Financing Innovative Activity and the Endogeneity of Patenting*

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Abstract

Debt financing plays an important role in the funding of small, innovative firms, and patents have been increasingly used as collateral. We examine how innovative firms without sufficient assets can be financed ex-ante if these firms may acquire patents in the future. We show that the endogeneity of the patenting decision creates an upper bound on the long-term payments that can be credibly promised to a lender. This introduces a unique inefficiency, which also affects the terms and feasibility of financing contracts. We discuss how patent protection and creditor-rights policies have a different impact on the growth of innovative firms.

Keywords: financing of innovation, endogeneity of patenting decision, debt, incomplete contracts.

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1 Introduction

Innovative activity is widely considered more difficult to finance by traditional lenders (Hall and Lerner (2010)) since it requires investments in specialized assets (Williamson (1988)) and usually gives rise to intangible assets that are difficult to possess and liquidate (Dell’Ariccia et al. (2021)) and therefore have a lower value as collateral. Despite these difficulties, there is substantial evidence that bank lending plays a more important role in the financing of small innovative firms than is suggested by existing theory (Robb and Robinson (2014); Cornaggia et al. (2015); Chava et al. (2013); Kerr and Nanda (2015); Mann (2018)). There is also emerging empirical evidence that intangible assets such as patents have been increasingly used as collateral for loans (Hochberg et al. (2018); Mann (2018); Ma and Wang (2019)). It is perhaps not that surprising that existing patents whose property rights can be reassigned can be pledged as collateral and facilitate debt financing. A critical question that, to our knowledge, has not been addressed so far is whether and how debt instruments can be used to finance investments in innovation *ex-ante* for firms that have yet to develop any innovations.

In this vein, an important and often overlooked consideration is that the decision of whether to patent an innovation is itself endogenous. While future innovations could provide a lender with collateral through blanket liens on all assets (Mann (1997)), firms may prefer not to patent an innovation if the cost associated with increasing a lender’s *ex-post* bargaining power outweighs any benefits obtained from patent protection. We show that the endogeneity of the patenting decision creates important and novel friction which has relevant implications for the feasibility of financing, the shape of financial contracts and policy.

In our model, an innovative firm with some existing assets and cash flows requires financing to invest in a two-period project which may produce an innovation at the end of the first period. However, existing assets and cash flows are not sufficient to support the financing that is needed for the new project. Moreover, the cash flows and the innovation are not verifiable (as in Grossman and Hart (1986); Hart and Moore (1990)) so that contracts that directly depend on them are not enforceable in courts. At the end of the first period, the
firm learns whether the project produced an innovation and how much cash was generated and decides on whether and how much to repay the lender. If the project is not liquidated in the first period second period cash flows are realized. We allow for renegotiation of the original contract in order to improve efficiency, allowing for early repayment, debt writedowns ("haircuts"), and debt rollovers.

A key feature of our framework is that when an innovation has been realized, the firm must then decide whether to patent the innovation. The firm’s patenting decision hinges on the tradeoff between getting a higher value for the innovation versus creating an asset that can be liquidated by the lender in the second period. The firm may find it optimal to strategically decline to patent the innovation and default if its long-term debt obligations are larger than the expected gain from patenting. As a result, the endogeneity of the patenting decision imposes an upper limit on the long-term repayment that the firm can credibly offer. However, under the optimal contract the firm never actually chooses to eschew patenting and, hence, defaults for strategic reasons. This is because the firm’s decision to not patent reduces the joint payoff by lowering the value of the innovation for the firm.

Nonetheless, the endogeneity of patenting affects the optimal financial contract through two channels. First, the constraint imposed on long-term promised payments may bind ex-ante, meaning that the long-term component of debt is limited and the firm as a result must increase the short-term component to ensure the lender’s participation. Second, even when the constraint on long-term payments does not bind ex-ante, it nevertheless affects the financial contract because it limits the amount that can be deferred to the second period when the firm cannot meet its first period obligation in full. In other words, this constraint may bind ex-post even if it doesn’t bind ex-ante. This second effect reduces the expected payoff to the lender, which must then be compensated through an increase in the total promised payment to the lender. At the same time, the need for greater promised payments results in a higher probability of liquidation of existing assets and/or the patent.

As a result, the constraint on long-term payments, introduced by the endogeneity of
the patenting decision, results in fewer projects being financed compared to what would happen if patenting were not a choice for the firm. A novel prediction is that feasibility of projects crucially depends on the difference between the value of innovation with and without patent. Perhaps, more importantly the infeasibility introduced by endogeneity of patenting is different from infeasibility introduced by outside financing due to inefficient liquidation, and is at least as large. For example, the endogeneity of the patenting decision still introduces significant infeasibility even when there is no social loss due to liquidation. In addition, the endogeneity of patenting reduces the payoff of the lender from riskier projects with the same expected payoff. As a result, when patenting is endogenous only projects with a higher probability of success will be financed, even if the expected value of the innovation for these projects are relatively low.

Firm, or industry, characteristics affect the terms of the financial contract primarily through their effect on the constraint on long-term payments. When the constraint binds, relaxing the constraint increase the duration of the debt contract. For instance, increases in the value of innovation, patent protection, and the expected long-term cash flows relaxes the constraint, and as a result increases the duration of the debt contract. On the other hand, for example, an increase in the value of innovation without patenting tightens the constraint and hence decreases the duration of the financial contract. The last prediction is novel and could potentially be tested in the cross-section by proxying the value of unpatented innovations with industry-level variables such as competition.

Our paper is related to the literature on patent protection policy, and our model yields a number of implications for policy. An important trade-off from a policy perspective is that while patent protection may provide incentives for innovation, it also creates a temporary monopoly (Nordhaus (1969); see Moser (2013), for a review). Our framework helps us to identify two complementary policy tools for patent policy. We show that, as expected, improving creditor rights and patent protection makes financing feasible for a greater range of projects. More specific to our model, we show that these policies are complementary: changes
in one have a greater effect on feasibility if the other one is higher. Another unique prediction of our model is the different impact of patent protection and creditor rights on feasibility as a function of innovation’s value when not patented: improving patent protection has a larger impact on feasibility when the value of an unpatented innovation is higher, whereas the opposite is true for improving creditor rights. These predictions could be tested by using changes in patent protection and enforcement over time (see Hall (2004)). For example, our model predicts that improving patent protection results in higher growth in industries with higher values of unpatented innovations.

Our focus is primarily on debt financing of younger, entrepreneurial firms with limited cash flows, where contracting possibilities on future innovations are limited. A natural setting to study such issues comes from the literature on incomplete contracting. (Grossman and Hart (1986); Hart and Moore (1990); Aghion and Bolton (1992) Hart and Moore (1994)), where debt like contracts naturally emerge. Like in much of that literature, when it is impossible to contract directly on cash flows, financing is only feasible if the investor receives the right to liquidate and can (credibly) threaten to do so if not repaid. However, our finding that the endogenous patenting decision imposes an incentive feasibility constraint, which limits what can credibly be repaid in the long-term, is not specific to the incomplete contracting framework and should arise in other frameworks as long as the patenting decision itself is not contractible.

While much of the literature that studies innovative firms focuses on venture capital, equity-based financing, such financing is relatively rare in practice. We show that even if unable to obtain such financing, or even if equity is not a feasible security to use, debt financing may nevertheless be possible through the reassignment of property rights over patented innovations, even if those innovations are yet to be developed. From this perspective, our paper is related to the literature on the financing of innovation through debt instruments (see, e.g., Mann (2018), Robb and Robinson (2014), Cornaggia et al. (2015), or Chava et al.)

\footnote{A recent contribution to this literature is Huang et al. (2019), who study the trade-off between the termination threat and the desire to avoid early liquidation in a dynamic, multi-period setting.}
Apart from offering a theoretical framework, our work differs by taking an ex-ante perspective and studying the financing possibilities for firms that do not yet have any patented innovations, but may develop some in the future.

Our main theoretical contribution is to study how the endogeneity of patenting decision affects financial contracting for innovative activities. We show that the inefficiency introduced by the endogeneity of the patenting decision is different from other well-known inefficiencies introduced by outside financing, and is at least as large. Our model also provides several unique empirical predictions on project feasibility, contract terms, and policy choices that may affect the funding and growth of small innovative firms.

The paper proceeds as follows. Section 2 introduces the model. Section 3 solves the model and provides cross-sectional empirical predictions. Section 4 analyzes the inefficiency introduced by the endogeneity of patenting. Section 5 discusses policy implications and Section 6 concludes.

2 The model

A firm has a two-period project that requires an initial investment of $I > 0$ at $t = 0$, and may deliver innovation at $t = 2$. The firm has no internal funds but has assets in place which produce a cash flow of $\tilde{C}_1$ at $t = 1$, which is uniformly distributed between 0 and $\bar{C}_1$ and, if allowed to continue, a further cash flow of $\tilde{C}_2$ at $t = 2$, uniformly distributed between 0 and $\bar{C}_2$. All cash flows are observable but not verifiable and, hence, cannot be contracted upon. The firm may use these cash flows, as well as the value of its innovation (described below) to repay investors, with any excess being consumed by the firm. The firm’s existing assets have a liquidation value $L_1$ at $t = 1$.

We assume that the liquidation value of the existing assets and expected cash flows from the first period are insufficient to cover the cost of the investment, i.e., $L_1 + E[C_1] < I$, and in particular the firm is cash-constrained at time 1: $C_1 < L_1 < E[C_2]$. This also means
that the continuation of the firm’s existing activities is always optimal, even in the absence of innovation. Partial liquidation of assets is not possible and the liquidation value of the assets in place decreases over time – for simplicity, we assume it is equal to zero at $t = 2$.

The investment has a chance of producing innovation, which arrives at time 1 with probability $p > 0$. The innovation is observable but not verifiable in court and hence cannot be contracted upon. The innovation, if successful, generates a positive cash flow with an expected value $V$ at some later date (beyond $t = 2$). $V$ represents the social value of the innovation and, as above, $V$ is not directly contractible. If the firm protects its innovation by patenting it between $t = 1$ and $t = 2$, it can appropriate a portion $\omega$ of the total value $V$ of the innovation (i.e., the firm obtains a value $\omega V$) such that $0 \leq \omega \leq 1$. Investment is efficient for the firm, i.e., $I \leq p\omega V$.

A patented innovation represents an asset that in principle can be liquidated at $t = 2$ for a fraction $\beta \leq 1$ of its value. In other words, rights to the patent can be reassigned to someone else who can realize a value $\beta \omega V$ for it. In the absence of patenting, or if no innovation occurs, there are no more assets that can be liquidated at $t = 2$.

