Pseudo-Market for Military Logistics
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Abstract

Ensuring the readiness of the US military has traditionally entailed positioning as many resources (‘iron mountains’) as far forward as possible. This had the advantage that units at the tactical edge did not have to ‘wait’ for what they needed but resulted in costly excess inventory. Budget pressures and changes in the operating environment have pushed the Military to consider more agile and responsive delivery. The result is that not everyone gets what they want right away, raising the challenge of whose requests should be prioritized. In this paper, we propose a model to capture the readiness of military units. We analyze the plausibility of using a pseudo-market to determine these priorities. We identify several attractive properties of our pseudo-market mechanism in terms of the system’s overall readiness, incentive compatibility, and computational complexity.

1 Introduction

The responsibility of supplying beans, bandages, and bullets to US military personnel belongs to the Department of Defense (DoD)’s Joint Logistics Enterprise (JLE). The JLE also provides relief supplies to victims of natural disasters, as well as humanitarian aid to refugees

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and internally displaced persons. Unsurprisingly, the scale of these sustainment operations is Brobdingnagian. John Paschkewitz of DARPA observes that

“Most people don’t realize that the Air Force alone operates a fleet of aircraft four times the size of one of the largest U.S. airlines.”

The objective of this sustainment operation is to ensure the readiness of US forces to conduct operations. Traditionally this has resulted in positioning as many resources (‘iron mountains’) as far forward as possible. Changes in the operating environment (larger numbers of independently operating units that are widely dispersed) and budget constraints made this impractical (see Thompson (2010)). This prompted a decade of a long effort to improve the operational efficiency of the US military’s supply chain (see Parlier (2005) and Peltz and Robbins (2012)). Among the changes was a substitution away from iron mountains towards more agile and responsive delivery methods. This is the motivation for considering a pseudo-market approach to this problem. But first, we describe important components of the current system.

Replenishment plans are typically made in advance of actual circumstances on the ground. An unanticipated shock will cause some units (e.g., the European Combatant Command) to consume resources at a level that exceeds what was planned for. In this case, the unit would like to expedite deliveries of scarce resources. Transportation capacity, however, is limited and there can be many such requests for expedited delivery. These requests are received by personnel at Operations Center for US Transportation Command who decide which requests to honor in an ad hoc way. These decisions have knock-on effects that ripple through the system. On the flip side, claims for priority are ‘cheap talk’ which gives incentives to units to label all their requests a high priority. Absent a transparent and principled procedure for allocating scarce resources, military personnel resort to padding supply orders and stockpiling extra items. Multiple requisitions are sometimes placed for

items already in theater, while others are procured locally. The chief virtue of the current system of deciding on priorities is the discretion and concomitant flexibility it affords the Operations Center for US Transportation Command.

If the Operations Center were able to monitor the available inventory at each demand node it serves, this problem evaporates. Knowing the relative importance of each demand node’s mission and the resources needed to accomplish them would allow the Operations Center, in a transparent way, to determine whose supply requests should be given priority. The Operations Center, however, tracks what is delivered not what is consumed (see Peltz and Robbins (2012)).

The other component of the problem is the incentives of units at the tactical edge such as an individual ship (which we will call demand nodes). When a demand node is assigned a mission there is a discussion about the resources that will be required for the mission. The demand node is also allocated a dollar budget to cover the cost of the needed resources.

The military recognizes that mission success is not just a function of ‘own’ effort but circumstances beyond ‘own’ control. In combat settings, for example, the effort of the enemy is relevant as well. However, ensuring readiness is considered to be the responsibility of the relevant agent. If failure can be attributed to a lack of readiness, then, the demand node is held responsible for mission failure. Therefore, each demand node is focused on making sure that the resources it has on hand are at the level needed to ensure readiness.

Hence, we take as given that the objective is to maximize overall ‘readiness’ (to be defined later) subject to limited supply (because of logistic constraints) and the fact that information about readiness is both dispersed and privately held. To overcome the last, informational constraint, we need to incentivize the agents to reveal how truly urgent their needs are. We do that by presenting them with a trade-off: how much are you willing to give up in the future in return for getting what you want in the present? This trade-off arises in the context of a

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3On large bases, knowing an item is in stock is not enough to find it. In this case, it is not unusual for someone to just order the item again rather than hunt for it.
pseudo-market to allocate priorities. Pseudo-markets use artificial currency and competitive equilibrium allocations to guide the allocation of a collectively owned endowment of goods. They are used in settings where monetary transfers are impermissible or where the agents have objectives (such as readiness) that cannot be measured in monetary terms, see Parlier (2005). Both apply in the context of the US Military.

A budget of artificial currency forces each agent to ask and answer the trade-off question posed above.\(^4\) We examine the plausibility of using a pseudo-market mechanism to allocate delivery priorities within the context of a stylized model. Our focus is on whether such a mechanism gives agents the incentives to reveal the information necessary to determine the urgency of their needs are. The idea is not alien in the Military context. One of the recommendations in Peltz and Robbins (2012) is the use of monetary credits to encourage the returns of excess but serviceable inventory from the field.\(^5\)

The problem of prioritizing supply orders is long-standing and one might expect the Military to have generated, endogenously, informal market mechanisms to accomplish the same ends. We discuss the difficulties of three other possibilities.

1. Why not use real money?

The military does allocate actual dollars to demand nodes which they use to pay for the cost of supplies. These dollars come from a budget set by Congress. Why not use the dollars to pay for priority? One could, but this only shifts the incentive problem to the planning stage. Now, when a demand node negotiates the dollars need to support its mission it must incorporate the actual cost of goods as well as a budget to prioritize its deliveries when and if needed. It must do this well in advance of what actual conditions will be.

2. Why not let demand nodes engage in barter with each other?

Presently this is illegal, although it does happen, it raises new problems. A number

\(^4\)The crucial information needed is the marginal rate of substitution between present consumption and future consumption which is what budgets provide.

\(^5\)That report proposed setting the credit terms exogenously rather than through a pseudo-market.
of the supplies, like food, are perishable and it is hard to monitor the quality of materials that are exchanged in this way. Second, such exchanges consume scarce logistic resources that must be priced in.

3. Why not rely on long-term relationships and favor exchanges?

While some of this does happen, it is not viewed as robust or sustainable because of the frequency with which personnel is rotated in and out of various positions.

This paper makes four contributions. The first is a market model in which the agents are focused on maximizing readiness. The model of readiness we use is inspired by the classical attrition model of Lanchester (1956). Second, our market model has the property that a competitive equilibrium outcome can be computed using a simple and intuitive tatonment algorithm. Third, we show that the competitive equilibrium outcome compares favorably with a variety of benchmark ‘planners’ outcomes. Finally, we show that agents in this economy are discouraged from misreporting their preferences.

Section 2 of this paper introduces the model and the formalization of readiness that will be employed in the analysis. Section 3 discusses the competitive equilibria of a pseudo-market based on this formalization of readiness. Section 4 introduces different readiness benchmarks and analyzes the efficiency of our mechanism. The subsequent section discusses the incentive properties of this pseudo-market both analytically and via simulation. Section 6 concludes with a brief discussion of the main implementation challenges that such a pseudo-market would impose.

1.1 Related work

This paper is related to the extensive literature on models for managing military logistics systems. Washburn (1994) provides an overview of this literature. The focus is on developing tractable optimization models for managing logistics operations. Bertsimas et al. (2019) is a recent representative example. It develops an optimization model to decide how requirements
(passengers and cargo) will be assigned to the available aircraft fleet and the sequence of pickups and dropoffs that each aircraft will perform to ensure that the requirements are delivered with minimal delay and with maximum utilization of the available aircraft. These papers do not account for the possibility that the information about needs and urgencies is both dispersed and privately held. This, however, is the focus of our paper.

