Allocation with Weak Priorities and General Constraints

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Abstract

With COVID 19 prevalent in the USA and the world, efficient social distance seating became an option for sports venues. Social distance seating allows only a fraction of the capacity of the sports venues. Hence it necessitates reassigning spectators to games. We model this as a resource allocation problem. Its novelty is the combination of three features: complex resource constraints, weak priority rankings over the agents, and ordinal preferences over bundles of resources. We develop a mechanism based on a new concept called *Competitive Stable Equilibrium*. It has several attractive properties, unifies two different frameworks of one-sided and two-sided markets, and extends existing methods to richer environments. Our framework also allows for an alternative and more flexible tie-breaking rule by giving agents different budgets. We empirically apply our mechanism to reassign season tickets to families in the presence of social distancing. Our simulation results show that our method outperforms existing ones in both efficiency and fairness measures.

1 Introduction

Catastrophes, natural or human-made, often trigger reallocation of resources. Draughts and floods disrupt agricultural production and lead to a redistribution of the food supply. A financial crisis causes systemic bankruptcy and forces firms to redistribute their debt payment. The current pandemic is no exception, but it gives rise to a distinct constraint on social distancing. The required distance between people severely reduces the normal capacity of physical spaces. However, unlike the reduction of the food supply after floods, the actual physical spaces remain unchanged. Their new capacity depends *endogenously* on the way they are reallocated.

As an example, consider the reallocation of games to season ticket holders in college football. With COVID 19, six feet between individuals must be considered if the game has live audiences.

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Depending on the seats' dimensions, this would translate to a certain number of empty rows and empty seats in a row between the individuals. However, season tickets are grouped by families, and only a safe distance between two *different* families needs to be maintained. Members of the same family can sit next to each other. Therefore, a large family needs fewer empty seats per person to maintain social distancing. A football season has about six home games. If priority is given to larger families for all the games, then many people can watch the live games, but the outcome will be highly unfair. Striking a good balance between efficiency and fairness is a nontrivial task.

Another challenge is the limited use of money in the reallocation mechanisms. While monetary transfer helps redistribute resources in many markets, it is not practical in this setting. First, it is difficult to elicit fans' preferences in cardinal values. Second, sports clubs aiming to maintain fan support tend to shy away from such a "monetary" solution.[1] Instead, they often have some form of priority over the fans. The priority is captured by the type of membership or club points that fans have collected through the years.

Motivated by this, we consider a resource (re)allocation problem without transfer. The novelty of our work is the combination of three features. First, each resource has a general constraint captured by *any* monotonic function; second, each resource also has a *weak* priority ranking over the agents; and third, the agents are multi-unit demand and have ordinal preferences over bundles of resources.

Our main contribution is an allocation method based on a new concept called *Competitive Stable Equilibrium* (CSE). This is an extension of competitive equilibrium with endowed budgets that accommodates weak priorities. In particular, an agent only needs to pay for a resource if he belongs to the last tier among the agents currently consuming the resource. Furthermore, the price is positive only if the resource constraint binds: a market clearing condition as in a competitive equilibrium.

Thus, if agents are endowed with equal budgets, then a CSE is a stable and envy-free outcome, which is both fair and Pareto optimal when resource constraints are capacity constraints. Moreover, a CSE when agents are given different budget corresponds to a tie-breaking rule among agents of the same tier. Tie-breaking rules can improve efficiency, especially when resource constraints are

[1] In our private discussion with the director of the athletic department at Purdue University, auctions for ticket resale are avoided because they offend fans.

complex. Our method's advantage is that adjusting budgets is a more flexible way of changing the tie-breaking rule. Furthermore, when agents consume a bundle of goods, it allows agents to "distribute" their tie-breaking budget over the different resources. We illustrate this in the application of assigning seats in sports venues and compare our method with other tie-breaking alternatives.

However, CSE need not exist because of two different reasons: income effect of a fixed budget and complementarities of bundled preferences. To overcome them, we allow for an ε approximate solution of the budget as in Budish (2011), and a violation of resource constraints by accommodating few extra agents as in Nguyen et al. (2016). The violation of budget can be arbitrarily small, and the violation of resource constraints depend on the size of the largest bundle that an agent can consume.