If the firm does not patent innovation, then there is a chance that the innovation may be copied and revenues will be lost by the firm: without patenting the value of expected revenues that accrue to the innovating firm is $\alpha V$. We assume that the potential gain from patent protection is larger than the loss from liquidation of the patent, i.e., $\omega V - \alpha V > \omega V - \beta \omega V$, or, equivalently, $\beta \omega > \alpha$. This implies that patenting is always socially optimal, and is consistent with the idea that the purchaser of the patent also gets protection from competition even if there is some loss of value due to liquidation. We assume that the maximum possible cash flows at time 2 are larger than the social gain of avoiding liquidation, i.e., $\bar{C}_2 > (\omega - \alpha)V$.

As we discuss further below, this ensures that $\bar{C}_2$ does not artificially impose a constraint.

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2If the innovation is not patented, it is natural to consider what happens if the firm chooses to default on the loan and thus gets liquidated. In this case, the firm’s shareholders could set up another firm and use the unpatented innovation to generate future cash flows. Hence, deciding not to patent innovation is costly to the firm, but does not lead to complete loss even under liquidation, whereas the lender is unable to obtain any value from liquidating a firm whose innovations have not been patented.
on what can be repaid at time 2.

Non-verifiability of cash flows forces the lender to rely on control rights and liquidation threats to ensure repayment (e.g., Bolton and Scharfstein (1990); Bolton and Scharfstein (1996); Aghion and Bolton (1992); Hart and Moore (1994)). We focus therefore on debt-like contracts with liquidation rights for the lender if a promised payment is not made. Specifically, the financial contract has a short-term payment of \( D_1 \) to be made at \( t = 1 \) and a long-term payment of \( D_2 \) to be made at \( t = 2 \). If any of these payments are not made the lender is entitled to liquidate the firm’s assets to recover his promised payment \( D_1 + D_2 \).

The credit market is competitive at time zero. We allow for renegotiation of the original contract to improve efficiency, incorporating if mutually agreed upon, the possibility of early debt repayment, debt writedowns (“haircuts”), or rollovers of the amount due. Specifically, if the firm repays its time 1 obligation \( D_1 \) in full, it can then make a take-it-or-leave-it offer to the investor for the remaining amount. In particular, the firm may want to repay some of its long-term debt early when it has the resources to do so. On the other hand, if the firm is unable to repay its obligation, or if it simply refuses to do so even if able (i.e., we allow for strategic default), then the lender is entitled to make a take-it-or-leave-it offer to the firm, with the limitation that the lender cannot ask for more than the total payment \( D_1 + D_2 \) promised in the original contract. This captures that when in default, a lender is likely to have more bargaining power under renegotiation than the defaulting firm. It also allows for the possibility that the lender may prefer not to force liquidation, but may instead opt to roll over some of the shortfalls in repayment, or may even find it optimal to forgive a portion of the debt.\(^3\) If an offer is accepted, the new contract replaces the existing contract. If rejected, the original contract remains in place and the investor liquidates if the promised payment was not made in full. As discussed below, this renegotiation mechanism results in agents agreeing on socially efficient outcomes whenever possible at time 1. We also discuss results under alternative renegotiation

\(^3\) We assume that when indifferent, the lender chooses to make the offer that is preferred by the firm. In other words, the lender chooses a Pareto-improving offer when indifferent.
mechanisms in the Appendix.

At time 2, we also allow for renegotiation. However, since the game ends at time 2 there is no room for either debt forgiveness or deferral of payments, and therefore for simplicity, we assume that only the firm can make a take-it-or-leave-it offer to the lender.

Figure 1 shows the timeline of the model. The timeline is as follows. At $t = 1$, the cash flow $C_1$ is realized, and both parties learn whether or not an innovation has occurred. After contract renegotiation, and if the firm is not liquidated, the firm decides whether to patent or not when it has innovation. Finally, if the project is continued the time 2 cash flows are realized, agents again may renegotiate the payment, and the lender may liquidate if the promised payment is not made.

Figure 1: Timeline

3 Analysis

We examine the payoffs of the firm and the lender using backward induction under the original financing contract $(D_1, D_2)$, which can be renegotiated at time 1 and time 2.

3.1 No innovation

The analysis is simple when the firm has not successfully innovated. At time 2, there are no assets to liquidate and therefore the payoff of the lender is zero while the firm gets $C_2$. If at
time 1 $C_1 < D_1$, then the firm does not have the resources to repay and, moreover, cannot credibly promise to repay anything at time 2. The lender then has the right to liquidate and will choose to do so, obtaining $L_1$. If the lender liquidates, the total amount $D = D_1 + D_2$ becomes due and the investor’s payoff is equal to $\min\{L_1, D_1 + D_2\} = L_1$ since the liquidation value of the assets is low relative to the investment ($L_1 < I$) and at the same time the total promised payment has to be larger than or equal to the initial investment ($D_1 + D_2 \geq I$) for the lender to break even.

If $C_1 \geq D_1$, then the firm can either pay $D_1$ and continue, or allow liquidation. In the former case, the firm gets $C_1 + E[C_2] - D_1$, while in the latter case it gets $C_1$, with the proceeds from liquidation, $L_1$, going to the lender. Given that $E[C_2] > \bar{C}_1$ and the maximum payment that can be made out of time 1 cash flows is $\bar{C}_1$, we can rule out the possibility that the firm prefers liquidation.

Consider now the possibility that $D_1 > \bar{C}_1$, the highest possible time 1 cash flow. Since the most the lender can get under liquidation is $L_1$, and the probability that $C_1$ is less than $D_1$ would be equal to 1, such a contract could always be replaced with a contract that has $D_1 = \bar{C}_1$ and which would keep the total payment to the investor unchanged. It is thus without loss of generality to assume that $D_1 \leq \bar{C}_1$, and we make this assumption going forward.

The lender’s payoff is therefore

$$
\begin{cases}
  L_1 & \text{for } C_1 < D_1 \\
  D_1 & \text{for } C_1 \geq D_1
\end{cases}
$$

Ex-ante, the expected payoff for the lender in case of no innovation can be written as a function of $D_1$:

$$
\mathcal{F}_0(D_1) = L_1 \cdot \Pr(C_1 < D_1) + D_1 \cdot \Pr(C_1 \geq D_1)
= L_1 \cdot \frac{D_1}{\bar{C}_1} + D_1 \cdot \left(1 - \frac{D_1}{\bar{C}_1}\right).
$$  (1)
Now consider a renegotiation of the original contract at time 1. When \( C_1 \geq D_1 \), the firm offers to pay \( D_1 \) and continue, which is exactly equal to the lender’s outside option. On the other hand, when \( C_1 < D_1 \) the firm will refuse to pay anything given that the firm will be liquidated regardless, and there is no welfare-improving renegotiation offer that is possible since \( \bar{C}_1 < L_1 \). Thus, ex-ante the joint payoff when there is no innovation is equal to:

\[
G_0(D_1) = \frac{1}{C_1} \int_0^{D_1} (C_1 + L_1) dC_1 + \frac{1}{C_1} \int_{D_1}^{C_1} (C_1 + E[C_2]) dC_1 \\
= E[C_1] + E[C_2] - (E[C_2] - L_1) \frac{D_1}{C_1}.
\] (2)

Note that \( G_0(D_1) \) decreases in \( D_1 \): increasing \( D_1 \) reduces the total payoff when there is no innovation. On the other hand, \( F_0(D_1) \) increases when \( D_1 \) increases. Therefore, increasing \( D_1 \) creates a trade-off between feasibility of financing and the total payoff when there is no innovation.

### 3.2 The case of innovation

Next, we describe payoffs when the firm has successfully innovated. We start with time 2 and use backward induction to calculate the payoffs.

#### 3.2.1 Second period

First, suppose that the firm has patented its innovation after paying the lender \( D_1 \) at time 1. After the realization of the time 2 cash flows \( C_2 \), the firm will pay \( \min\{D_2, \beta \omega V\} \) to the lender if \( C_2 \geq \min\{D_2, \beta \omega V\} \) to avoid liquidation of the project. The lender will accept this payment given that it is equivalent to what they will get in the case of liquidation. However, if \( C_2 < \min\{D_2, \beta \omega V\} \), then the firm will be unable to repay the promised amount and the lender will liquidate to receive \( \min\{D_2, \beta \omega V\} \). In other words, the lender’s payoff at time 2, if there is a patented innovation, is always equal to \( \min\{D_2, \beta \omega V\} \). Note as well that the lender’s payoff varies with \( D_2 \) only when \( D_2 < \beta \omega V \). This gives us the following preliminary
Lemma 1 The payoffs of the firm and the investor from a contract with $D_2 > \beta \omega V$ are equivalent to the corresponding payoffs from a contract with $D_2 = \beta \omega V$.

Given that $\beta \omega V$ is known at time zero, going forward we will, without loss of generality, assume that agents always agree on a time 2 payment which is no greater than the liquidation value of the patent, i.e., $D_2 \leq \beta \omega V$. Therefore, at $t = 2$, after the firm has paid $D_1$ and patented its innovation, the investor’s payoff is always $D_1 + D_2$.

Suppose instead now that the firm has innovation, paid $D_1$ at time 1 to the lender, but decided not to patent the innovation. If the firm did not patent, then the investor cannot obtain any payments at time 2 because there are no assets that can be liquidated or threatened to be liquidated to force the firm to pay. If the firm does not patent, then the investor only obtains $D_1$ and the firm’s payoff is

$$C_1 + C_2 + \alpha V - D_1. \quad (3)$$

3.2.2 First period

Given the set of possible outcomes and payoffs at time 2, we now analyze what happens at time 1, starting with the patenting decision.

Patenting and the constraint on long-term payments

The firm’s decision to patent at time 1 depends on its payoff when it patents versus when it does not, which it may choose not to do for strategic reasons. Suppose that $C_1 > D_1$, so that the firm can make the time 1 promised payment. In this case, if the firm chooses to patent its expected payoff is

$$C_1 + E[C_2] + \Pr(C_2 \geq D_2) \cdot \omega V + \Pr(C_2 < D_2) \cdot \beta \omega V - (D_1 + D_2). \quad (4)$$

This payoff reflects that, if the firm makes its time 1 payment and patents, its payoff equals
the first period cash flow $C_1$ plus the expected time 2 cash flow, as well as the (patented) value of the innovation when able to meet its obligation at time 2, but only the liquidation value $\beta \omega V$ when not. The firm will patent if and only if (4) is at least as large as (3), or in other words if

$$(\omega \cdot \Pr(C_2 \geq D_2) + \omega \beta \cdot \Pr(C_2 < D_2))V - \alpha V \geq D_2.$$  

(5)

Note that the value of $D_2$ that satisfies (5) is less than $\bar{C}_2$ because $C_2$ is larger than the maximum possible gain from patenting, i.e., $\omega V - \alpha V < \bar{C}_2$. Consequently, the left-hand side of (5) is decreasing in $D_2$, while the right-hand side is increasing in $D_2$.

We can now establish the following result.