Given this, our paper is related to the literature on pseudo-markets. They have been implemented to allocate courses in colleges (Yekta (2018)), computing resources (Altmann et al. (2008)), and food to food banks (Prendergast (2017)). In the first two examples, there is a planner who wishes to allocate a collectively owned asset. In course allocation, for example, it is generally true that no one student has a greater claim than another for a seat in a particular course. The same is true of participants who share a common data center. Furthermore, the planner is indifferent to who gets which resource. Their goal is to match the resources to the needs of the agent in an equitable way. In the third example, the resources are not collectively owned and there is no planner. Each food bank operates independently of the other and controls the donated food it receives. The goal of the pseudo-market is to replicate what an omniscient central planner might do, i.e., reallocate resources to reduce waste, i.e., swap an excess of fruit in one location with a surfeit of potatoes in another.

The application under consideration here differs from the first two examples in important ways. While there is a planner, it has preferences over who receives what and when. These are reflected in the fact that resources are committed in advance to the participants. The problem is to reallocate these resources in the light of changes in the underlying environment. Our setting differs from the food bank example in that agents cannot directly trade resources that are in their physical possession. They can only trade future deliveries.

There is also a recent stream of work that seeks to minimize the difference between what is achievable with money and without. Balseiro et al. (2019) and Gorokh et al. (2021) are representative examples. However, the settings considered do not apply here. In those papers, agents consume what they are given in each period and their preferences change over
Finally, this paper is related to the literature on computing general equilibria in exchange economies. For a survey, see Codenotti et al. (2004). Our contribution is a new class of exchange economies in which preferences satisfy gross substitutes. This implies the existence of an intuitive price adjustment procedure that converges to an equilibrium outcome.

2 Model

This section introduces and justifies the way we will model readiness. To abstract away from the problems associated with scheduling the delivery of supplies we assume the Planner decides on an initial delivery schedule (quantities, routes, mode of transportation, etc) over a planning horizon of \( T \). This determines the endowment of each agent. For analytical tractability, we assume a single divisible good is being allocated and we normalize its supply in each period to be one unit. We ignore additional constraints that will be present due to the capacity constraints associated with vehicles used to transport supplies. Nothing of conceptual importance changes with their inclusion.

As time unfolds, actual consumption is realized which may depart from the predicted consumption used to determine the endowment. We model this as a one-time shock to consumption at the start of the planning horizon. Thereafter consumption follows a predictable path until the end of the horizon.\(^6\) The question is how to adapt the original delivery schedule to account for these departures. One could allow the agents to trade the supplies they already have in their possession between themselves but that is not considered here. One, as noted above, it is not permitted (although it may happen). Two, the transportation system is configured for ‘drop-off’ only and not drop-off and pickup. Instead, we will allow them to trade that portion of their endowment they have not yet received, i.e., they can postpone current delivery to the future or borrow against future deliveries. In this context, prices are simply an accounting device to record exchange rates across time periods, i.e., a unit of

\(^6\)Section B contains a discussion of the issues raised by relaxing this assumption.
supply delivered today is equivalent to two units delivered tomorrow. In this way, we avoid complications faced in the food bank application (Prendergast (2017)): introduction of an exogenous artificial currency whose supply needs to be managed.

We wish to know if a market mechanism based on prices for goods in each period until the end of the planning horizon, will encourage the truthful revelation of each agent’s readiness. The conceptual challenge will be to identify a model of preferences that captures the notion of readiness.

### 2.1 Preferences

We propose to treat the amount of the resource available to an agent in each period as a measure of their readiness by relating it to the probability of the agent ‘surviving’ to the next period.

Let \( N \) denote the set of agents and denote by \( I_i \) agent \( i \)’s ‘target’ level of inventory in each period. The target inventory level represents the inventory needed at the start of each period for an agent to be considered in full readiness. In reality, this is decided on in advance (in meetings between the agent and the planner). The agent’s incentives are to maintain inventory close to the target level. Recall, mission failure that can be attributed to insufficient resources at hand will be punished. Hence, the farther below the target level an agent’s current inventory, the more impatient they will be.

Let \( x_{it} \) be the number of units of the resource delivered to agent \( i \) in period \( t = 1, \ldots, T \), where \( T \) is the duration of the planning horizon. In the absence of any shocks, we assume that the consumption of resources is perfectly predictable. Thus, one should think of \( T \) as being chosen so that the probability of an unanticipated shock in the interval \([1, T]\) is negligible.

We think of period 1 as beginning with an unanticipated’ shock that reduces resources to a level below what was planned for. Hence, agent \( i \) has a shortfall of \( s_i \) at the beginning of period 1, so that her available inventory at the start of that period is \( I_i - s_i \), which we
denote $x_{i0}$. This information is known only to agent $i$.

2.1.1 Attrition Model of Consumption

We assume that resources are depleted by the same fixed percentage each period. Let $c_i$ be agent $i$’s conservation rate. Hence if agent $i$ has $r$ units of the resource entering a period, it will have $c_ir$ units at the end of it. The term $1 - c_i$ would be an attrition rate. Attrition rates are commonly used to model the depletion of personnel and equipment such as tanks in combat. Their use was inspired by the Lanchester model of combat (Lanchester (1956)). A feature of the model is that the amount of a resource consumed increases with the amount supplied. If $d$ units of supply are allocated to agent $i$ at the beginning of the period, $(1 - c_i)d$ units will be consumed by the end of the period. This is reasonable in contexts where the agent chooses actions based on their currently available resources. As their resources increase, they choose to simply do more, so increasing consumption.

One can allow a time-dependent conservation rate $c_{it}$, where $t$ denotes the time period. For economy of notation only, we choose not to allow it. All the claims made in this section and Section 3 would still hold true with time-dependent conservation rates. In Section 4, which summarizes some simulations of the model, we incorporate time-dependent conservation rates.

Suppose $x_{it}$ units arrive in period $t = 1, \ldots, T$. In period 1 there will be $c_ix_{i0} + x_{i1}$ units in inventory. In period 2, there will be $c_i(c_ix_{i0} + x_{i1}) + x_{i2}$ etc. Therefore, by period $t$ there will be

$$c_i^t x_{i0} + \sum_{j=1}^{t} c_i^{t-j} x_{ij}$$

units in inventory. We treat the fraction of the target inventory level in period $t$ as the probability of surviving to period $t + 1$, given one has survived periods $1, \ldots, t_1$. Notice that $Pr(\text{surviving all periods}) = \prod_{t=1}^{T} Pr(\text{surviving period } t | \text{survived } 1, \ldots, t - 1)$, the probability
of surviving all $T$ periods is

$$
\Pi_t^T = \left( c_i^t \frac{x_{i0}}{I_i} + \sum_{j=1}^{t} c_i^{t-j} \frac{x_{ij}}{I_i} \right).
$$

Consider the log function, which suggests a period $t$ 'utility' or readiness of

$$
\log\left[ c_i^t \frac{x_{i0}}{I_i} + \sum_{j=1}^{t} c_i^{t-j} \frac{x_{ij}}{I_i} \right].
$$

Hence, the utility of agent $i$ for a vector $(x_{i1}, \ldots, x_{iT})$ will be

$$
u_i(x_{i1}, x_{i2}, \ldots, x_{iT}) = \sum_{t=1}^{T} \log\left[ c_i^t \frac{x_{i0}}{I_i} + \sum_{j=1}^{t} c_i^{t-j} \frac{x_{ij}}{I_i} \right] = \sum_{t=1}^{T} \log\left( \sum_{j=0}^{t} c_i^{t-j} x_{ij} \right) - T \log I_i.
$$

Observe the utility function is strictly concave.

In a pseudo-market each agent $i$ will be assigned a budget $b_i$ of money and prices, $p_t$ for each period $t$ will be set for the sale or purchase of the resource. Given prices, each agent will choose the quantity to be delivered in each period to maximize their utility subject to their budget constraint.

