, 1\}$ vector.

Let a_i be the number of non-zero coordinates of $\alpha(i)$ that are free with respect to F. Among the active constraints in S, choose the constraint $i \in S$ with smallest a_i.

Update $Q := F$ and $S := S \setminus \{i\}$.

Step 3: Return to **Step 1**.

are disjoint. Now, at most $2\Delta \cdot |S|$ free coordinates with respect to F are partitioned among $|S|$ constraints. Hence, the constraint that has the least number of free coordinates, selected for deletion in Step (2b) of the algorithm, can have at most 2Δ free coordinates with respect to F. We now show that if the algorithm has not terminated, then the equality cannot occur, thus the supply constraint selected for deletion in Step (2b) has at most $2\Delta - 1$ free coordinates.

Equality occurs when there are exactly $2\Delta \cdot |S|$ free coordinates, each supply con-

straint contains exactly 2Δ of them, and the dimension of F is equal to $|S|$. According to Lemma D.2, the face F is affinely independent of the $|S|$ supply constraints. In Lemma D.3, if there are $2\Delta \cdot dim(F)$ free coordinates with respect to F, then F is in the affine linear space spanned by its $dim(F)$ linearly independent edges. Furthermore, each of these edges has exactly Δ number of $+1$ coordinates and Δ number of -1 coordinates. Notice that these edges are perpendicular to $\vec{1}$. Hence, F lies in the hyperplane perpendicular to $\vec{1}$ that contains x^*. Mathematically, for all $x \in F : \vec{1} \cdot x = \vec{1} \cdot x^*$. Notice that if we add all the supply constraints, we also obtain $\vec{1} \cdot x = \sum_i s_i$. This shows that F cannot be affinely independent of the $|S|$ supply constraints unless $F = \{x^*\}$. But this means that the algorithm has terminated.

Now, consider the supply constraint, i, to be deleted at this iteration. This constraint contains at most $2\Delta - 1$ free coordinates. By Lemma D.4 these coordinates correspond to ones where x^* is *strictly* between 0 and 1. In the later iteration of the algorithm, these coordinates will be rounded in the final outcome z to either 0 or 1. The deviation caused by the rounding of each of these coordinates is *strictly* less than 1. Hence, the total deviation from this is strictly less than $2\Delta - 1$. Thus, we have

$$|\sum_{j=1}^{n} \mathcal{T}_i(x^{*j}) - \sum_{j=1}^{n} \mathcal{T}_i(z^j)| = |s_i - \sum_{j=1}^{n} \mathcal{T}_i(z^j)| < 2\Delta - 1.$$

This is what we need to show.

Notice that when $\sum_{j=1}^{n} \mathcal{T}_i(x^{*j}) = s_i$ is integral, because $\sum_{j=1}^{n} \mathcal{T}_i(z^j)$ is also integral, so we have $|\sum_{j=1}^{n} \mathcal{T}_i(z^j) - s_i| \leq 2\Delta - 2$.

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