**Lemma 2** Let $D_2^{max}$ be the maximum value such that (5) holds. The firm finds it optimal to patent its innovation only if $D_2 \leq D_2^{max}$.

Lemma 2 puts a constraint on the amount that can be credibly pledged to an investor, with the result that any contract which leads to patenting must have a promised time 2 payment no greater than $D_2^{max}$. Given that $C_2$ is uniform in $[0, \bar{C}_2]$, the solution for $D_2^{max}$ is:

$$D_2^{max} = \frac{\omega - \alpha}{V + \frac{(1-\beta)\omega}{\bar{C}_2}}.$$  

(6)

This upper bound represents an important constraint on contracting when patenting decision is endogenous. We present comparative statics on $D_2^{max}$ later when discussing some of the implications of the model.

Note that, as shown above, there is another upper bound on the payment that can be made at time 2. Specifically, the time 2 payment has to be less than the liquidation value of the patent, i.e., $D_2 \leq \omega \beta V$, as otherwise the firm could always renegotiate the payment down to $D_2 = \omega \beta V$. As a result, we will focus only on the cases when $D_2^{max} \leq \omega \beta V$ going forward.

**First period payoffs**
Given the firm’s patenting decision, we can now characterize the payoffs of both parties at time 1. There are two cases depending on whether $D_2$ is larger or smaller than $D_{2}^\text{max}$.

Let’s first consider the case where $D_2 \leq D_{2}^\text{max}$ and $C_1 \geq D_1$, so that the firm is able to make the time 1 payment and afterward finds it optimal to patent the innovation. In this case, the outside option of the lender is equal to $D_1 + D_2$ as described above. The firm also can make a take-i-or-leave-it offer, and finds it optimal to pay the maximum amount possible at time 1, $C_1$, paying as much of its outstanding debt obligation as possible early in order to minimize the probability of liquidation at time 2. Therefore, the firm offers a new contract such that $D'_1 = C_1$ and $D'_2 = D_1 + D_2 - C_1$. This contract keeps the payoff of the lender the same but increases the payoff to the firm by reducing the probability of liquidation at time 2.

Next consider the case where $D_2 \leq D_{2}^\text{max}$ and $C_1 < D_1$. In this case, the firm cannot make the time 1 payment and, under the original contract, the lender would liquidate. If the lender liquidates the total amount $D = D_1 + D_2$ becomes due and the investor’s payoff is equal to $\min\{L_1, D_1 + D_2\} = L_1$.

However, when the firm fails to the make the promised payment, the lender has an option to make a take-it-or-leave-it offer to the firm. Being aware that the firm will only patent if $D'_2 \leq D_{2}^\text{max}$, the lender will offer the contract $(C_1, \min\{D_{2}^\text{max}, D_1 + D_2 - C_1\})$, with payoff to the lender of $C_1 + \min\{D_{2}^\text{max}, D_1 + D_2 - C_1\}$. The lender will find it optimal to offer this contract when $C_1 + \min\{D_{2}^\text{max}, D_1 + D_2 - C_1\} \geq L_1$, rolling over a portion of the promised payment rather than forcing early liquidation. Importantly, when $D_{2}^\text{max} < D_1 + D_2 - C_1$, the lender also prefers to forgive a portion of the remaining debt obligation equal to $D_1 + D_2 - C_1 - D_{2}^\text{max} > 0$ rather than asking for full repayment and thus creating disincentives for the firm to patent its innovation.

We now consider the case when $D_2 > D_{2}^\text{max}$. In this case, the firm is not willing to patent the innovation under the terms of the original contract. Let’s assume that $C_1 \geq D_1$. In this case, the firm can make the payment $D_1$, which prevents the lender from liquidating,
and continue without patenting the innovation. On the other hand, if \( C_1 < D_1 \), then the lender has the right to liquidate and make a take-it-or-leave-it offer. The lender can only offer \((C_1, D_2^{\text{max}})\) because in this case, \( \min\{D_2^{\text{max}}, D_1 + D_2 - C_1\} \) is always equal to \( D_2^{\text{max}} \). The lender chooses to roll over part of the loan, but also forgives a portion, rather than either forcing liquidation or asking for full repayment.

The actions and payoffs of agents differ depending on whether \( D_2 \) or \( D_2^{\text{max}} \) is bigger. Different payoff structures result in two optimization problems and two (potentially) different optimal contracts, which need to be compared. To avoid repetition, we first focus on the case when \( D_2 \leq D_2^{\text{max}} \) and later show that a contract with \( D_2 > D_2^{\text{max}} \) can never be optimal.

3.3 Feasibility of financing

We have assumed throughout that the project is positive NPV for the firm, meaning that \( p \omega V > I \) so that if the firm had internal cash it would always undertake the project. The need for outside financing can make the project infeasible because it introduces inefficiencies. The project may become infeasible in two ways. First, it may not be possible to satisfy the participation constraint of the lender. Second, the possibility of liquidation introduced by outside financing may make the project negative NPV from the perspective of the firm. We focus on the former possibility first and postpone the discussion of firm’s decision to invest. For the former, we have the following characterization.

**Theorem 1** There exists a feasible financing contract if and only if

\[
\frac{\bar{C}_1}{2} + D_2^{\text{max}} \geq \frac{I - (1 - p)L_1}{p},
\]

(7)

where \( D_2^{\text{max}} \) is given by (6). Moreover, when financing is feasible, there will be no liquidation at time 1 when there is innovation: The firm always pays all time 1 cash flows \( C_1 \) to the lender and patents the innovation.

The formula for feasibility, (7), shows that the sum of the maximum expected payment
from time 1 cash flows, $E[C_1]$, and the maximum expected payment from time 2 cash flows, $D_2^{\text{max}}$, has to be greater than the initial investment, adjusted for the fact that the lender does not get the promised payment when there is no innovation but instead the firm is liquidated. Theorem 1 also establishes that, for any feasible contract, the innovation will never be liquidated at time 1. This is because liquidation is never efficient and, for any feasible contract, renegotiation ensures that both parties will always prefer to continue rather than liquidating the project. In the proof of Theorem 1 we also show that feasibility requires that $D_2^{\text{max}} > L_1$, as otherwise, the total expected repayment to the lender would be insufficient to cover the cost $I$ of the investment.

Furthermore, for a given feasible contract $(D_1, D_2)$, if $D_2^{\text{max}} > D_1 + D_2$, meaning that the constraint on the firm’s optimization problem implied by Lemma 2 is slack, then $D_2^{\text{max}}$ would not have any effect on the contract. In particular, the limit on the maximum time 2 payment $D_2$ would never be binding and the firm would find it optimal to patent all innovations. Therefore, going forward to characterize the optimal contract we focus on contracts in which $D_1 + D_2 \geq D_2^{\text{max}} > L_1$.

Figure 2 displays the lender’s payoff, $f_1$, as a function of the terms of an original feasible contract assuming $D_2 \leq D_2^{\text{max}}$ and $D_1 + D_2 > D_2^{\text{max}}$. The proof of Theorem 1 provides an explicit expression for this payoff, and establishes that $f_1(C_1, D_1 + D_2) = \min\{D_2^{\text{max}} + C_1, D_1 + D_2\}$.

Note that the payoff of the lender is a function of the total promised payment $D_1 + D_2$ when there is innovation. The lender’s expected payoff can then be obtained after taking the expectation over the time 1 cash flow as

$$F_1(D_1 + D_2) := \frac{1}{C_1} \int_0^{C_1} f_1(C_1, D_1 + D_2) dC_1 = (D_1 + D_2) - \frac{(D_1 + D_2 - D_2^{\text{max}})^2}{2C_1}. \quad (8)$$

We can now derive the participation constraint of the lender. Given that financing is competitive at time 0, the lender participates as long as its expected payoff from the contract
is equal to the initial investment $I$. The expected payoff of the lender can be written as

$$
(1 - p) \cdot \left( L_1 \cdot \frac{D_1}{C_1} + D_1 \cdot (1 - \frac{D_1}{C_1}) \right) + p \cdot \left( (D_1 + D_2) - \frac{(D_1 + D_2 - D_2^{max})^2}{2C_1} \right) = I
$$

The first part of (9) shows the expected payment to the lender when there is no innovation, $\mathcal{F}_0(D_1) = L_1 \cdot \frac{D_1}{C_1} + D_1 \cdot (1 - \frac{D_1}{C_1})$. Given that $L_1 \geq \bar{C}_1$, the payment to the lender when there is no innovation, $\mathcal{F}_0(D_1)$, strictly increases in $D_1$ and attains its maximum when $D_1 = \bar{C}_1$, at $\mathcal{F}_0(\bar{C}_1) = L_1$. The expected payment to the lender in case of innovation, $\mathcal{F}_1(D_2^{max}, D_1 + D_2) = (D_1 + D_2) - \frac{(D_1 + D_2 - D_2^{max})^2}{2C_1}$, increases as the constraint on the time 2 payment, $D_2^{max}$, increases.

It is interesting to note, from Theorem 1, that the time 2 cash flows do not play a direct role on feasibility, only affecting feasibility through the constraint on the time 2 promised payment. In other words, time 2 cash flows matter for feasibility only because of the endogeneity of the patenting decision. For example, if the firm could commit to patent any innovation, then $C_2$ would be irrelevant for the feasibility of financing. This is because the time 2 promised payment could always be made by liquidating the patent.
3.4 Optimal contract

A contract \((D_1, D_2)\) is optimal if it satisfies the participation constraint of the lender and maximizes the firm’s payoff. Given that the participation constraint will be binding, we can equivalently view the optimal contract as maximizing total surplus. We therefore focus on the maximization of total surplus. We first consider the case when \(D_2 \leq D_2^{\text{max}}\). In Theorem 2 we show that contracts where \(D_2 > D_2^{\text{max}}\) cannot be optimal.