Given the budget $b_i$, and a price vector $(p_1, \ldots, p_T)$, the choice problem for agent $i$ is

$$
\max \sum_{t=1}^{T} \log\left( \sum_{j=0}^{t} c_i^{t-j} x_{ij} \right) - T \log I_i
$$

s.t. $c_i^t x_{i0} + \sum_{j=1}^{t} c_i^{t-j} x_{ij} \leq I_i \ \forall t$

$$
\sum_{t=1}^{T} p_t x_{it} \leq b_i \ \forall t
$$

$$
x_{it} \geq 0 \ \forall t
$$

The first constraint in this optimization problem does not permit an agent to hold more than their target level of inventory. In practice agents can and do hold more than their target level of inventory. For this reason we will dispense with this constraint. Concavity of
the utility function being optimized shows that in an optimal solution one is unlikely to violate this constraint.\textsuperscript{7}

Therefore, the budget problem for agent $i \in N$ becomes:

$$\max_{\sum_{t}^{T}} \sum_{t=1}^{T} \log \left[ \sum_{j=0}^{t} c_{i}^{t-j} x_{ij} \right] - T \log I_{i}$$

(1)

s.t. $\sum_{t=1}^{T} p_{i} x_{it} \leq b_{i}$

(2)

$x_{it} \geq 0 \forall t$

(3)

It is straightforward to see that this optimization problem is well defined.

We illustrate, numerically, how an agent’s readiness or utility varies with its conservation rate and initial inventory. These are based on a constant conservation rate. Figure 1 illustrates how readiness varies with initial inventory for various conservation rates. These figures were generated under the assumption that the target inventory level is 1 and the agent receives in each period an amount necessary to maintain its target level of inventory. As apparent, agents with a high conservation rate are more sensitive to the initial level of

Figure 1: Readiness - Initial Inventory

\textsuperscript{7}There is also a technical benefit from ignoring this constraint. Its presence means that we violate the non-satiation assumption often invoked to prove the existence of a competitive equilibrium.
inventory. This is because because a high conservation rate means that current choices, as well as the initial inventory have a larger long term impact. To see how an agent’s purchasing behavior changes with their initial inventory level and conservation rate, we suppose a constant price per period.

![Figure 2: Pricing: Demand - Initial Inventory](image)

Figure 2 illustrates that agents with smaller initial inventory prefer to accelerate delivery, while those with a larger initial inventory slightly prefer to delay delivery.

### 3 Market Mechanism

In this section we describe the proposed market mechanism for allocating resources over time.

Each agent \( i \in N \) is endowed with a fraction \( \alpha_i \) of available resources in each period.\(^8\)

Given a price vector \( p = (p_1, \ldots, p_T) \), the ‘wealth’ or budget of agent \( i \in N \) is \( \alpha_i \sum_{t=1}^{T} p_t \).

\(^8\)If we allowed time dependent conservation rates, to be consistent we should allow the promised fraction to vary with time as well.
The mechanism’s goal is to determine a competitive equilibrium of the exchange economy just described. A price vector \( p = (p_1, \ldots, p_T) \) and an allocation \( \{x_{it}\}_{t=1}^T \) denoted \( (p, x) \) is a competitive equilibrium of the economy if the following are true:

- Each agent is allocated her most preferred bundle subject to her budget constraint, i.e., \( x_i(p) = \arg\max_{p y \leq \alpha_i p \cdot 1} u_i(y) \).
- The market clears, i.e., \( \sum_{i \in N} x_i = 1 \) for all \( t = 1, \ldots, T \).

We propose a mechanism \( \mathcal{M} \) which outputs a competitive equilibrium allocation under the assumption that the \( c_i \)'s for all \( i \in N \) are common knowledge.

**Definition 3.1.** Mechanism \( \mathcal{M} \) proceeds in two steps:

1. Each agent \( i \) reports her initial inventory \( x_i^0 \). Given \( c_i \) is known, this parameter (assuming truthful reporting) reveals the agent’s demand in each period as a function of the entire vector of prices.

2. The mechanism computes a competitive equilibrium with respect to the reported demand.

Under the attrition model of preferences, standard results (e.g. Arrow and Debreu (1954)) imply that the set of competitive equilibrium allocations is non-empty and Pareto Optimal. Competitive equilibrium allocations also satisfy a certain fairness condition: no two agents with the same endowment will envy each others equilibrium allocation. We now give a non-linear inequality formulation of the set of competitive equilibria under the attrition model. We use it in the subsequent simulations to compute competitive equilibrium allocations.

**Theorem 3.1.** A price vector \( p \) and allocation \( x \) form a competitive equilibrium if and only if:

\[
\sum_{r=t}^{T} \sum_{j=0}^{r} c_{r-j} x_{ij} - \frac{p_t}{\alpha_i} \sum_{j=1}^{T} p_j \sum_{s} x_{is} \sum_{r=s}^{T} \sum_{j=0}^{r} c_{r-j} x_{ij} \leq 0 \ \forall i, t
\]

\[
\sum_{i} x_{it} \leq 1 \ \forall t
\]
\[ p_t, x_{it} \geq 0 \ \forall i, t \]

**Proof.** Recall the budget problem for agent \( i \in N \):

\[
\begin{align*}
\max_{t=1}^{T} & \sum_{t=1}^{T} \log \left( \sum_{j=0}^{t} c_{i}^{t-j} x_{ij} \right) - T \log I_i \\
\text{s.t.} & \sum_{t=1}^{T} p_t x_{it} \leq b_i \\
& x_{it} \geq 0 \ \forall t
\end{align*}
\]

The budget of agent \( i \) depends upon her endowment, i.e., \( b_i = \alpha_i \sum_{t=1}^{T} p_t \).

Let \( \lambda^i \) be the Lagrange multiplier associated with the budget constraint and \( \nu^i_t \) the multipliers associated with the non-negativity constraints. Agent \( i \)'s first order conditions for optimality (which are both necessary and sufficient) will be:

\[
\sum_{r=t}^{T} \frac{c_{i}^{r-t}}{\sum_{j=0}^{r} c_{i}^{r-j} x_{ij}} + \nu^i_t - \lambda^i p_t = 0 \ \forall t \geq 1.
\]

Multiplying both sides by \( x_{it} \) and invoking complementary slackness yields:

\[
x_{it} \sum_{r=t}^{T} \frac{c_{i}^{r-t}}{\sum_{j=0}^{r} c_{i}^{r-j} x_{ij}} - \lambda^i p_t x_{it} = 0.
\]

Summing over \( t \) and using the fact that the budget constraint will bind at optimality:

\[
\sum_{i} x_{it} \sum_{r=t}^{T} \frac{c_{i}^{r-t}}{\sum_{j=0}^{r} c_{i}^{r-j} x_{ij}} - \lambda^i b = 0.
\]

Now, we write down a system of equations that characterizes a competitive equilibrium:

\[
\sum_{r=t}^{T} \frac{c_{i}^{r-t}}{\sum_{j=0}^{r} c_{i}^{r-j} x_{ij}} - \lambda^i p_t \leq 0 \ \forall i, t
\]
We can eliminate the $\lambda^i$ terms and substitute in the expression for $b_i$ to yield the conclusion:

$$\sum_{i} x_{it} \leq 1 \forall t$$

$$p_t, x_{it} \geq 0 \forall i, t$$

For larger scale problems, determining a solution to this non-linear system will be impractical. However, we show next, that competitive equilibrium allocations can be computed by a straightforward tâtonnement algorithm.

Formally, the problem of computing a competitive equilibrium of an exchange economy belongs to the class PPAD- complete (see Codenotti et al. (2008)). Under some conditions a competitive equilibrium can be determined using a tâtonnement algorithm which is similar to gradient descent and easy to implement. In a tâtonnement algorithm a price vector is announced and demands computed. If the demand for a good exceeds its supply, the price of that good is increased by a small amount. If the demand for a good is strictly below its supply, its price is decreased slightly. The algorithm terminates once the demand matches supply for each good. Tâtonnement algorithm are implemented in auction markets where the auctioneer adjusts prices according to the algorithm and contracts are executed when the algorithm terminates.