Our new allocation method unifies existing frameworks of one-sided and two-sided markets and extends them to a more prosperous environment. In particular, on the one extreme, when the priority of resources over the agents is strict, then our allocation corresponds to the standard notion of stability. On the other extreme, if all agents belong to the same tier, our allocation corresponds to an approximate competitive equilibrium outcome with an equal budget of Budish (2011), extending it to general resource constraints.

Because of its generality, our framework applies to settings beyond sport and entertainment events. For example, even in the context of social distancing, daycare facilities face a similar problem of reallocating slots to children and their siblings to different days of the week. Homeless shelters and refugee camps also need to reallocate families to limit the spread of the virus.

1.1 Related Literature

Our paper connects two different resource allocation frameworks that depend on whether or not the resources have preferences over the agents (two-sided or one-sided markets, respectively). As illustrated in this paper, this distinction is, in fact, blurry because of possible ties in the resources' preferences.[2] Hence, for our purpose, it is more useful to summarize the literature by the methodologies of the allocation mechanisms. There are two main types of allocation mechanisms: the

[2]Furthermore, in settings like kidney exchange or roommate problems, there is no separation of resources and agents.

greedy type and the *equilibrium* type. In a greedy mechanism, the agents obtain resources by choosing their objects in a predetermined order, usually consistent with their preferences. On the other hand, in an equilibrium style mechanism, the allocation is determined based on a competitive equilibrium.

The Deferred Acceptance (D.A) mechanism (Gale and Shapley, 1962) is perhaps the most influential greedy mechanism. It has since inspired Fleiner (2003), Hatfield and Milgrom (2005), Ostrovsky (2008), Hatfield and Kojima (2010) and others that seeks to enlarge domains in which the DA algorithm is applicable. The school choice literature (Abdulkadiroğlu and Sönmez (2003), Erdil and Ergin (2008), Ashlagi and Nikzad (2020) just to name a few) also considers ties in schools' preferences. These papers analyze and compare different tie-breaking rules used in the implementation of D.A. Recently, Kamada and Kojima (2019) also extends D.A to an environment with general resource constraints similar to ours. However, all of the papers above assume agents are single-unit demand. This is simply because the nature of the D.A approach algorithm excludes the possibility of accommodating complementarities in preferences.

If one allows allocations to be randomized, then there are few other options. The Probabilistic Serial (P.S) (Bogomolnaia and Moulin, 2001) is such a mechanism. Another example is Budish et al. (2013), which examine the implementation of randomized allocations for a broad class of resource constraints. With new techniques in approximation algorithms, Nguyen et al. (2016) extend P.S to preferences over bundles, while Ashlagi et al. (2020) and Akbarpour and Nikzad (2020) use linear programming and probability methods to accommodate additional constraints.

The first equilibrium-based allocation mechanism is by Hylland and Zeckhauser (1979), in which agents are endowed with the same budget and obtain a randomized allocation according to the allocation of a competitive equilibrium. To ensure the existence of an equilibrium, an equilibrium-based approach often assumes cardinal preferences and randomization (i.e., von Neumann-Morgenstern utilities). For example, He et al. (2018) use this approach to solve an allocation problem with priorities. Echenique et al. (2019) consider a problem with general constraints. However, these papers maintain von Neumann-Morgenstern utilities, as opposed to our work that assumes the agents have ordinal preferences over bundles of goods, and the allocations are deterministic.

The difficulty of using a competitive equilibrium (CE) without randomization is that a CE need not exist. Budish (2011) is one of the first to propose a solution by perturbing budgets and allowing approximation in resource constraints. Our paper advances this line of work by incorporating priorities and general resource constraints.

Even though our paper follows the theme of Budish (2011), the approach we use is different. Our solution is based on Scarf's lemma, an approach pioneered by Biró and Fleiner (2016); Nguyen and Vohra (2018) and advanced by Nguyen et al. (2019). Compared with these papers, we make two innovations. The first is a significant generalization of Scarf's lemma to incorporate nonlinear resource functions. The second is how we use this new lemma to create a hybrid mechanism that works for both one-sided and two-sided markets.