Let \(X\) denote the time 2 renegotiated payment. The probability of liquidation at time 2, conditional on innovation, is then \(\Pr(C_2 < X)\). The expected value of the project, conditional on innovation, is

\[
\Pr(C_2 \geq X)\omega V + \Pr(C_2 < X)\beta \omega V = (\Pr(C_2 \geq X) + \beta \Pr(C_2 < X))\omega V.
\]

To simplify the notation, denote

\[
\gamma(X) := \Pr(C_2 \geq X) + \beta \Pr(C_2 < X). \tag{10}
\]

The total surplus when there is no innovation is given by (2). By contrast, when there is innovation, the surplus is

\[
\mathcal{G}_1(D_1 + D_2) = \frac{1}{C_1} \int_{0}^{D_1 + D_2 - D_2^{\text{max}}} (C_1 + E[C_2] + \gamma(D_2^{\text{max}})\omega V) dC_1 + \frac{1}{C_1} \int_{C_1}^{\bar{C}_1} (C_1 + E[C_2] + \gamma(D_1 + D_2 - C_1)\omega V) dC_1.
\]

The two regions in the payoffs above can be matched to two intervals in Figure 2. The first interval corresponds to the region where the lender forgives a portion of the debt, and the contract has the time 2 payment renegotiated to \(D_2^{\text{max}}\). The second region corresponds to the case where the lender is fully repaid \(D_1 + D_2\), with the payment in the second period...
being \( D_1 + D_2 - C_1 \). Thus, the optimal contract is the solution to

\[
\max_{D_1, D_2} \quad (1 - p) \cdot G_0(D_1) + p \cdot G_1(D_1 + D_2)
\]

subject to

\[
(1 - p) \cdot F_0(D_1) + p \cdot F_1(D_1 + D_2) = I
\]

\[
0 \leq D_1 \leq \bar{C}_1
\]

\[
0 \leq D_2 \leq D_2^{\max}.
\]

Maximizing the total surplus is equivalent to minimizing the inefficiencies introduced by external financing. There are two inefficiencies. First, the firm may get liquidated at time 1 when there is no innovation and the firm’s cash flows are less than the promised payment: \( C_1 < D_1 \). The expected inefficiency of time 1 liquidation is

\[
(1 - p) \cdot \Pr(C_1 < D_1) \cdot (E[C_2] - L_1) = (1 - p) \cdot \frac{D_1}{\bar{C}_1} \cdot (E[C_2] - L_1).
\]

Second, the patent may get liquidated at time 2, and the expected inefficiency associated with the liquidation of the patent is

\[
p \cdot \omega(1 - \beta)V \cdot \Pr(\text{time 2 liquidation}),
\]

where the probability of liquidation is endogenous and depends on the contract. The payment to the lender is \( f_1(C_1, D_1 + D_2) = \min\{C_1 + D_2^{\max}, D_1 + D_2\} \), as in Figure 2. At equilibrium, conditional on innovation, the first payment is \( C_1 \), and the time 2 payment is \( f_1(C_1, D_1 + D_2) - C_1 \). Thus, the probability of liquidation at time 2 conditional on innovation is

\[
\Pr(\text{time 2 liquidation}) = \Pr(C_2 < f_1(C_1, D_1 + D_2) - C_1) = \Pr(C_1 + C_2 < f_1(C_1, D_1 + D_2)).
\]

(11)

The probability of liquidation at time 2 is a function of the realization of both time 1 and 2 cash flows. We compare these cash flows to \( D_1 + D_2 \) if the constraint on the time 2 payment
does not bind and to $C_1 + D_2^{max}$ if the constraint does bind.

We can now rewrite the optimization problem of the firm as follows:

$$
\begin{align*}
\min & \quad (1 - p) \cdot \frac{D_1}{C_1} \cdot (E[C_2] - L_1) + p \cdot \omega (1 - \beta) V \cdot \Pr(C_1 + C_2 < f_1(C_1, D_1 + D_2)) \\
\text{subject to} & \quad (1 - p) \cdot F_0(D_1) + p \cdot F_1(D_1 + D_2) = I, \\
& \quad 0 \leq D_1 \leq \bar{C}_1, \\
& \quad 0 \leq D_2 \leq D_2^{max}.
\end{align*}
$$

The firm aims to minimize the total inefficiency due to external financing subject to the lender’s participation constraint, the constraint that the short-term payment not exceed the maximum amount of cash the firm could possibly have at time 1, and the constraint $D_2^{max}$ on the long-term payment.

Recall that $F_1(.)$ is the expected payoff to the financier when an innovation has occurred and $D_2 \leq D_2^{max}$, while $F_0(.)$ is the payoff under no innovation. Notice that $F_0(.)$ and $F_1(.)$ are strictly increasing functions. Therefore, the short and long-term components of the financial contract $(D_1, D_2)$ are linked to each other through the lender’s participation constraint. In particular, there is a one-to-one correspondence between $D_1$ and $D_2$ for all contracts $(D_1, D_2)$ with $D_2 \leq D_2^{max}$ that satisfy (12). With this, we can convert the constraint that $D_2 \leq D_2^{max}$ to an equivalent constraint $D_1 \geq D_1^{min}$, where $D_1^{min}$ is defined in what follows.

Plugging $D_2 = D_2^{max}$ into the participation constraint, (12), we get

$$
(1 - p) \cdot \left( L_1 \cdot \frac{D_1}{C_1} + D_1 \cdot (1 - \frac{D_1}{C_1}) \right) + p \cdot \left( D_1 + D_2^{max} - \frac{D_1^2}{2C_1} \right) = I,
$$

which can be rewritten as a quadratic equation of $D_1$:

$$
\left( (1 - p) \frac{L_1}{C_1} + 1 + p \right) \cdot D_1 - \frac{2 - p}{2C_1} (D_1)^2 = I - pD_2^{max}.
$$

There are two values of $D_1$ that satisfy (14). One must be larger and the other smaller

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than $\tilde{D}_1$, where $\tilde{D}_1$ is the value of $D_1$ that maximizes the left-hand side of (14) given that this expression is strictly concave in $D_1$. In particular,
\[
\tilde{D}_1 = C_1 \frac{(1-p)\frac{L_1}{C_1} + 1 + p}{2 - p} \geq \frac{C_1 (1 - p) + 1 + p}{2 - p} > C_1.
\]

Hence, we will define $D_1^\text{min}$ as the lower value of $D_1$ that satisfies (14). Observe that this lower value of $D_1$ is at most $\bar{C}_1$. This is because the left-hand side of (14) is increasing in $D_1$ when $D_1 < \tilde{D}_1$. Furthermore, at $D_1 = \bar{C}_1$, the feasibility condition implies that the left-hand side of (14) is at least equal to its right hand side. However, it is possible that its value is negative. We modify $D_1^\text{min}$ as follows:
\[
D_1^\text{min} = \max \left\{ 0, \bar{C}_1 \frac{(1-p)\frac{L_1}{C_1} + 1 + p - \Delta}{2 - p} \right\}, \tag{15}
\]
where
\[
\Delta = \sqrt{((1-p)\frac{L_1}{C_1} + 1 + p)^2 - 2(2-p)\frac{I - pD_2^\text{max}}{C_1}}.
\]

Because of the one-to-one correspondence between $D_1$ and $D_2$, for any $D_1 \in [D_1^\text{min}, \bar{C}_1]$ there exists a unique $D_2 \in [0, D_2^\text{max}]$ such that the contract $(D_1, D_2)$ satisfies the participation constraint in (12). Denote such $D_2$ as $\Delta(D_1)$. Using the participation constraint, we obtain
\[
\Delta(D_1) = \bar{C}_1 \left( 1 - \sqrt{1 - \frac{2A}{\bar{C}_1}} \right) + D_2^\text{max} - D_1, \tag{16}
\]
where
\[
A := \frac{I}{p} - \frac{(1-p)}{p} \cdot \left( L_1 \cdot \frac{D_1}{\bar{C}_1} + D_1 \cdot \frac{1 - D_1}{\bar{C}_1} \right) - D_2^\text{max}.
\]

Note that with $D_1^\text{min} \leq D_1 \leq \bar{C}_1$, the time 2 promised payment $D_2 = \Delta(D_1) \leq D_2^\text{max}$ and by definition of $D_1^\text{min}$, if $D_1 = D_1^\text{min}$, then $D_2 = \Delta(D_1^\text{min}) = D_2^\text{max}$.
We can now rewrite the optimization problem as follows and solve for the optimal $D_1$.

$$
\min_{D_1} \quad (1 - p) \frac{D_1}{C_1} (E[C_2] - L_1) + p \omega (1 - \beta) V \cdot Pr(C_1 + C_2 < f_1(C_1, D_1 + D_2)) \\
\text{subject to} \quad (1 - p) \cdot F_0(D_1) + p \cdot F_1(D_1 + D_2) = I.
$$

The following result solves for the optimal contract when $D_2 \leq D_2^{max}$. In the proof we establish that any contract with the feature that $D_2 > D_2^{max}$ is dominated. In other words, the optimal contract below is globally optimal as well.

**Theorem 2** The solution to the firm’s optimization problem, ignoring the constraint that $D_1^{min} \leq D_1 \leq \bar{C}_1$, is given by

$$
D_1^* = \frac{L_1}{2} - \frac{\bar{C}_2}{2 \omega (1 - \beta) V} (\bar{C}_2 - \bar{C}_1 - L_1).
$$

A contract with $D_2 > D_2^{max}$ cannot be optimal and the optimal contract, incorporating the constraint, can be characterized as

- If $D_1^* \geq \bar{C}_1$, then $D_1 = \bar{C}_1$ and $D_2^* = \Delta(\bar{C}_1)$

- If $D_1^{min} < D_1^* < \bar{C}_1$, then $D_1 = D_1^*$ and $D_2^* = \Delta(D_1^*)$

- If $D_1^* \leq D_1^{min}$, then $D_1 = D_1^{min}$, $D_2^* = \Delta(D_1^{min}) = D_2^{max}$.

The solution for $D_2 = \Delta(D_1)$, (16), indicates that the long-term promised component of the total payment is affected by the endogeneity of patenting even when the constraint on long-term payments does not bind ex-ante. This occurs because even if ex-ante the constraint is not binding, ex-post it may nevertheless bind if the realization of time 1 cash flows is low and the promised time 1 payment needs to be rolled over to time 2. The possibility that the lender gets less than $D_1 + D_2$ when the constraint binds ex-post reduces the expected
payoff of the lender so that the total promised payment needs to go up to satisfy the lender’s participation constraint.

3.5 When does the constraint on long-term payments bind ex-ante?

As discussed above, the terms of the optimal contract are affected by the constraint on the time 2 payment, $D_{2}^{max}$. It is useful to see for what type of projects $D_{2}^{max}$ binds ex-ante, so that the time 1 payment takes on its minimum value, $D_1 = D_1^{min}$, and the firm defers as much as possible to time 2 (i.e., $D_2 = D_2^{max}$) while still ensuring that patenting takes place. We illustrate this through an example where we vary $\alpha$ and $V$ and plot the regions where the constraint binds. Changes in $\alpha$ vary the strictness of the constraint since it affects the incentive to patent, but it does not directly affect the total payoff given that the firm always chooses to patent in equilibrium. Changes in $V$, by contrast, directly relate to the value of the project, and also affect feasibility by relaxing the constraint on the long-term payment.

Figure 3 shows different areas based on $\alpha$ and $V$ where other parameters are set as follows: $\omega = 1$, $\beta = 0.8$, $I = 0.7$, $L_1 = \bar{C}_1 = 0.2$, $\bar{C}_2 = 1$, $p = 0.9$. The bottom right corner represent projects where the constraint on time 2 payments is very strict and the value of the innovation is relatively low. Therefore, debt financing is not feasible. At the opposite corner, we have a set of projects where the constraint on long-term payments is relaxed and the value of the projects are very high. As a result, the constraint on long-term payments does not bind ex-ante and the short-term component of debt is equal to $D_1^*$. In between, we have a region where the debt financing is feasible but the constraint on long-term payments binds ex-ante, so that $D_1 = D_1^{min}$ and $D_2 = D_2^{max}$.