A well known sufficient condition on preferences to guarantee the convergence of a
tatônement is (weak) gross substitutes (see Arrow and Hurwicz (1960)). To define it, let \( x_i(p) = (x_{i1}(p), \ldots, x_{iT}(p)) \) denote the optimal solution to the budget problem (1-3) when the price vector is \( p \). As the budget depends on \( p \), it does not appear in this notation. Gross substitutes means that for each \( t = 1, \ldots, T \) that \( \frac{\partial x_{it}(p)}{\partial p_t} < 0 \) and \( \frac{\partial x_{ij}(p)}{\partial p_t} \geq 0 \) for all \( j \neq t \). In words, if we increase \( p_t \), the demand for delivery in period \( t \) will fall while the demand in other periods will either stay the same or increase. In fact, Codenotti et al. (2005) show that under gross substitutes a discrete price tatônement converges to an approximate equilibrium in polynomial time.

We show that preferences in the attrition model satisfy the gross substitutes property.

**Lemma 3.2.** The utility function of the attrition model satisfies the gross substitutes property. That is, for all \( t = 1, \ldots, T \), \( \frac{\partial x_{it}(p)}{\partial p_t} < 0 \) and \( \frac{\partial x_{ij}(p)}{\partial p_t} \geq 0 \) for all \( j \neq t \).

**Proof.** As \( x_{it}(p) \geq 0 \) for all \( t \), the first order optimality condition is as follows:

\[
\sum_{t=1}^{T} \frac{c_i^{T-t}}{\sum_{j=0}^{T} c_i^{T-j} x_{ij}(p)} = \lambda^t p_t - \mu_t \quad \forall t \geq 1.
\]

Where \( \mu_t = 0 \) for \( x_{it}(p) > 0 \) and \( \mu_t \geq 0 \) otherwise.

Let \( \hat{p}_t = p_t - \mu_t/\lambda^t \) (observe that \( \lambda^t > 0 \) as the budget constraint will bind). For \( t = T \) we get

\[
\frac{1}{\sum_{j=0}^{T} c_i^{T-j} x_{ij}(p)} = \lambda^T \hat{p}_T.
\]

For \( t = T - 1 \) we get

\[
\frac{1}{\sum_{j=0}^{T-1} c_i^{T-1-j} x_{ij}(p)} + \frac{c_i}{\sum_{j=0}^{T} c_i^{T-j} x_{ij}(p)} = \lambda^T \hat{p}_{T-1}.
\]

Scaling equation (4) by \( c_i \) and subtracting it from equation (5) produces:

\[
\frac{1}{\sum_{j=0}^{T-1} c_i^{T-1-j} x_{ij}(p)} = \lambda^T (p_{T-1} - c_i \hat{p}_T).
\]
More generally,
\[
\frac{1}{\sum_{j=0}^{T-k} c_i^{T-k-j} x_{ij}(p)} = \lambda^i (p_{T-k} - c_i p_{T-k+1}).
\]
Substituting \( k = T - 1 \) yields \( x_{i1}(p) = \frac{1}{\lambda^i (\hat{p}_1 - c_i \hat{p}_2)} - c_i x_{i0} \). Proceeding inductively in this way we deduce that:
\[
x_{it}(p) = \frac{1}{\lambda^i (\hat{p}_t - c_i \hat{p}_{t+1})} - \frac{c_i}{\lambda^i (\hat{p}_{t-1} - c_i \hat{p}_t)} - c_i^t x_{i0} \quad \forall t = 1, \ldots, T. \quad (6)
\]

By assumption the left hand side of (6) is positive. As \( \lambda^i \) is the Lagrange multiplier of the binding budget constraint, it follows that \( \lambda^i > 0 \). Hence, \( \frac{1}{(\hat{p}_1 - c_i \hat{p}_2)} - \frac{c_i}{(\hat{p}_{t-1} - c_i \hat{p}_t)} > 0 \).
Therefore, for \( t = 1 \) we get that \( \hat{p}_1 - c_i \hat{p}_2 > 0 \). For \( t = 2 \) we get
\[
\frac{1}{(\hat{p}_2 - c_i \hat{p}_3)} - \frac{c_i}{(\hat{p}_1 - c_i \hat{p}_2)} > 0 \Rightarrow \frac{1}{(\hat{p}_2 - c_i \hat{p}_3)} > 0.
\]
In general, \( \hat{p}_t \geq c_i \hat{p}_{t+1} \) for all \( t = 1, \ldots T - 1 \). Here we have to make an important remark.
If \( x_{it}(p) = 0 \) then, if we increase \( p_t \) and obtain a new price vector \( p' \), the optimum demand, \( x(p') \) will be unchanged, i.e. \( x(p) = x(p') \). The proof is straightforward. If we increase \( p_t \) by \( \epsilon \), by increasing \( \mu_t \) by \( \lambda^i \epsilon \) (keeping the rest of the dual variables unchanged) the KKT conditions will still hold for \( x(p) \) hence it will stay optimal. In other words if demand for a time period is zero then increasing its price further will not change the demand in any other time period. By complementary slackness \( b_i = \sum_{t=1}^{T} p_t x_{it}(p) = \sum_{t=1}^{T} \hat{p}_t x_{it}(p) \). Therefore,
\[
b_i = \sum_{t=1}^{T} \hat{p}_t x_{it}(p) = \sum_{t=1}^{T} \left[ \frac{\hat{p}_t}{\lambda^i (\hat{p}_t - c_i \hat{p}_{t+1})} - \frac{c_i \hat{p}_t}{\lambda^i (\hat{p}_{t-1} - c_i \hat{p}_t)} \right] - x_{i0} \sum_{t=1}^{T} c_i^t \hat{p}_t = \frac{T}{\lambda^i} - x_{i0} \sum_{t=1}^{T} c_i^t \hat{p}_t.
\]
\[
\Rightarrow \frac{1}{\lambda^i} = \frac{x_{i0} \sum_{j=1}^{T} c_i^j \hat{p}_j}{T}.
\]
Hence,
\[
x_{it}(p) = \frac{b_i + x_{i0} \sum_{j=1}^{T} c_i^j \hat{p}_j}{T} \left( \frac{1}{(\hat{p}_t - c_i \hat{p}_{t+1})} - \frac{c_i}{(\hat{p}_{t-1} - c_i \hat{p}_t)} \right) - c_i^t x_{i0}
\]
17
\[
\frac{\partial x_{it}(p)}{\partial \hat{p}_t} = \frac{\alpha_i + x_{i0}c_i^t}{T} \frac{1}{(\hat{p}_t - c_i\hat{p}_{t+1})} - \frac{c_i}{(p_{t-1} - c_i\hat{p}_t)} = \frac{\sum_{j=1}^{T} (\alpha_i + c_i^j x_{i0}) \hat{p}_j}{T} \left( \frac{1}{(\hat{p}_t - c_i\hat{p}_{t+1})} - \frac{c_i}{(p_{t-1} - c_i\hat{p}_t)} \right) - c_i^t x_{i0} \forall t.
\]

For any \( j \neq t \) it is easy to see that \( x_{ij}(p) \) is non-decreasing in \( p_t \) (if \( x_{it}(p) = 0 \) it will stay the same as explained earlier). Now,

\[
\frac{\partial x_{it}(p)}{\partial \hat{p}_t} = \frac{\alpha_i + x_{i0}c_i^t}{T} \frac{1}{(\hat{p}_t - c_i\hat{p}_{t+1})} - \frac{c_i}{(p_{t-1} - c_i\hat{p}_t)} = \frac{\sum_{j=1}^{T} (\alpha_i + c_i^j x_{i0}) \hat{p}_j}{T} \left( \frac{1}{(\hat{p}_t - c_i\hat{p}_{t+1})} - \frac{c_i}{(p_{t-1} - c_i\hat{p}_t)} \right)
\]