Our paper is also related to the literature on stable matching with indifference (see Manlove et al. (2014) and citations therein). However, this literature mostly focuses on one-to-one matching and different stability criteria (weak, strong, and super stability). The main questions here are the computational complexity and approximation algorithms for finding matching satisfying these criteria.

In Table 1, we list the most related papers and their properties. These papers are divided by their environment: two-sided, one-sided markets and hybrid ones (models with priorities).

	ordinal preferences	preferences over bundles	general constraints	deterministic outcomes
Gale and Shapley (1962)	✓	✗	✗	✓
Kamada and Kojima (2019)	✓	✗	✓	✓
Nguyen and Vohra (2018)	✓	✓	✗	✓
Nguyen et al. (2019)	✓	✗	✓	✓
Hylland and Zeckhauser (1979)	✗	✗	✗	✗
Bogomolnaia and Moulin (2001)	✓	✗	✗	✗
Budish (2011)	✓	✓	✗	✓
Echenique et al. (2019)	✗	✓	✓	✗
He et al. (2018)	✗	✗	✗	✗
This paper	✓	✓	✓	✓

Table 1: Relationship with prior literature. Papers in the top part of the table are for two-sided markets. The ones in the middle are for one-sided and at the bottom are for models with priorities.

1.2 Organization

The remaining of the paper is organized as follows. Section 2 lays out the notations and the model. Section 3 develops the generalization of Scarf's. Section 4 provides the main mechanism. Section 5 discusses the implementation of our mechanism in the context of sports events. Proofs are given in Appendix.

2 Model

For ease of presentation, we describe the model in the context of sports venues and season ticket holders and families.

Suppose we have m families that purchase football season tickets, and there are n games held in the season. Let F be the set of all the families, and let G be the set of the games. Each family $f \in F$ has b_f members and let $\bar{b}$ be the largest family size. Each family f is endowed with a budget d_f. Each game $g \in G$ can accommodate C_g people prior to social distancing.

A game $g \in G$ has a *weak* ranking $\succeq_g$ over the families.[3] We write $f_1 \succ_g f_2$ if family f_1 is strictly ranked above f_2 and $f_1 =_g f_2$ if they belong to the same priority group of g. As discussed in the introduction, having a weak ranking allows us to model a variety of scenarios. A strict ranking will correspond to the usual setting of two-sided matching. A trivial ranking where all families belong to the same priority group corresponds to the standard one-sided market.

A family can be allocated a bundle of games S, which is a subset of G. A family f has a *strict* ordinal preference, denoted by $\succ_f$, over bundles of games. Let G_f denote the collection of game bundles that f is interested in. We write $S_1 \succ_f S_2$ if family f prefers bundle S_1 to bundle S_2.

Before the social distancing constraint is introduced, each person is endowed with a ticket. Thus, a family f is endowed with b_f tickets. The original capacity constraint is satisfied for each game, which implies

$$\sum_{f \in F} b_f \leq C_g.$$

How does social distancing influence this constraint? Let $x_{f,S} \in \{0, 1\}$ denote a decision variable with $x_{f,S} = 1$ corresponding to assigning family f to game bundle S. Let z_b^g denote the number of

[3]This ranking in many cases corresponds to "club points" of the families. In another case, priority is given to a family of graduating senior students or the type of tickets and membership.

families with size b assigned to the game g. That is,

$$z_b^g := \sum_{\substack{f,S:\ S \in G_f \\ g \in S,\ b_f = b}} x_{f,S}. \tag{1}$$

To illustrate, consider a stadium with a rectangular seating. If families are required to sit s seats far apart in a row, and leaving r rows empty in between, then a family of size b occupies roughly $(b+s)\cdot(r+1)$ seats. Hence, the capacity constraint can be approximated as follow:

$$\sum_{f \in F} \sum_{S \in G_f} x_{f,S} \cdot (b_f + s)(r+1) = \sum_{b=1}^{\bar{b}} (b+s)(r+1) \cdot z_b^g \leq C_g. \tag{2}$$