For the set of parameters given above, this latter region is quite large. For example, if we fix $\alpha = 0.4$, then we need to increase the value of the patented innovation about 25% to move from the line of feasibility to the line where constraint on the long-term payment no longer binds ex-ante.
Figure 3: The figure displays the region where the constraint on the long-term payment, $D_2^{\text{max}}$, binds ex-ante. The area below the orange curve, of low project values ($V$) coupled with large values for the innovation even when not patented ($\alpha$), represents cases where financing is not feasible. Above the blue curve, where $V$ is relatively large and/or $\alpha$ is relatively low, $D_2^{\text{max}}$ does not bind ex-ante and the optimal contract has an interior solution: $D_1^* > D_1^{\text{min}}$ and $D_2 < D_2^{\text{max}}$. In between the two curves, the constraint binds ex-ante: $D_1^* = D_1^{\text{min}}$ and $D_2 = D_2^{\text{max}}$. Other parameters are set to: $\omega = 1$, $\beta = 0.8$, $I = 0.7$, $L_1 = \bar{C}_1 = 0.2$, $\bar{C}_2 = 1$, $p = 0.9$.

4 Cross-sectional predictions on loan terms

The model provides a set of predictions regarding the duration of optimal contracts for the cross-section of firms. The set of firms where the upper bound on the time 2 payment introduced by the endogeneity of the patenting decision binds ex-ante (i.e., $D_2 = D_2^{\text{max}}$), are those for which concerns related to patenting matter most in shaping the financial contracts they will use. We, therefore, focus on these firms to establish predictions unique to our model.

When the constraint on long-term payment binds, the optimal contract is given by $D_1 = D_1^{\text{min}}$ and $D_2 = D_2^{\text{max}}$, where $D_2^{\text{max}} = \frac{\omega - \alpha}{V + (1 - \beta)\bar{C}_2}$. Plugging $D_2^{\text{max}}$ into the participation constraint, as in (13), and simplifying, yields

$$p \cdot D_2^{\text{max}} = I + (2 - p) \frac{D_1^2}{2\bar{C}_1} - ((1 - p)\frac{L_1}{\bar{C}_1} + 1)D_1.$$
Changes in $I$, $\bar{C}_1$, and $L_1$ do not affect $D_{2}^{\text{max}}$. Therefore, changes in these variables have to be balanced by changes in $D_{1}^{*}$ in order to satisfy the lender’s participation constraint. For example, an increase in the level of investment $I$ must be compensated with an increase in the short-term promised payment. As a result, increases in $I$ or $\bar{C}_1$ increase $D_{1}^{*}$ (this can also be seen directly from (15), which characterizes $D_{1}^{\text{min}}$). On the other hand, increases in $L_1$ result in lower $D_{1}^{*}$. This is somewhat surprising and driven by the fact that the normal trade-off between the short and long-term components of debt is not in play when the constraint on long-term debt binds.

Changes in other variables, such as the value of the innovation $V$ and time 2 cash flows $\bar{C}_2$, affect both the short and long-term components of debt by changing $D_{2}^{\text{max}}$. Increases in $V$ and $\bar{C}_2$ decrease $D_{1}^{*}$ as can be seen from Theorem 2 and increase $D_{2}^{\text{max}}$ as can be seen from (6).

A unique prediction of our model is how changes in the value of innovation that can be captured without patenting, $\alpha$, affect the terms of the financial contract. This is because $\alpha$ does not directly affect the value of the innovation, but matters through its effect on the constraint on long-term payments. As a result, this variable would typically play no role in models where the patenting decision is not endogenous. Here, increases in $\alpha$ reduce the incentive to patent and make the constraint on long-term payment tighter ($D_{2}^{\text{max}}$ decreases), which then reduces the promised payment $D_2$. At the same time, the reduction in $D_{2}^{\text{max}}$ increases the time 1 payment $D_1$ since, when the constraint binds, $D_1 = D_{1}^{\text{min}}$ (see (15)). Therefore, increases in $\alpha$ reduces the duration of the debt contract.

The lemma below summarizes comparative statics on the duration of the optimal contract.

**Lemma 3** When the constraint on long-term payments binds ex-ante, increasing investment $I$, time 1 cash flows $\bar{C}_1$, and the value of innovation that can be captured without patenting, $\alpha$, decreases the duration of the debt contract. On the other hand, increasing the value of the innovation $V$, the liquidation value of the existing assets $L_1$, or time 2 cash flows $\bar{C}_2$,
increases the duration of the debt contract.

5 Endogenous patenting and feasibility

As discussed above, when whether to patent is a choice for innovating firms, there is a constraint imposed on long-term payments that can be credibly promised to a lender. This constraint introduces inefficiencies by limiting feasibility and increasing the probability of liquidation. One way of illustrating the effect of the endogeneity of the patenting decision is to consider what happens when patenting occurs automatically rather than being under the control of the firm. This provides a benchmark that allows us to isolate the effect coming from this choice for the firm rather than from the more widely studied frictions arising from the need to obtain outside financing.

5.1 The benchmark case when patenting is automatic

We assume here that any innovation is always patented if agents continue to time 2, but otherwise keep the model the same as before. When there is no innovation, the payoff of the agents is exactly the same as in the case studied above.

However, in the case of innovation, the model is simpler since strategic non-patenting is not possible. Consider the case where \( C_1 \geq D_1 \). If the firm refuses to pay, or pays less than \( D_1 \), the investor has the right to make a take-it-or-leave-it offer. In that case, the investor can simply offer again the same original contract, which maximizes its payoff if the investor decides to pay. The firm will then either pay \( D_1 \) or let the investor liquidate. Given that \( D_1 + D_2 \leq E[C_2] + \gamma(D_2) \omega V \), the firm always prefers to pay \( D_1 \) and continue.\(^4\)

Alternatively, the firm can pay \( D_1 \) and retain the right to make a renegotiation offer. If the firm just pays \( D_1 \), then it needs to pay \( D_2 \) at time 2, and the investor will liquidate in states where \( C_2 < D_2 \). To reduce this possibility, the firm prefers to pay as much as possible at time 1 and leave as little as possible remaining for time 2. By assumption, \( \bar{C}_1 < I \), so that

\(^4\)Recall that \( \gamma(.) \) is defined in (10).
paying off the entire debt using just $C_1$ is not possible. Therefore, the firm offers renegotiated payments \{\hat{D}_1, \hat{D}_2\} that maximize the firm’s payoff: $\hat{D}_1 = C_1$ and $\hat{D}_2 = D_1 + D_2 - C_1$.

Now consider the case where there is an innovation but $C_1 < D_1$. In this case, the firm cannot make the required payment and the investor either liquidates or makes a take-it-or-leave-it offer. If the investor liquidates it only gets $L_1$, which is also socially inefficient. Instead, the payoffs of both agents can be improved if the investor offers the following contract: $\hat{D}_1 = C_1$ and $\hat{D}_2 = D_1 + D_2 - C_1$. This contract maximizes both the investor’s and the firm’s payoff subject to the constraint that $\hat{D}_1 \leq C_1$.

Given the payoffs discussed above, we can now describe the optimization problem of the firm when patenting occurs automatically. As usual, at time 0 the firm chooses the initial financial contract that maximizes the joint payoff while ensuring that the investor’s participation constraint is satisfied. The joint payoff at time 0 is

$$
\max_{D_1, D_2} E[C_1] + p \cdot \left( E[C_2] + \frac{1}{C_1} \int_0^{\tilde{C}_1} \gamma(D_1 + D_2 - C_1) \omega V dC_1 \right) \\
+ (1 - p) \cdot \frac{1}{C_1} \left( \int_0^{\hat{D}_1} L_1 dC_1 + \int_{\hat{D}_1}^{C_1} E[C_2] dC_1 \right)
$$

subject to the participation constraint of the investor,

$$
I = p \cdot (D_1 + D_2) + (1 - p) \cdot \left( \frac{D_1}{C_1} L_1 + (1 - \frac{D_1}{C_1}) D_1 \right). 
$$

The last term in (17), $\frac{D_1}{C_1} L_1 + (1 - \frac{D_1}{C_1}) D_1$, increases in $D_1$ given that $D_1 < \tilde{C}_1$ and $D_1 < L_1$, and therefore $D_1 < (\tilde{C}_1 + L_1)/2$. As a result, there is a trade-off in changing $D_1$: increasing $D_1$ increases the investor’s payoff in case of no innovation and allows the firm to reduce the total payment $D_1 + D_2$. However, this change has two opposing effects. First, increasing $D_1$ increases the probability of liquidation in the case of no innovation. Second, reducing $D_1 + D_2$ lowers the probability of liquidation in case of innovation.

Again we can transform the problem into one involving the minimization of the ineffi-
ciency introduced by outside financing, as follows:

\[
\min \ (1 - p) \frac{D_1}{\bar{C}_1} (E[C_2] - L_1) + p \omega (1 - \beta) V \cdot Pr(C_1 + C_2 < D_1 + D_2) \\
\text{s.t} \quad (1 - p) \cdot (L_1 \cdot \frac{D_1}{\bar{C}_1} + D_1 \cdot (1 - \frac{D_1}{\bar{C}_1})) + p \cdot ((D_1 + D_2)) = I \\
0 \leq D_1 \leq \bar{C}_1.
\]

This optimization problem is different from the one for the case with endogenous patenting in a number of ways. First, the objective function is different because, when patenting is endogenous, the constraint may bind ex-post, resulting in debt forgiveness as described above, whereas that does not occur here. Second, the participation constraint simply equals the total promised payments when an innovation has occurred and patenting occurs automatically. Finally, there is no constraint on \(D_2\) as in the case of endogenous patenting.

We can compare the terms of the optimal contract. In the case of endogenous patenting, the maximum payoff the lender can get at time 2 is limited. As a result, to satisfy the participation constraint, the expected payment the lender receives from time 1 cash flows must increase so that the contract with endogenous patenting will generally have a higher short-term component unless the constraint on long-term repayment does not bind.

Having described a benchmark where patenting occurs automatically, we next discuss how the feasibility of financing is affected by the endogeneity of the patenting decision.

### 5.2 How endogeneity of patenting affects feasibility

When the project is internally financed all that matters is that the expected value of the innovation is greater than the initial investment. It will therefore undertake the project if

\[
\omega V \geq \frac{I}{p}.
\]

This is also the condition for the first best because there is no inefficient liquidation.