The term \( \frac{1}{(\hat{p}_t - c_i\hat{p}_{t+1})} - \frac{c_i}{(p_{t-1} - c_i\hat{p}_t)} \) is positive. To complete the proof it suffices to show that

\[
\frac{\alpha_i + x_{i0}c_i^t}{T} - \frac{\sum_{j=1}^{T} (\alpha_i + c_i^j x_{i0}) \hat{p}_j}{T} \left( \frac{1}{(\hat{p}_t - c_i\hat{p}_{t+1})} + \frac{c_i}{(p_{t-1} - c_i\hat{p}_t)} \right) < 0. \tag{7}
\]

Given \( \hat{p}_j \geq c_i p_{j+1} \) for \( j = 1, \ldots, T - 1 \) it follows that \( \hat{p}_t \left( \frac{1}{(\hat{p}_t - c_i\hat{p}_{t+1})} + \frac{c_i}{(p_{t-1} - c_i\hat{p}_t)} \right) > 1 \). Thus,

\[
\frac{\sum_{j=1}^{T} (\alpha_i + c_i^j x_{i0}) \hat{p}_j}{T} \left( \frac{1}{(\hat{p}_t - c_i\hat{p}_{t+1})} + \frac{c_i}{(p_{t-1} - c_i\hat{p}_t)} \right) > \frac{(\alpha_i + c_i^t x_{i0})}{T}.
\]

Inequality (7) now follows. \( \square \)

Using Lemma 3.2 and Arrow and Hurwicz (1960), we obtain the following result.

**Theorem 3.3.** In the attrition model, the tâtonnement algorithm converges to a competitive equilibrium.

## 4 Benchmark for Readiness & Simulations

Our readiness interpretation of preferences is useful for providing a benchmark against which to evaluate competitive equilibrium outcomes. If \( x_i \) is the vector of deliveries to agent \( i \) with
utility function $u_i(x_i)$, then, recall, that $e^{u_i(x_i)}$ can be interpreted as the readiness of agent $i$. If the planner has perfect information about initial inventories and conservation rates it is natural to think they would choose the $x_is$ so as to maximize the total readiness, i.e., $\sum_{i \in N} e^{u_i(x_i)}$, of all agents. As readiness in our formulation corresponds to the probability of survival over the horizon, summing these corresponds to the expected number of agents who survive until the end of the horizon. We propose three different benchmarks based on this idea for planning the supply of agents. Unlike our stylized model, we allow the conservation rate to vary over time to check the robustness of the qualitative insights obtained from the case of a constant conservation rate. Let $c_{it}$ denote the conservation rate of agent $i$ in period $t$. We will use the conservation rates of an agent to determine the share $\alpha_{it}$ of the available supply it is entitled to in each period. Specifically, $\alpha_{it} = 1 - c_{it}$.

**Benchmark 1:** Here the planner assumes that all agents have their target level of inventory at the beginning of the planning horizon and delivers to each agent $i$, $\alpha_{it} = 1 - c_{it}$ units of supply in each period. We can interpret this as the planner ignoring the agent’s initial inventory and blindly following the ‘forecast’. Benchmark 1 is obviously incentive compatible.

**Benchmark 2:** Here we assume the planner has full information about each agent’s initial inventory and time dependent conservation rates, and allocates the supply in order to maximize the overall readiness. The only constraint is that the available supply cannot be shifted across time periods. In each period there is at most one unit of supply. The planner’s optimization problem is as follows:

$$\max \sum_{i \in N} e^{u_i(x_i)}$$

s.t. $\sum_{i \in N} x_{it} \leq 1 \ \forall t = 1, \ldots, T$ \hspace{1cm} (B2)

$$x_i \geq 0 \ \forall i \in N.$$
Benchmark 3: Benchmark 2 does not guarantee any one agent that they will be replenished. This benchmark modifies benchmark 2 to ensure that each agent $i$ is entitled to a fraction $\sum_{t=1}^{T} \alpha_{it}$ of the total supply over all periods. The planner’s problem can be stated as follows:

$$\text{max } \sum_{i \in N} e^{u_i(x_i)}$$

s.t. $\sum_{i \in N} x_{it} \leq 1 \quad \forall t = 1, \ldots, T$

$$\sum_{t=1}^{T} x_{it} \geq \sum_{t=1}^{T} \alpha_{it} \forall i \in N$$

$$x \geq 0.$$

(B3)

The allocations produced under benchmarks 2 and 3 are the solutions to a concave programming problem. By the properties of such programs we know that the solutions essentially ‘follow the gradient’, i.e., favor agents with high marginal utilities for supply. The marginal utility for supply delivered at period $t$ to agent $i$ (under a constant conservation rate for convenience) is:

$$\frac{\partial u_i}{\partial x_{ik}} = \sum_{t=k}^{T} \frac{c_i^{t-k}}{\sum_{j=0}^{t-k} c_i^{t-j} x_{ij}}.$$  (8)

The right hand side of equation (8) increases as $x_{i0}$ decreases. This is true for all $i$ and all $t$. Hence, both benchmarks give agents a strict incentive to misreport their initial inventories. Indeed, it is in the incentive of each agent to report that they have zero initial inventory! Thus, when comparing mechanism $M$ against the above benchmarks in terms of overall readiness, we are assuming the agents will truthfully report their initial inventories under benchmarks 2 and 3. Even under this assumption it is not obvious how they compare with each other. Benchmark 3, for example, guarantees that each agent receives at least $\sum_{t=1}^{T} \alpha_{it}$ of the total resources. $M$, however is not so constrained but it is unclear that it will produce an outcome that maximizes total readiness.
4.1 Readiness

For illustrative purposes only, we compare the readiness delivered under benchmark 1 to that of $\mathcal{M}$ on a randomly generated instance consisting of ten agents and six time periods. The initial inventory of each of the ten agents is selected uniformly at random from the interval $[0.2, 0.8]$. The conservation rate of each of the ten agents in each period is selected uniformly at random from the interval $[0.2, 0.8]$. Assuming an agent’s initial inventory was equal to its target level, this choice of $\alpha_{it}$ ensures that each agent is replenished up to their target level in each period. This is consistent with the idea of the $\alpha$s being chosen based on forecasted needs before any shocks are realized.

For these parameter settings, the readiness of each agent lies in $[0, 1]$. A competitive equilibrium was computed using a tatonnement algorithm. Figure 3 illustrates the competitive equilibrium outcome for a single instance.

The panel on the left displays the equilibrium prices in each period. Notice, they decline over time. If all agents prefer early to later delivery, this is what one would expect.

In the right hand panel we see the demands of each of the agents (numbered from 1 to 10 on the horizontal) for each of the 6 periods. The small horizontal black lines indicate each agent’s $\alpha_{it}$. The corresponding conservation rate will be $1 - \alpha_{it}$. As can be seen all agents choose to either delay or expedite their supply instead of receiving a fixed portion $\alpha_{it}$ in each period.

For the same instance we exhibit the delivery plans under benchmarks 2 and 3 in Figure 4. Under benchmark 2, the planner prioritizes some agents over others to boost the overall readiness. This follows from the nature of the optimization problem associated with benchmark 2.

Benchmark 3, recall, guarantees each agent a minimum amount of supply over the entire horizon. The resulting delivery plan is less flexible at redistributing supplies between agents.

---

9This can be thought of as a yearly schedule implemented with bimonthly deliveries.
Figure 3: Demand in Equilibrium

Figure 4: Demand in the Benchmarks

Figure 5 displays the readiness of each agent under each of the three benchmarks on this instance as well as mechanism $\mathcal{M}$. While benchmarks 2 and 3 lead to larger overall readiness, they do so at the expense of certain agents (see agent #2’s readiness under benchmark 2 for example). Mechanism $\mathcal{M}$ cannot achieve the same overall levels of readiness as benchmarks 2 and 3, but it does ensure a more ‘equitable’ distribution of readiness.