In practice, depending on the shape of the seating chart and other considerations, the knapsack constraint above might not accurately capture the actual implementation. In other applications, the resources might have multi-dimensional constraints (see Nguyen et al. (2019)). For this reason, for each game $g \in G$, we will consider a general constraint of the form

$$\Omega_g(z_1^g, .., z_{\bar{b}}^g) \leq C_g, \tag{3}$$

where $\Omega_g(.)$ is a non-negative and strictly increasing function on each of the coordinates z_b^g, and $\Omega(\mathbf{0}) = 0$. Our general resource constraints are consistent and similar to Kamada and Kojima (2019) and Echenique et al. (2019). Specifically, Kamada and Kojima (2019) model feasibility by a downward closed set system, and Echenique et al. (2019) consider a constraint set as lower contour polytope.

For ease of notion, we denote $\vec{z}^g := \{z_b^g\}_{b \in [\bar{b}]}$. Without loss of generality, we assume that $\Omega_g(.)$ is defined on vectors of real numbers, $\mathbb{R}_{\geq 0}^{\bar{b}}$, and continuous.[4]

Throughout the paper, we will use the following conventional notations. Given a vector $\vec{v} \in \mathbb{R}^k$ and a set of indices $A \subset \{1, .., k\}$, we will denote $\vec{v}_A$ as the projection of $\vec{v}$ on the set of coordinates A.

[4]This is because otherwise, we can take a natural extension of Ω from $\mathbb{Z}_{\geq 0}^{\bar{b}}$.

Interpretation of the model to other settings

We provide the interpretation of our model to the assignment of children to daycare. Under COVID constraints, part-time daycare services could be a solution to accommodate more families and still maintaining social distancing constraints. To adapt to this application, the games correspond to different days of the week. Families correspond to siblings attending the daycare facility. Each family has preferences over which days to attend. Daycare might or might not have priority over the families.

Another example is in the context of refugee resettlement. Here games correspond to localities that have weak priority and complex constraints over the families (see for example Delacrétaz et al. (2019) for a more detailed description of the application). Families' preferences are over the localities (bundles are of size 1 in this case).

In the course allocation of many universities, the courses are often identified as "cores" or "core electives" for specific majors or programs. Priorities are usually given to students in those majors and programs. Our model can be adapted to this application where the games correspond to courses, and families correspond to students, who have preferences over bundles of courses. Even with simple capacity resource constraints, our model offers an extension of (Budish, 2011) to accommodate priorities.

3 Technical Tools

This section establishes the main technical tool of our mechanism. We follow Nguyen and Vohra (2018) and use Scarf's lemma to show the existence of a dominating solution, which is the key to the allocation. However, the original Scarf's lemma (Scarf, 1967), which shows the existence of a dominating solution in a *linear system*, is not strong enough to capture our general resource constraints.

To accommodate general constraints, in this section, we develop a generalization of Scarf's lemma (Theorem 1). We will reuse some of the notations from Section 2. In the next section, we will apply it to our model, where the same notation will match each other. However, to maintain mathematical rigor, the notations used in this section should be treated independently from the rest of the paper.

3.1 Non-linear Scarf's Lemma

Let $\mathcal{F}$ and $\mathcal{G}$ be 0-1 matrices of size $m \times \ell$ and $n \times \ell$, respectively. We use F and G to denote the set of the rows of $\mathcal{F}$ and $\mathcal{G}$, respectively, and $f \in F$ and $g \in G$ to denote a single row of $\mathcal{F}$ and $\mathcal{G}$, respectively. $\mathcal{F}$ and $\mathcal{G}$ share the same set of columns denoted H. Let $h \in H$ be one of these columns.

For every row $f \in F$ and $g \in G$, let $H_f \subseteq H$ and $H_g \subseteq H$ denote the columns whose entry at row f and g is non-zero, respectively. Each row $f \in F$ and $g \in G$ has a *strict* ranking, denoted by $\gg_f$ and $\gg_g$, over the columns in H_f and H_g, respectively.

We assume that $\mathcal{F}$ has the following property:

$$\{H_f | f \in F\} \text{ is a partition of } H. \tag{4}$$

Associated with every row $g \in G$ is a continuous function $\Phi_g : \mathbb{R}^\ell \rightarrow \mathbb{R}$, such that $\Phi_g(.)$ is strictly increasing in each of the coordinates in H_g and is independent of the coordinates in $H \setminus H_g$ In our applications, this has a natural meaning that a resource is not influenced by how agents consuming *other* resources. We also assume $\Phi_g(\mathbf{0}) = 0$.