When patenting occurs automatically, but the firm requires outside financing to undertake
the investment, the condition for a lender to participate is

\[ \omega \beta V + \frac{\bar{C}_1}{2} \geq \frac{I - (1 - p)L_1}{p}. \]  

(19)

Not all projects satisfying the first best, (18), will be taken because of the agency problem and associated inefficient liquidation. There are two types of inefficient liquidations, which happen at either time 1 or 2. To simplify the analysis, we will focus on the inefficiency at time 2 stemming from the potential liquidation of the patent. Thus, we assume \( \bar{C}_1 = L_1 = 0 \), which eliminates the inefficiency of liquidation of the existing assets at time 1. The participation constraint simplifies and becomes \(^5\)

\[ \omega \beta V \geq \frac{I}{p}. \]  

(20)

For this case, the only limit on the time 2 promised payment is the liquidation value of the patent, \( \omega \beta V \). By comparing (18) to (20) we can see that more projects get infeasible due to outside financing as \( \beta \) gets smaller, i.e. as the inefficiency introduced by liquidation becomes larger.

Assume now a project is a positive NPV, and is feasible with external financing when patenting is automatic. When the patenting decision is endogenous, we know from Theorem 1 that financing is feasible if

\[ D_2^{max} + \frac{\bar{C}_1}{2} \geq \frac{I - (1 - p)L_1}{p}. \]

Maintaining the assumption that \( \bar{C}_1 = L_1 = 0 \), and substituting for \( D_2^{max} \), we obtain the following combined condition for participation and positive NPV:

\[ \frac{\omega - \alpha}{V} + \frac{(1 - \beta)\omega}{c_2} \geq \frac{I}{p}. \]

(21)

\(^5\)Notice this also guarantees that (18) will be satisfied and that the project has positive NPV under external funding even though there might be liquidation at time 2.
One simple way to compare the impact of endogenous patenting on the feasibility of financing is to consider the case when $\beta = 1$, i.e., when there is no inefficiency in the liquidation of the patent. The condition for financing to be feasible when patenting automatically occurs, (19), coincides with the first best, while feasibility when patenting is endogenous is stricter. In particular, it becomes $(\omega - \alpha)V \geq \frac{I_p}{\beta}$, which highlights how the incremental value to the firm from patenting, $\omega - \alpha$, is an important driver of financing feasibility, in contrast to the case where the firm can either finance internally, or when there is no question as to whether innovations will be patented. In particular, as $\alpha$ increases, projects become more difficult to finance externally, with no projects being feasible as $\alpha \to \omega$. This also illustrates that the inefficiency introduced by the endogeneity of the patenting decision is not entirely driven by the existence of an inefficiency arising from liquidation, but rather from the ex-post hold-up due to endogeneity of patenting decision.

Figure 4 shows how the feasibility of the three cases varies with $\beta$, the efficiency of the secondary market for the firm’s patent(s), illustrating that the inefficiency resulting from the endogeneity of patenting is unique and does not disappear as $\beta$ gets larger. Moreover, from the figure, it is apparent that the economic value of this inefficiency can be at least as large as that of the more commonly studied inefficiencies that often arise when outside financing is needed (this can be measured by comparing the distance between the red and blue curves, to that between the blue and black dashed curve). Note as well that the feasibility of obtaining financing when patenting is endogenous depends on $\alpha$, the value of the firm’s innovation when not patented. The higher is $\alpha$, the more difficult it is to finance a given project.

Figure 5 illustrates how the inefficiency arising from the endogeneity of patenting varies with $\omega$, the value of patenting to the firm. As the value of patent protection increases, this inefficiency decreases. By contrast, the inefficiency stemming from outside financing largely remains the same, with a similar difference between the marginal project obtaining financing (in terms of the project’s value $V$) and the marginal project under the first best. As above, the magnitude of the inefficiency is always at least as large as that stemming just from the
need to obtain outside financing. Likewise, the magnitude of the inefficiency depends on the innovation’s value when not patented, $\alpha$, being larger the greater is this value.

### 5.3 Project risk and feasibility

An aspect that we have so far not emphasized is how the distribution of a project’s payoffs affects the feasibility of obtaining financing for a project when patenting is endogenous. Since the project’s payoff is $V$ with probability $p$ and 0 with probability $1 - p$, to analyze how endogeneity of patenting affects the riskiness of projects that obtain financing, here we fix the project’s expected payoff to be constant, $pV := \mathcal{V}$, and vary $p$. For simplicity, we also maintain the assumptions that $\bar{C}_1 = L_1 = 0$, as in our study of feasibility from the previous section. Given this, for a given probability of innovation $p$, the variance of the project’s payoff is

$$\mathcal{V}^2 \cdot \frac{1 - p}{p}, \quad (22)$$
Figure 5: Feasibility according to $\omega$. Vertical axis represents $V$. The project is feasible if $V$ is above the curve of each scenario: “commit” refers to the case where patenting automatically occurs and is not a choice for the firm. “no-commit” represents the case where the decision to patent is endogenous. ($\beta = 0.8, I = 1, p = 0.9, \bar{C}_2 = 2, L_1 = \bar{C}_1 = 0$)

which is a strictly decreasing function of $p$. Thus, a higher $p$ corresponds to a project with lower risk.

The feasibility conditions for both the first best, $\omega pV = \omega V \geq I$ (see (18)), and for the benchmark case where patenting occurs automatically, $\omega \beta pV = \omega \beta V \geq I$ (see (20)), only depend on the expected return of the project $V$. By contrast, when patenting is endogenous, the condition for feasibility, (21), can be written as

$$\frac{\omega - \alpha}{V} + \frac{(1-\beta)\omega}{pC_2} \geq I.$$

When $pV = V$ is fixed, the left-hand side is an increasing function in $p$. Therefore, for projects with the same expected payoff, only those that are less risky (i.e., higher $p$) will be feasible. Figure 6 shows the feasibility region for different scenarios. For either the first best case or the case where patenting occurs automatically, the variance of project payoff related
to changes in \( p \), \((22)\), does not affect which projects are feasible since only the expected project payoff matters, and we assume \( pV = \mathcal{V} \) is fixed. By contrast, when patenting is endogenous, the figure shows that it is easier to obtain debt financing for projects that are safer and have a higher probability of delivering innovation, even if that innovation is less valuable when successful.

Figure 6: Feasibility according to project expected payoff \((pV)\) and success probability \((p)\). The vertical axis represents \( \mathcal{V} = pV \). The horizontal axis represents the probability of innovation \( p \). The area above each curve represents feasible projects for each scenario. \((\omega = 0.9, \beta = 0.8, I = 1, C_2 = 2, L_1 = C_1 = 0)\)

6 Policy implications

There are two variables associated with policy decisions in our model: the degree of patent protection \( \omega \) and the extent of creditor liquidation rights, which would be one of the main determinants of \( \beta \) in the model.\(^6\) The feasibility constraint \((21)\) becomes more relaxed as \( \omega \)

\(^6\)The efficiency of the secondary market for the firm’s assets should influence \( \beta \) as well, which may vary by industry or geographic region. We view this aspect as providing important cross-sectional variation, separately from policy initiatives that could be taken to increase the liquidation value \( \beta \) of the firm’s patents.
or \( \beta \) increases. This is natural, and likely consistent with other related findings, because it means that as patent protection and the liquidation value of patents increase, \( D_2^{max} \) increases and as a result, more projects will be feasible. These implications may well arise in other settings as well.

More specific and novel to our model is how changes in policy variables affect the upper limit on long-term payments, \( D_2^{max} \), and feasibility in the cross-section of firms. In particular, note that the way \( D_2^{max} \) increases as \( \omega \) or \( \beta \) change is different. \( D_2^{max} \) is concave in \( \omega \) but convex in \( \beta \) (see Figure 7 and formal results in Lemma 4). This means that it is more efficient to improve patent protection \( \omega \) when protection is low, while the reverse is true for the liquidation value \( \beta \) of patents.

![Figure 7: \( D_2^{max} \) changes according to \( \omega \), \( \beta \)](image)

Consider now two industries with different values of \( \alpha \), the value of the innovation to the firm if it is not patented. For \( \alpha_1 < \alpha_2 \), we have

\[
D_2^{max}(\alpha_1) - D_2^{max}(\alpha_2) = \frac{\alpha_2 - \alpha_1}{\frac{1}{V} + \frac{(1-\beta)\omega}{\rho_2}} > 0.
\]

Since \( D_2^{max} \) is essentially a measure of feasibility, the difference \( D_2^{max}(\alpha_1) - D_2^{max}(\alpha_2) \) can be interpreted as the number or proportion of projects that are feasible in industries with \( \alpha_1 \) but not feasible in industries with \( \alpha_2 \). The positive difference implies that all things equal, the industry with lower \( \alpha \) will have more projects that are feasible. Moreover, this difference
is decreasing in $\omega$ but increasing in $\beta$, meaning that as we improve patent protection the difference in feasibility across industries increases, but the reverse is true for creditor rights, $\beta$. An implication of these findings is that, since higher $\alpha$ industries have fewer projects being financed, they should then be smaller. Since the difference $D_{2}^{\max}(\alpha_{1}) - D_{2}^{\max}(\alpha_{2})$ is decreasing in $\omega$, however, improvements in patent protection should lead to relatively higher growth in high “$\alpha$” industries.

The result below provides a more complete characterization of comparative statics related to feasibility and illustrates how different industries with different characteristics are affected by changes in policy choices.

**Lemma 4** The following comparative statics on $D_{2}^{\max}$, the upper limit on long-term payments, hold:

$$\frac{\partial^{2} D_{2}^{\max}}{\partial \beta^{2}} > 0, \quad \frac{\partial^{2} D_{2}^{\max}}{\partial \omega^{2}} < 0,$$

$$\frac{\partial^{2} D_{2}^{\max}}{\partial \alpha \partial \omega} > 0, \quad \frac{\partial^{2} D_{2}^{\max}}{\partial \alpha \partial \beta} < 0, \quad \frac{\partial^{2} D_{2}^{\max}}{\partial \beta \partial \omega} > 0.$$

The first part of the lemma establishes the convexity and concavity of $D_{2}^{\max}$ as a function of $\beta$ and $\omega$, respectively. The second part of the lemma establishes that improving creditor rights and patent protection makes financing feasible for a greater range of projects. More interestingly, these policies are complementary, i.e., changes in one have a greater effect on feasibility if the other one is higher. We speculate that these predictions can be tested by using changes in patent protection and enforcement over time (see Hall (2004)) and proxying for the value of innovation without patenting by using the industry level of competition.

**7 Conclusion**

Growing empirical evidence shows that debt financing plays an important role in funding the growth of small and innovative firms (Robb and Robinson (2014); Cornaggia et al. (2015);
Chava et al. (2013); Kerr and Nanda (2015); Mann (2018)), despite a perception that such financing is not suitable for firms that invest in specialized assets with low liquidation values (Williamson (1988)). One reason why debt financing for innovative firms is difficult is because of their paucity of assets that could be useful as collateral. While recent research has shown that intellectual property assets, in the form of patents, can potentially be used as collateral (Hochberg et al. (2018); Mann (2018); Ma and Wang (2019)), many small, innovative firms may not have such assets at the time of financing. Moreover, the decision to patent future innovations is endogenous, and firms may sometimes prefer not to patent their innovations for strategic reasons, hoping to reduce the ex-post bargaining power of their lenders in the event of renegotiation. This key friction – that patenting is itself an endogenous choice for firms – shapes financial contracts and has implications for what types of projects can be financed through debt instruments.