Table 1 presents summary statistics on overall readiness over 100 simulations involving 10 agents and six timer periods. In each iteration we measure the readiness gains (in percentage terms) of mechanism $\mathcal{M}$, benchmarks 2 and 3 benchmark over benchmark 1. The first row records the average percentage gain over all 100 simulations. The second and third row reports the minimum and maximum over the same runs. The results are as one might expect. Again, it is important to recall that benchmarks 2 and 3 assume truthful reporting of initial inventory. The allocation returned by benchmark 1 does not depend on the initial
inventories.

<table>
<thead>
<tr>
<th>-</th>
<th>Mechanism $\mathcal{M}$</th>
<th>Benchmark 2</th>
<th>Benchmark 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>2.73%</td>
<td>10.92%</td>
<td>4.02%</td>
</tr>
<tr>
<td>Maximum</td>
<td>5.00%</td>
<td>13.71%</td>
<td>5.97%</td>
</tr>
<tr>
<td>Minimum</td>
<td>1.34%</td>
<td>7.58%</td>
<td>1.84%</td>
</tr>
</tbody>
</table>

Table 1: Estimated Readiness Improvements

5 Incentives

In this section we examine the incentives of agents to misreport their initial inventory levels, $x_{i0}$, for strategic advantage assuming that the conservation rates are common knowledge.

Misreporting one’s initial inventory level corresponds to a misreport of one’s demand as a function of price. We know from Roberts and Postlewaite (1976) that in markets with a large number of agents, no one agent is in a position to significantly influence the competitive equilibrium price by misreporting their demand. However, in this application, it is unreasonable to rely entirely on the large markets assumption. Our examination of the incentives to misreport initial inventory levels will not rely on this assumption. However, for analytical tractability we will restrict attention to the case of two periods, i.e., $T = 2$. See Section 5.1, for simulations over more than two periods.
We will show that the agent with the largest conservation rate is the one with the biggest impact on competitive equilibrium outcomes. This indicates the robustness of a pseudo-market mechanism against misreporting private information because the agent with the largest conservation rate is the most patient of all agents and has the least incentive to misreport.

To begin with we need to determine each agent’s demand at the price vector \((p_1, p_2)\). For economy of notation, we drop the that index \(i\) the identifies an agent. Our ‘generic’ agent has a conservation rate of \(c\) and a utility function denoted

\[
u(x_1, x_2) = \log(cx_0 + x_1) + \log(c^2x_0 + cx_1 + x_2),\]

where \(x_i\) is the amount delivered in period \(i = 1, 2\). The constant term involving the target inventory level has been dropped. The agent is endowed with a share \(\alpha\) of the total supply in each period and has an initial inventory level of \(x_0\). If \(p_i\) is the price in period \(i = 1, 2\), her budget problem is

\[
\max u(x_1, x_2)
\]

\[
\text{s.t. } p_1x_1 + p_2x_2 = \alpha(p_1 + p_2)
\]

\[
x_1, x_2 \geq 0
\]

As is standard we can normalize \(p_2\) to take the value 1 and take \(p_1 = p\).

**Lemma 5.1.** An interior optimal solution \((x_1, x_2)\) to the budget problem satisfies

\[
x_1 = \frac{\alpha(1 + p) - (p - 2c)x_0}{2(p - c)}
\]

and

\[
x_2 = \alpha(1 + p) - px_1.
\]

See Appendix A.1 for the proof.
We use Lemma 5.1 to determine the competitive equilibrium value of $p$. We re-introduce the index $i$. Each agent $i$ has a conservation rate of $c_i$ and their reported initial inventory is $y_i$. Note, $y_i$ need not be equal to their actual initial inventory of $x_{i0}$. The share of supply that agent $i$ is endowed with is $\alpha_i$.

At equilibrium, $p$ must be chosen so that the supply in period 1 equals the demand in period 1. The supply balance constraint for the second period is redundant. Hence,

$$\sum_i \frac{\alpha_i(1 + p) - (p - 2c_i)c_iy_i}{2(p - c_i)} = 1$$

(9)

Using equation (9) we will be able to determine the effect on $p$ from agent $i$, say varying $y_i$. The following straightforward lemmas will do this. See Appendix A.2 for their proofs.

**Lemma 5.2.** Let $f_i(p, y) = \frac{\alpha_i(1+p)-(p-2c_i)c_iy_i}{p-c_i}$. $f_i$ is decreasing in $p$ and decreasing in $y$ when $p \geq 2c_i$.

The equilibrium condition can be restated as $\sum_i f_i(p_1, y_i) = 2$. If agent $i$ lowers the value of $y_i$, then, it is easy to see by Lemma 5.2 that $p$ must increase in order to satisfy the equilibrium condition. This means that when an agent under-reports her initial inventory the equilibrium price goes up. The next theorem identifies the agent with largest price impact when the equilibrium price is sufficiently large.

**Theorem 5.3.** Denote the vector of initial inventories by $y = (y_1, \ldots, y_n)$. Suppose, the corresponding equilibrium period one price, $p \geq \max_{j \in N} 2c_j$. Then, the agent with the largest conservation rate can influence the equilibrium price $p$ by changing the report of her initial inventory the least.

**Proof.** Fix a $\delta > 0$. Let $\epsilon_j$ be the amount by which agent $j$ must lower their report of initial inventory $\epsilon_j$ to increase the equilibrium price by, $\delta$. Therefore, for each agent $j$ the following must hold

$$\sum_{i \in N} \frac{\alpha_i(1 + p + \delta) - (p + \delta - 2c_i)c_iy_i}{2(p + \delta - c)} + \frac{(p - 2c_j)c_j\epsilon_j}{2(p - c)} = 1$$

(10)
The term in the summation is independent of \( \epsilon_j \) and less than 1. Set

\[
t = 1 - \sum_{i \in N} \frac{\alpha_i(1 + p + \delta) - (p + \delta - 2c_i)c_y_i}{2(p + \delta - c_i)} \geq 0.
\]

Then,

\[
\frac{(p - 2c_j)c_j \epsilon_j}{2(p - c_j)} = t \iff \epsilon_j = \frac{2t(p - c_j)}{c_j(p - 2c_j)}.
\]

The expression \( \frac{2t(p-c_j)}{c_j(p-2c_j)} \) is decreasing in \( c_j \) as can be verified by examining the derivative. Therefore, the agent with the largest \( c_i \) needs the smallest under-report \( \epsilon_j \) to raise the equilibrium price by \( \delta \).

Thus, a high equilibrium price is particularly sensitive to the initial inventory of the agent with the largest conservation rate. This is consistent with Figure 1. The agent with the highest conservation rate, has the least incentive to misreport precisely because they don’t rapidly consume their supplies.

### 5.1 Simulations on Incentives

We examine the incentives that an agent might have to misreport their initial inventory under mechanism \( \mathcal{M} \) in a multi-period setting. To do so we recall how \( \mathcal{M} \) will be implemented. Agents don’t ‘trade’ directly. Rather, agent \( i \in N \) submits demand curves \( \{d_{it}(p)\}_{t=1}^T \) that specify their demand in each period as function of the vector of prices. The practicality of this is discussed in Section 6.

We start with a single instance as described in Section 4.1 and vary the conservation rate and initial inventory of agent 1 only. In the matrix of panels exhibited in Figure 6, each row corresponds to a different conservation rate and each column corresponds to a different level of initial inventory. The horizontal axes in each panel corresponds to reported initial inventory while the vertical axes is records utility. So, the the second row and fourth column
correspond to agent 1’s actual conservation rate and initial inventory being set at 0.5 (the pair of numbers in the upper right hand corner of the relevant panel). We then examine the impact on their utility of misreporting their initial inventory level.