Our analysis shows that the difference between the patented value of an innovation and its value in the absence of patent protection is particularly important since it determines how large an incentive a firm may have to subsequently patent, and thus determines whether outside financing is feasible. To the best of our knowledge, the result that the value of innovation without patenting playing a role in determining a firm’s ability to obtain financing is novel.

Our analysis also derives unique implications for patent policy and creditor liquidation rights. Although improving both creditor liquidation rights and patent protection improves the feasibility of financing innovation, these two policies have different implications on industry growth depending on the value of the innovation without patenting. Moreover, the model is flexible in that it does not assume there are large benefits to patenting, but rather offers predictions based on the size of any such benefits, and their interaction with other variables of interest. These implications we believe are new to the literature, and offer a novel perspective on aspects that may be important to consider when revising policies related to patent protection, as well as to ease with which patents may be liquidated by creditors.
References


A Appendix: Proofs

To prove Theorem 1, we first establish the following preliminary result, which also establishes the second part of the theorem, that liquidation does not occur in equilibrium when financing is feasible.

**Lemma 5** Financing is feasible only if $D_{2}^{max} > L_{1}$. Moreover, when financing is feasible, there will be no liquidation at time 1 when there is innovation: The firm always pays all time 1 cash flows $C_{1}$ to the lender and patents the innovation.

### A.1 Proof of Lemma 5

The expected payment the investor receives conditional on no innovation is

$$L_{1} \cdot \frac{D_{1}}{C_{1}} + D_{1} \cdot (1 - \frac{D_{1}}{C_{1}}) \leq L_{1} + D_{1} \cdot (1 - \frac{D_{1}}{C_{1}}) \leq L_{1} + \frac{C_{1}}{4} < L_{1} + E[C_{1}] .$$

Given that $L_{1} + E[C_{1}] < I$ by assumption, the expected payment to the lender when there is no innovation is clearly less than $I$.

Consider now the case where there is innovation. If $D_{2}^{max} \leq L_{1}$, and assume that $D_{2} \leq D_{2}^{max}$, then the payment conditional on innovation, as a function of $C_{1}$, is

$$f_{1}(C_{1}, D_{1} + D_{2}) = \begin{cases} 
L_{1} & \text{if } C_{1} \leq L_{1} - D_{2}^{max} \\
D_{2}^{max} + C_{1} & \text{if } L_{1} - D_{2}^{max} < C_{1} \leq D_{1} + D_{2} - D_{2}^{max} \\
D_{1} + D_{2} & \text{if } C_{1} \geq D_{1} + D_{2} - D_{2}^{max}
\end{cases} \quad (23)$$

If, on the other hand, $D_{2} > D_{2}^{max}$, then the payment conditional on innovation is only $D_{1}$ when $C_{1} > D_{1}$ as the firm, which can make a take-it-or-leave-it offer, would always offer $D'_{2} = 0$. The lender would accept since that is what they would get under the original contract which, since $D_{2} > D_{2}^{max}$, would not lead to patenting. The payment conditional on innovation is, therefore, at most $f_{1}(C_{1}, D_{1} + D_{2})$. 

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Hence, like in the no innovation case, the payment to the lender is at most $f_1(C_1, D_1 + D_2) \leq L_1 + C_1$. Therefore, the expected payment is at most $L_1 + E[C_1] < I$. Thus, for any feasible contract it must be the case that $D_2^{max} > L_1$.

Finally, we show that if financing is feasible, there will be no liquidation conditional on innovation. We focus on the case where $D_2 \leq D_2^{max}$. If $C_1 < D_1$, then the firm patents only if $D_1 + D_2 - C_1 \leq D_2^{max}$. Otherwise, the firm would not patent and the lender’s outside option would be equal to $L_1$ under the original contract. However, if $D_2^{max} \geq L_1$, then the firm could promise to pay up to $D_2^{max}$ even when the time 1 cash flow, $C_1$, equals 0 and avoid liquidation. The firm clearly finds it optimal to do so, so that liquidation never occurs when financing is feasible.

A.2 Proof of Theorem 1

As argued earlier, we may restrict the analysis to contracts where $D_1 \leq \bar{C}_1$. Next, we establish that a contract with $D_2 = D_2^{max}$ yields a higher payoff for the lender than a contract with $D_2 > D_2^{max}$, for the same $D_1$. To see this, consider what happens under the original contract when $D_2 > D_2^{max}$: if $C_1 > D_1$, the firm can repay $D_1$, and has the right to make a take-it-or-leave-it offer $D'_2$ to the lender for the time 2 payment. If the lender rejects the new offer, the firm will prefer not to patent the innovation, and hence the payment to the lender at time 2 will be zero. Hence, the lender’s outside option is equal to zero. By contrast, if $D_2 = D_2^{max}$ under the original contract, any offer $D'_2$ less than $D_2^{max}$ would be rejected by the lender since, with the original contract in place, the firm will prefer to patent the innovation, leading to a payment of $D_2 = D_2^{max}$ for the lender.

Likewise, the payoff to the lender for any contract with $D_2 < D_2^{max}$ can again be increased by instead setting $D_2 = D_2^{max}$, for the same $D_1$. To see this, consider the case where $C_1 < D_1$, so that the firm cannot make the time 1 payment. This allows the lender to make a take-it-or-leave-it offer to the firm, where the optimal such offer is $(C_1, \min\{D_2^{max}, D_1 + D_2 - C_1\})$. The payoff from this contract is weakly increasing in $D_2$ for $D_2 < D_2^{max}$, and is strictly
increasing for $C_1$ close to $D_1$. Hence, the expected payment to the lender is maximized for $D_2 = D_2^{\text{max}}$, given a fixed value for $D_1$.

We now show that, given $D_2$, the payoff to the lender is maximized for $D_1 = \bar{C}_1$. To see this, suppose that $D_1 < \bar{C}_1$, and assume first that there is no innovation. When $C_1 < D_1$, the lender can liquidate to obtain $\min\{L_1, D_1 + D_2\} = L_1$. However, when $C_1 \geq D_1$, the firm can simply make the time 1 payment $D_1$ and pay nothing at time 2. Hence, the lender’s payoff can be strictly increased by raising $D_1$, as long as $D_1 < \bar{C}_1$.

Consider then the case of innovation. As argued above, for any given $C_1$, and assuming $D_2 \leq D_2^{\text{max}}$, the lender’s payoff is $D_1 + D_2$ when the firm has the power over renegotiation, and is $C_1 + \min\{D_2^{\text{max}}, D_1 + D_2 - C_1\}$ when instead the lender can propose the renegotiated contract. Either way, the lender’s payoff is increasing in $D_1$, for a given $D_2$. Together with the argument above, this implies that the contract that yields the highest expected payoff for the lender is $(D_1, D_2) = (\bar{C}_1, D_2^{\text{max}})$.

Given this, we can now substitute the contract $D_1 = \bar{C}_1$, $D_2 = D_2^{\text{max}}$ into the lender’s participation constraint to obtain

$$(1 - p) \cdot L_1 + p \cdot (D_2^{\text{max}} + \frac{\bar{C}_1}{2}) \geq I,$$

which, after some rearranging, yields the expression in the statement of the Theorem.
A.3 Proof of Theorem 2

Case 1: $D_2 \leq D_2^{max}$

We first characterize the optimal contract with the assumption that $D_2 \leq D_2^{max}$. Observe that according to (11),

$$\Pr(\text{time 2 liquidation}) = \Pr(C_2 < f_1(C_1, D_1 + D_2) - C_1)$$

$$= \frac{1}{C_1} \int_0^{C_1} \frac{1}{C_2} \int_0^{C_2} (f_1(C_1, D_1 + D_2) - C_1) dC_2 dC_1$$

$$= \frac{1}{C_1 C_2} \int_0^{C_1} f_1(C_1, D_1 + D_2) dC_1 - \frac{1}{C_1} \int_0^{C_1} C_1 dC_1$$

$$= \frac{1}{C_1 C_2} \int_0^{C_1} f_1(C_1, D_1 + D_2) dC_1 - \frac{\bar{C}_1}{2 C_1}.$$

Thus, we obtain

$$\Pr(\text{time 2 liquidation}) = \frac{F_1(D_1 + D_2)}{C_2} - \frac{\bar{C}_1}{2 C_2}.$$

Using the participation constraint, (9), this probability is the same as

$$\Pr(\text{time 2 liquidation}) = \frac{I - (1 - p) F_0(D_1)}{p C_2} - \frac{\bar{C}_1}{2 C_2}. \quad (24)$$

Thus, the total inefficiency is

$$(1 - p) H_0(D_1) + p \cdot \omega(1 - \beta) V \cdot \left( \frac{I - (1 - p) F_0(D_1)}{p C_2} - \frac{\bar{C}_1}{2 C_2} \right).$$

We can write the total inefficiency as a quadratic function of $D_1$.

$$\mathcal{T}(D_1) := (1 - p) \cdot \left( D_1 \cdot \frac{E[C_2] - L_1}{C_1} - \frac{\omega(1 - \beta)V}{C_2} F_0(D_1) \right) + \omega(1 - \beta) V \left( \frac{2I - p C_1}{2 C_1} \right)$$

$$= (1 - p) \cdot \left( D_1 \cdot \frac{E[C_2] - L_1}{C_1} - \frac{\omega(1 - \beta)V}{C_2} \left( L_1 \frac{D_1}{C_1} + D_1 \left( 1 - \frac{D_1}{C_1} \right) \right) \right) + \omega(1 - \beta) V \left( \frac{2I - p C_1}{2 C_1} \right)$$
\[
= (1-p)(D_1\left(\frac{E[C_2] - L_1}{C_1} - \frac{\omega(1-\beta)V}{C_2}\frac{L_1}{C_1} - 1\right) + \frac{\omega(1-\beta)V}{C_1C_2}D_1^2) + \frac{\omega(1-\beta)V}{2C_1}(2I - p\bar{C}_1) \tag{25}
\]

From this, we can characterize the equilibrium because it is function of only one variable, \(D_1\). The derivative is 0 at

\[
D^*_1 = \frac{\bar{C}_2}{2\omega(1-\beta)V}(\bar{C}_1 + L_1 - E[C_2]) + \frac{L_1}{2}
\]

\[
= \frac{L_1}{2} - \frac{\bar{C}_2}{2\omega(1-\beta)V}(\frac{\bar{C}_2}{2} - \bar{C}_1 - L_1). \tag{26}
\]

This establishes the result under the assumption that \(D_2 \leq D_2^{max}\). However, a feasible contract may exist with \(D_2 > D_2^{max}\). Therefore, we next show that the optimal contract characterized here is also better than any contract with \(D_2 > D_2^{max}\).