![Graphs showing incentives to misreport initial inventory](image)

**Figure 6: Incentives to Misreport Initial Inventory**

Notice, agent 1 does not have an incentive to misreport her initial inventory, since her readiness peaks where the reported initial inventory coincides with her actual initial inventory level.

Next, we consider the impact on readiness from agents misreporting their initial inventory level.
levels. The results are displayed in Table 2. We compute allocations under \( M \), benchmarks 1 and 2 *supposing* the initial inventory of each agent is zero. In other words, all agents report untruthfully that they have no initial inventory. This will affect prices and the subsequent allocations. However, we evaluate their readiness with respect to their *actual* initial inventories. We compare this with readiness under benchmark 1 which *ignores* the initial inventory position of each agent and so does not depend on what the agents report. Mechanism \( M \) is only ‘approximately’ incentive compatible. It discourages misreporting but does not eliminate it entirely.

<table>
<thead>
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<th>Mechanism ( M )</th>
<th>Benchmark 2</th>
<th>Benchmark 3</th>
</tr>
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<tbody>
<tr>
<td>Average</td>
<td>-1.64%</td>
<td>-3.93%</td>
<td>-6.41%</td>
</tr>
<tr>
<td>Maximum</td>
<td>1.52%</td>
<td>7.14%</td>
<td>-0.41%</td>
</tr>
<tr>
<td>Minimum</td>
<td>-5.25%</td>
<td>-11.94%</td>
<td>-12.29%</td>
</tr>
</tbody>
</table>

Table 2: Estimated Readiness Improvements under Misreporting

The outcome of benchmark 1 dominates the others because it generates an allocation that is *independent* of the initial inventories. The interesting observation is that mechanism \( M \) now does better than benchmarks 1 and 2. This is consistent with the idea that mechanism \( M \) discourages misreporting of the initial inventories, i.e., the agents do not gain much from misreporting. This is consistent with Section 5.1. In fact, it can be seen that our assumption that under mechanism \( M \), agents will report an initial inventory of zero is unlikely. Hence, the numbers in table 2, understate the overall readiness that mechanism \( M \) will deliver.

6 Conclusion

First, and most important is that the participating agents have to be convinced that mechanism \( M \) solves a problem they face without introducing new ones. The obvious new problem that may arise is the demands placed on agents in ‘playing’ the mechanism. Importantly agents need to be convinced through experimentation that the gains from strategizing are small. Put differently, ‘gaming’ the mechanism should not become a full-time job.
While mechanism $M$ admits a market metaphor, it does not follow that when implemented, the agents will be actively trading future deliveries. They have better things to do. As in other applications the determination of prices and allocations are determined in the background (see Budish et al. (2017)) after agents have submitted their preferences. The prices serve as a vehicle for explaining why an agent received the allocation they did, i.e., a more preferred allocation was unaffordable given their budget. The main technical challenge is to design a flexible and intuitive interface through which agents can report their preferences. In our stylized model, for example, each agent’s preferences could be described by just two parameters: the initial inventory and the conservation rate. It suffices for the agents to report that to determine equilibrium prices and allocations. Given the range of goods delivered (some consumable and others durable, for example) thought has to be given to what an intuitive way to express own demand would be. If we set $T = 2$ in mechanism $M$ a natural candidate emerges: how much of a particular good is one willing to give up ‘tomorrow’ for an additional unit of the good ‘today’.

In moving to mechanism $M$, the planner loses discretion and the ability to shape outcomes according to their preferences. One might be tempted to allow the planner to intervene in $M$ under ‘special’ circumstances. If the planner does this often enough it will come to resemble the status quo. The pseudo-market will become an appendix that serves no useful function. Planner intervention must be constrained by a commitment that it be used in a transparent and principled way. This can be done by including the planner as an agent with an endowment. The endowment would represent the supply the planner keeps in reserve to allocate (think loan or bailout) to agents when and if it thinks warranted. In particular, how the planner chooses to allocate her endowment can depend upon the relative importance she assigns to the various agents. What is important is that similar to the agents, she is budget constrained. This limits her ability to intervene and forces her to make trade-offs.
Acknowledgements

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References


Appendix

A Missing Proofs

A.1 Proof of Lemma 5.1

Using the first order conditions for optimality from Section 2.1.1 we have:

\[ p = 2c + \frac{x_2}{cx_0 + x_1} \]
\[ px_1 + x_2 = \alpha(p + 1) \]

We substitute \( x_2 \) from the second equation into the first to get

\[ p = 2c + \frac{\alpha(1 + p) - px_1}{cx_0 + x_1} \iff \]
\[ (p - 2c)(cx_0 + x_1) = \alpha(1 + p) - px_1 \iff \]
\[ 2(p - c)x_1 = \alpha(1 + p) - (p - 2c)cx_0 \iff \]
\[ x_1 = \frac{\alpha(1 + p) - (p - 2c)cx_0}{2(p - c)} \]

Therefore, the demand in period 2, \( x_2 \) is equal to \( \alpha(1 + p) - px_1 \).
A.2 Proof of Lemma 5.2

For the first, it suffices to determine the derivative of $f_i$ with respect to $p$:

$$\frac{\partial f_i}{\partial p} = \frac{(p - c_i)[\alpha_i - c_i y] - [\alpha(1 + p) - (p - 2c_i)c_i y]}{(p - c_i)^2}$$

$$= \frac{\alpha p - c_i \alpha_i - c_i yp + c_i^2 y - \alpha_i - \alpha_i p + c_i yp - 2c_i^2 y}{(p - c_i)^2}$$

$$= \frac{-c_i \alpha_i - \alpha_i - c_i^2 y}{(p - c_i)^2}$$

$$\leq 0$$

For the second we determine the derivative with respect to $y$:

$$\frac{\partial f_i(p)}{\partial y} = -\frac{(p - 2c_i)c_i}{p - c_i} \leq 0$$

The last inequality follows from the assumption that $p \geq 2c_i$.

B Uncertainty in Consumption

So as to focus on the problem of truthful revelation of initial inventory, we assumed both the planner and agents could accurately forecast consumption over the planning horizon. One may wonder how robust mechanism $M$ is to errors in these forecasts. After all, prices are set in period 1 for all subsequent periods in the planning horizon. Thus, there is no opportunity to revise prices and allocation as the uncertainty about consumption is resolved.

If the errors in forecasts of future consumption grow with $T$, then, the planner may choose $T$ to be small and break time up into a sequence of blocks of length $T$. Within a $T$-period block we might expect forecast error not to be an issue. However, $T$ small, means that an agent’s demand within the current block will very sensitive to their forecasts about consumption in the next block.

Note however, that the current procedure for allocating resources is subject to the same
forecast error. Mechanism $\mathcal{M}$ gives agents an incentive to reduce forecast error. Simply assuming the worst tomorrow, will only raise demand today so increasing prices.

The textbook answer to the question of how to modify a market to handle future uncertainty is to use contingent pricing (see Arrow (1964)) i.e., prices are conditioned on future circumstances. Roughly, if the realized aggregate demand in the next period is ‘large’, then, the price for delivery in the next period should be high and low otherwise. Thus, prices are allowed to adjust to future circumstances as they are realized.

A simple way to do this is to run a sequence of spot markets period by period. Hence, the period $t$ price is determined in period $t$ rather than in advance. In principle, this allows prices to be determined in ‘real’ time. Below we examine a simple two period model that illustrates both a peril as well as an opportunity in this approach.

### B.1 An example

Assume two periods numbered 1 and 2 as well as two agents called $a$ and $b$. Each agent is endowed with a half a unit of supply in each period.

The amount of supply that the agents will consume in period 2 is uncertain. We model this uncertainty as period 2 being in one of two possible states called $H$ and $L$. One can interpret the possible states as corresponding to different levels of consumption. The probability of state $H$ is denoted $\pi_H$ while the probability of state $L$ is $\pi_L$. The period 2 state $s \in \{H, L\}$ is realized at the end of period 1.