**Case 2: \(D_2 > D_2^{max}\)**

Let \((D_1, D_2)\) be a feasible contract with \(D_2 > D_2^{max}\). First, we show that it must be that \(D_1 \geq D_1^{min}\).

\[\text{Figure 8: Payment to the lender when } D_1 < D_1^{min}, D_1 > D_2^{max}\]

Assume to the contrary that \(D_1 < D_1^{min}\). Then the payment to the lender conditional on innovation is

\[
h(C_1, D_1) = \begin{cases} 
D_2^{max} + C_1 & \text{if } C_1 \leq D_1 < D_1^{min} \\
D_1 & \text{if } C_1 \geq D_1
\end{cases}
\]
Notice that \( h(C_1) \leq f_1(C_1, D_1^{min} + D_2^{max}) \), which is the payment function under the contract \((D_1^{min}, D_2^{max})\). Thus, the expected payment of such a contract conditional on innovation is at most that of the contract \((D_1^{min}, D_2^{max})\) (see Figure 8). Furthermore, when there is no innovation, the expected payment is a strictly increasing function of \(D_1\). Therefore, the total expected payment of the contract \((D_1, D_2)\) is less than that of the contract \((D_1^{min}, D_2^{max})\). This shows that \((D_1, D_2)\) cannot be feasible because \((D_1^{min}, D_2^{max})\) satisfies the participation constraint with equality. Hence, for \((D_1, D_2)\) to be feasible, it must be that \(D_1 \geq D_1^{min}\).

Now let \(D'_2 := \Delta(D_1)\), as defined in \((16)\). Because \(D_1^{min} \leq D_1 \leq C_1\), \(D'_2 \leq D_2^{max}\) and the participation constraint of the contract \((D_1, D'_2)\) binds. Next, we will show that the probability of liquidation at time 2 under the contract \((D_1, D'_2)\) is smaller than that of \((D_1, D_2)\). Because both contracts satisfy the participation constraint with equality, this implies that the firm’s payoff is larger under \((D_1, D'_2)\) and thus the contract \((D_1, D_2)\) with \(D_2 > D_2^{max}\) cannot be optimal.

According to \((24)\), the probability of liquidation at time 2 for the contract \((D_1, D'_2)\) is

\[
\Pr(\text{time 2 liquidation}) = \frac{I - (1 - p)\mathcal{F}_0(D_1)}{pC_2} - \frac{\bar{C}_1}{2C_2},
\]

where \(\mathcal{F}_0(D_1) = L_1 \cdot \frac{D_1}{C_1} + D_1 \cdot (1 - \frac{D_1}{C_1})\).

To compute the probability liquidation at time 2 for the contract \((D_1, D_2)\), we start from the participation constraint. Recall that for \(D_2 > D_2^{max}\), we defined the payoff to the lender when there is innovation as \(h(C_1, D_1)\). Thus, the expected payoff of the lender conditional on innovation is

\[
\int_0^{D_1} (C_1 + D_2^{max})dC_1 + \int_{D_1}^{\bar{C}_1} D_1 dC_1 = \frac{D_1}{C_1}(D_2^{max} + \frac{D_1}{2}) + D_1(1 - \frac{D_1}{C_1}) = \frac{D_1}{C_1}(D_2^{max} + \bar{C}_1 - \frac{D_1}{2}).
\]

Hence, the participation constraint becomes

\[
(1 - p) \cdot (L_1 \cdot \frac{D_1}{C_1} + D_1 \cdot (1 - \frac{D_1}{C_1})) + p \cdot \frac{D_1}{C_1}(D_2^{max} + \bar{C}_1 - \frac{D_1}{2}) = I
\]

(28)
\[ = (1 - p)F_0(D_1) + pF'_1(D_1), \]

where we define \( F'_1(D_1) = \frac{D_1}{C_1}(D_2^{\text{max}} + \bar{C}_1 - \frac{D_1}{2}) \) as the lender's expected payoff conditional on innovation under this contract. Notice that

\[
\Pr(\text{time 2 liquidation}) = \Pr(C_1 < D_1 \& C_2 < D_2^{\text{max}}) = \frac{D_1 D_2^{\text{max}}}{C_1 C_2} = \frac{F'(D_1)}{C_2} - D_1 \frac{2\bar{C}_1 - D_1}{2C_1C_2}. 
\]

Using the participation constraint, (28), to substitute for \( F'_1(D_1) \), we have

\[
\Pr(\text{time 2 liquidation}|(D_1, D_2)) = \frac{I - (1 - p)F_0(D_1)}{pC_2} - D_1 \frac{2\bar{C}_1 - D_1}{2C_1C_2}. \tag{29} 
\]

From (29) and (27), we obtain

\[
\Pr(\text{time 2 liquidation}|(D_1, D_2)) - \Pr(\text{time 2 liquidation}|(D_1, D_2')) = \\
\frac{\bar{C}_1}{2C_2} - D_1 \frac{2\bar{C}_1 - D_1}{2C_1C_2} = \frac{(\bar{C}_1 - D_1)^2}{2C_1C_2} \geq 0. 
\]

Equality is only obtained when \( D_1 = \bar{C}_1 \), in which case the two contracts are the same. This shows that a contract with \( D_2 > D_2^{\text{max}} \) cannot be optimal.

### A.4 Proof of Lemma 4

The lemma can be established from simple differentiation of \( D_2^{\text{max}} \), as follows:

\[
\frac{\partial^2 D_2^{\text{max}}}{\partial \beta^2} = 2V^3 \omega^2 C_2 \frac{\omega - \alpha}{(V\omega + \bar{C}_2 - V\beta\omega)^3} > 0 
\]

\[
\frac{\partial^2 D_2^{\text{max}}}{\partial \omega^2} = 2V^2 C_2 (\beta - 1) \frac{V\alpha + \bar{C}_2 - V\alpha\beta}{(V\omega + \bar{C}_2 - V\beta\omega)^3} < 0 
\]

\[
\frac{\partial^2 D_2^{\text{max}}}{\partial \alpha \partial \omega} = V^2 C_2 \frac{1 - \beta}{(V\omega + \bar{C}_2 - V\beta\omega)^2} > 0 
\]
\[
\frac{\partial^2 D_{2}^{\text{max}}}{\partial \alpha \partial \beta} = -V^2 \omega \frac{C_2}{(V \omega + C_2 - V \beta \omega)^2} < 0
\]

\[
\frac{\partial^2 D_{2}^{\text{max}}}{\partial \beta \partial \omega} = -V^2 \frac{C_2}{(V \omega (1 - \beta) + C_2)^2} \left( C_2 (\alpha - 2 \omega) - V \omega (1 - \beta) \right) > 0.
\]

\section*{B Robustness}

In this section, we consider a variation of the renegotiation game where the firm has all the bargaining power, and as a result, always makes the offer in the bargaining process. Our main goal is to show that the endogeneity of patenting remains as significant friction for lending. Specifically, we show the following result.

\textbf{Theorem 3} Under the assumption that the firm always has the right to make a take-it-or-leave-it offer to the lender, a feasible contract exists if and only if \(pD_{2}^{\text{max}} \geq I\).

To prove this result, first consider the case when there is no innovation. If \(C_1 < D_1\), the lender gets \(L_1\), and the firm liquidates at time 1 with probability 1. If \(C_1 \geq D_1\), the lender gets \(D_1\) and the firm does not liquidate. This is the same as before.

When there is innovation, the outcome of the contract depends on whether \(D_2 \leq D_{2}^{\text{max}}\) or \(D_2 > D_{2}^{\text{max}}\).

- If \(D_2 \leq D_{2}^{\text{max}}\):
  - If \(C_1 < D_1\), the lender gets \(L_1\) and the firm gets liquidated at time 2 with \(\Pr(C_2 < L_1 - C_1)\);
  - If \(C_1 \geq D_1\), the lender gets \(D_1 + D_2\) and the firm gets liquidated at time 2 with \(\Pr(C_2 < D_1 + D_2 - C_1)\).

- If \(D_2 > D_{2}^{\text{max}}\):
- If $C_1 < D_1$, the lender gets $L_1$ and the firm gets liquidated at time 2 with $\Pr(C_2 < L_1 - C_1)$;
- If $C_1 \geq D_1$, the lender gets $D_1$ and the firm does not patent.

Thus, when $D_2 > D_2^{\text{max}}$ the payoff of the lender is

$$\frac{D_1}{C_1}L_1 + (1 - \frac{D_1}{C_1})D_1 \leq L_1.$$

Therefore, such a contract is not feasible.

Now, when $D_2 \leq D_2^{\text{max}}$, the expected payment to the lender is

$$p \cdot \left( \frac{D_1}{C_1}L_1 + (1 - \frac{D_1}{C_1})(D_1 + D_2) \right) + (1 - p) \cdot \left( \frac{D_1}{C_1}L_1 + (1 - \frac{D_1}{C_1})D_1 \right)$$

$$= \frac{1}{C_1} \left( (L_1 + \bar{C}_1 - pD_2)D_1 - D_1^2 \right) + pD_2. \quad (30)$$

**Lemma 6** If $(D_1, D_2)$ is feasible, then the expected payment to the lender in (30) is a decreasing function of $D_1$.

**Proof.** We need to show that $L_1 + \bar{C}_1 - pD_2 \leq 0$, which implies that (30) is a decreasing function of $D_1$. Assume that $L_1 + \bar{C}_1 - pD_2 > 0$. Then (30) is maximized when $D_1 = \min\{\frac{L_1 + \bar{C}_1 - pD_2}{2}, \bar{C}_1\}$, and the maximum value is either

$$\frac{1}{C_1} \left( \frac{L_1 + \bar{C}_1 - pD_2}{2} \right)^2 + pD_2$$

when $\frac{L_1 + \bar{C}_1 - pD_2}{2} \leq \bar{C}_1$

or

$L_1$ when $\frac{L_1 + \bar{C}_1 - pD_2^{\text{max}}}{2} > \bar{C}_1$.

In the former case, that value is at most

$$\left( \frac{L_1 + \bar{C}_1 - pD_2}{2} \right)^2 + pD_2 = \left( \frac{L_1 + \bar{C}_1 + pD_2}{2} \right) \leq L_1 + \bar{C}_1 < I.$$
So in both cases, our assumption rules out feasibility.

This implies that for a contract to be feasible, $L_1 + \bar{C}_1 - pD_2 \leq 0$, which establishes the result.

Because of Lemma 6, the expected payoff to the lender is maximized when $D_1 = 0, D_2 = D_2^{\text{max}}$, and therefore the maximum value is $pD_2^{\text{max}}$. This proves Theorem 3.