The net amount agent $a$ trades in period $t$ is denoted $x_{at}$. Similarly with agent $b$. Hence, vector of resources agent $a$ is allocated is $(\frac{1}{2} + x_{a1}, \frac{1}{2} + x_{a2})$. Similarly, the vector of resources agent $b$ is allocated is $(\frac{1}{2} + x_{b1}, \frac{1}{2} + x_{b2})$. The utilities of the agents will be state dependent and are given by

$$U_a((\frac{1}{2} + x_{a1}, \frac{1}{2} + x_{a2}), s) = U_a(\frac{1}{2}, \frac{1}{2}, s) + x_{a1} - \frac{(x_{a1})^2}{2} + a_s x_{a2}$$
and
\[ U_b((\frac{1}{2} + x_{b1}, \frac{1}{2} + x_{b2}), s) = U_b((\frac{1}{2}, \frac{1}{2}, s) + x_{b1} - \frac{(x_{b1})^2}{2} + b_s x_{b2}). \]

The parameters \(a_s, b_s\) are state dependent and can be interpreted as arising from different consumption rates in different states. Let \(\bar{a} = \pi_L a_L + \pi_H a_H\) and \(\bar{b} = \pi_L b_L + \pi_H b_H\).

In this setting we will compare the performance of mechanism \(M\) with a mechanism that allows for contingent pricing. In mechanism \(M\) prices for period 1 and 2 are determined in period 1. Hence, the period 2 price does not adjust to the realized state.

The alternative mechanism sets a period 1 price in period 1 before the period 2 state is realized. However, the period 2 price is only determined in period 2 after the period 2 state is realized. In the alternative mechanism, there is a period 1 price and in period 2 there are two possible prices depending upon the realized state denoted \(p_H\) and \(p_L\). The alternative mechanism is clearly more flexible than \(M\). As the example will show, the alternative can produce an outcome that is preferred by all agents over the outcome under \(M\). However, the same example also shows that the reverse can be true. Contingent pricing allows for a wider set of equilibrium outcomes not all of them superior to the equilibrium outcome under \(M\).

Consider first mechanism \(M\). Normalize the first period price to 1 and let \(p\) be the second period price. Agent \(a\) will choose her net trades \(x_{aj}\) for \(j = 1, 2\) to maximize her expected utility:
\[
\max \pi_L U_a((\frac{1}{2} + x_{a1}, \frac{1}{2} + x_{a2}), L) + \pi_H U_a((\frac{1}{2} + x_{a1}, \frac{1}{2} + x_{a2}), H)
\]
\[\text{s.t. } \frac{1}{2} + x_{a1} + p(\frac{1}{2} + x_{a2}) \leq \frac{1+p}{2}\]

As the budget constraint must bind at optimality, it follows that \(x_{a1} = -px_{a2}\). Substituting this into agent \(a\)'s utility function and invoking the first order condition for optimality we deduce that
\[
1 - x_{a1} - \frac{\bar{a}}{p} = 0 \Rightarrow 1 - \frac{\bar{a}}{p} = x_{a1}.
\]
Similarly, for agent \( b \) we have
\[
1 - \frac{b}{p} = x_{b1}.
\]
As \( x_{a1} \) and \( x_{b1} \) are the net trades in period 1, it must be that \( x_{a1} + x_{b1} = 0 \). Hence,
\[
p = \frac{\bar{a} + \bar{b}}{2}.
\]
Therefore,
\[
x_{a1} = \frac{\bar{b} - \bar{a}}{\bar{a} + \bar{b}}; \quad x_{b1} = \frac{\bar{a} - \bar{b}}{\bar{a} + \bar{b}}.
\]
Notice, when \( \bar{a} = \bar{b} \), there is no trade in period 1.

Now let us consider contingent pricing. Normalize the period 1 price to be $1. In period 2, the price will be \( p_L = \) if state \( L \) is realized and \( p_H = rp \) if state \( H \) is realized. Let \( x_{i2}(s) \) be the net-trade of agent \( i \) in period 2 contingent on the state being \( s \). The budget problem for agent \( a \) is:
\[
\max \pi_L \! U_a((\frac{1}{2} + x_{a1}, \frac{1}{2} + x_{a2}(L)), L) + \pi_H \! U_a((\frac{1}{2} + x_{a1}, \frac{1}{2} + x_{a2}(H)), H)
\]
\[
s.t. \quad \frac{1}{2} + x_{a1} + p(\frac{1}{2} + x_{a2}(L)) \leq \frac{1 + p}{2}
\]
\[
\frac{1}{2} + x_{a1} + rp(\frac{1}{2} + x_{a2}(H)) \leq \frac{1 + rp}{2}
\]

Notice, there are two budget constraints, one for each contingency. From the fact that each must bind:
\[
x_{a1} = -x_{a2}(L) \cdot p = -x_{a2}(H) \cdot rp.
\]
Agent \( a \)'s utility will be
\[
\pi_L \! U_a(\frac{1}{2}, \frac{1}{2}, L) + \pi_H \! U_a(\frac{1}{2}, \frac{1}{2}, H) + x_{a1} - \frac{(x_{a1})^2}{2} - a_L \pi_L \frac{x_{a1}}{p} - a_H \pi_H \frac{x_{a1}}{rp}.
\]
The first order condition for optimality implies:

\[ 1 - a_L \frac{\pi_L}{p} - a_H \frac{\pi_H}{rp} = x_{a1}. \]

Let \( \tilde{a} := \pi_L a_L + \frac{1}{r} \pi_H a_H \). Then,

\[ 1 - \frac{\tilde{a}}{p} = x_{a1}. \]

Similarly, for agent \( b \)

\[ 1 - \frac{\tilde{b}}{p} = x_{b1}. \]

From the market clearing condition \( x_{a1} + x_{b1} = 0 \) we deduce that

\[ p = \frac{\tilde{a} + \tilde{b}}{2} \]

\[ x_a = \frac{\tilde{b} - \tilde{a}}{\tilde{b} - \tilde{a}}; \quad x_b = \frac{\tilde{a} - \tilde{b}}{\tilde{b} - \tilde{a}}. \]

Now recall, when \( \tilde{a} = \tilde{b} \), there will be no trade under mechanism \( M \). However, when \( \tilde{a} \neq \tilde{b} \) there will be trade with contingent pricing. Trade is always mutually beneficial to the agents. Thus if if \( r \) is chosen carefully, the outcome under contingent pricing will be preferred by all agents to the outcome under \( M \). However, the reverse is also true. If we \( r \) is such that \( \tilde{a} = \tilde{b} \), but \( \tilde{a} \neq \tilde{b} \), then there will be no trade under contingent pricing but there will be in mechanism \( M \).

### B.2 General model of uncertainty

One can generalize the example above to more than two states to get a sense of the complexity involved. Consider a two period economy, where the states in period 2 are drawn at random from a finite state space \( S = \{s_1, \ldots, s_k\} \). For each \( s \in S \), the utility of agent \( i \) is \( U_i((x_1, x_2), s) \) where \( x_i \) is the amount consumed in period \( i = \{1, 2\} \). The goal of the planner
is to pick a set of prices \( p_1, \ldots, p_k \), one for each state, so as to clear the market in each period. If the state space is a singleton, this setting reduces to that of mechanism \( M \). If \( S \) is not a singleton, the planner can choose ‘multipliers’ \( (r_2, \ldots, r_k) \) rather than prices so that

\[
p_1 = p, p_2 = r_2 p, \ldots, p_k = r_k p.
\]

Thus, state \( s_1 \) is chosen as a benchmark, and prices for all other states are chosen relative to the benchmark. If state \( s_1 \) is labeled as ‘normal’, the other prices can be communicated by saying that under ‘such and such’ a condition the price will be twice that of the price set in the normal state.

Thus, with contingent pricing, the planner has to be actively involved in choosing the multipliers. Importantly, as the example above shows, the final outcome can be very sensitive to the choice of those multipliers.