We are interested in a vector $\vec{x} \in \mathbb{R}^\ell$ satisfying

$$\vec{x} \geq \vec{\mathbf{0}}, \tag{5}$$

$$\mathcal{F} \cdot \vec{x} \leq \vec{\mathbf{1}}, \tag{6}$$

$$\Phi_g(\vec{x}) \leq 1 \quad \forall g \in G. \tag{7}$$

We start with the following definition of domination. A vector $\vec{x}$ satisfying (5-7)*dominates* a column h when there is at least a binding constraint in $(6-7)$ such that in the ranking of this constraint, column h has a lower rank than all other columns where x is positive. Formally:

Definition 1 (Dominate). *An $\vec{x} \in \mathbb{R}^\ell$ satisfying (5-7)* dominates *a column $h \in H$ if there exists* either

- *$f \in F$ such that: $h \in H_f$;* (6) *binds at row f; if $h \neq h' \in H_f$ and $x_{h'} > 0$, then $h \ll_f h'$*
- or *$g \in G$ such that: $g \in H_g$; $\Phi_g(x) = 1$; if $h \neq h' \in H_g$ and $x_{h'} > 0$, then $h \ll_g h'$.*

An informal but intuitive way to understand the definition of domination above is by a game between the column h and the rows $F \cup G$. Column h would like to convince all the rows in $F \cup G$ to increase the value of x_h. If the constraint at a row already binds, then to increase x_h it has to decrease the value of another coordinate $x_{h'} > 0$. This row will not agree to do so if it prefers h' to h in its ranking. Next, we define a dominating solution, which says that no column can convince the rows to increase their value.

Definition 2 (Dominating solution). *An $\vec{x} \in \mathbb{R}^\ell$ satisfying (5-7) is a* dominating solution *if $\vec{x}$ dominates h for every column $h \in H$.*

Our main result in this section is the following.

Theorem 1. *There exists a dominating solution for (5-7).*

Proof. The proof's intuition will follow the informal description of the definition of domination above. We will define a two-person (non-zero-sum) game and show that the equilibrium of this game corresponds to a dominating solution.

The first step is, however, technical. We will introduce new variables to (5-7) by adding a new variable y_f for each row $f \in F$. Let $\mathcal{I}$ be the identity matrix of size $m \times m$. Consider the following expanded system of inequalities.

$$\vec{x}, \vec{y} \geq \vec{\mathbf{0}}, \tag{8}$$

$$\mathcal{F} \cdot \vec{x} + \mathcal{I} \cdot \vec{y} \leq \vec{\mathbf{1}}, \tag{9}$$

$$\Phi_g(\vec{x}) \leq 1 \quad \forall g \in G, \tag{10}$$

$$\frac{1}{m+1} \sum_{f \in F} y_f \leq 1. \tag{11}$$

Notice that because of the assumption (4) if we add up all the constraints in (9), we obtain $\sum_{h \in H} x_h + \sum_{f \in F} y_f \leq m$. Therefore, (11) never binds. Hence, we call this constraint *dummy.*

We next define the following two-person game associated with (8-11):

The column player chooses $\vec{\beta} \in \mathbb{R}^{\ell+m}$. With slight abuse of notation, we use β_f for $f \in F$ to denote the added columns associated with y_f. The action $\vec{\beta} \in \mathbb{R}^{\ell+m}_{\geq 0}$ of the column player need to satisfy:

$$\sum_{h \in H} \beta_h + \sum_{f \in F} \beta_f = m. \tag{12}$$

The row players chooses $\vec{\alpha} \in \mathbb{R}_{\geq 0}^{m+n+1}$. We denote the indices of $\vec{\alpha}$ as $\{\alpha_f | f \in F\}$, $\{\alpha_g | g \in G\}$ and $\{\alpha_d\}$, which correspond to each of the constraints in (9), (10), and (11), respectively. The action $\vec{\alpha} \in \mathbb{R}_{\geq 0}^{m+n+1}$ of the row player need to satisfy:

$$\sum_{f \in F} \alpha_f + \sum_{g \in G} \alpha_g + \alpha_d = 1.$$

The payoff of the row player is tied with (8-11). In particular, the payoff for playing row f is $\sum_{h \in H} \mathcal{F}_{f,h} \cdot \beta_h + \beta_f$; for playing row g is $\Phi_g(\vec{\alpha}_H)$; and for playing row d is $\frac{1}{m+1} \sum_{f \in F} \beta_f$. The overall payoff is

$$\sum_{f \in F} \alpha_f \cdot \left(\sum_{h \in H} \mathcal{F}_{f,h} \cdot \beta_h + \beta_f \right) + \sum_{g \in G} \alpha_g \cdot \Phi_g(\vec{\alpha}_H) + \alpha_g \cdot \frac{1}{m+1} \sum_{f \in F} \beta_f. \tag{13}$$

Next, define the payoff matrix, $\mathcal{C}$, of size $(m+n+1) \times (\ell + m)$ for the column player. $\mathcal{C}$ will depend on the rankings $\ll_f$ and $\ll_g$. First, pick an integer $N > \ell + 1$, and let

- For $f \in F$ and $h \in H_f$, $\mathcal{C}_{f,h} = -N^k$ if h is ranked k^{th} according to $\ll_f$.
- For $f \in F$, $\mathcal{C}_{f,f} = -N^{\ell+1}$. (Notice here $\mathcal{C}_{f,f}$ denotes the entry of $\mathcal{C}$ at row f and the column corresponding to β_f.)
- For $g \in G$ and $h \in H_g$, $\mathcal{C}_{g,h} = -N^k$ if h is ranked k^{th} according to $\ll_g$.
- $\mathcal{C}_{d,f} = -N^{\ell+1}$ for all $f \in F$, where d is the new dummy row beside F and G.
- All other entries of $\mathcal{C}$ is 0.

The payoff of the column player is

$$\vec{\alpha}^T \cdot \mathcal{C} \cdot \vec{\beta}. \tag{14}$$

In the game defined above, each player's decision space is convex, closed, and bounded, and the payoff of each player is a linear function of their decision variables. They are continuous in the other player's strategy. Hence, a pure equilibrium exists Rosen (1964).

Let $\vec{\alpha}, \vec{\beta}$ be an equilibrium. In the remaining of the proof, we will show that $\vec{\beta}_H$ is a dominating solution of (5-7). Let r and c be the payoffs of the row player and the column player at this equilibrium, respectively.

First, we have the following observation.

Claim 1. *$r > 0, c < 0$.*

Proof. For any $\vec{\beta} \geq \vec{\mathbf{0}}$ satisfying (12), there is at least a row that gives the row player a positive payoff. Thus, $r > 0$.

Because of the way we set up payoff matrix $\mathcal{C}$, whenever a variable is in a constraint, the entry of the payoff matrix at the corresponding column and row is nonzero. Thus, at any equilibrium, there will be a row and a column at which $\vec{\alpha}$ and $\vec{\beta}$ are positive and the entry of the payoff matrix $\mathcal{C}$ is not zero. All entries of $\mathcal{C}$ are at most 0. Thus, $c < 0$ □

Next, we show the following claim.

Claim 2. *$r = 1$.*

Proof. Suppose $r < 1$, then the payoff of the row player at any row $f \in F$ is less than 1, which means the payoff from row $f \in F$ is

$$\sum_{h \in H} \mathcal{F}_{f,h} \cdot \beta_h + \beta_f < 1.$$

Because of assumption (4), summing all the $f \in F$, we obtain $\sum_{h \in H} \beta_h + \sum_{f \in F} \beta_f < m$, which contradicts the condition (12) of the column player.

With $r \geq 1$, we will show that the row's player payoff at every row $f \in F$ is r.

Assume not, let $f \in F$ row's player payoff is strictly less than r. This implies $\alpha_f = 0$. Notice that the payoff of the row player at row d is $\frac{1}{m+1} \sum_{f \in F} \beta_f \leq \frac{m}{m+1} < 1 \leq r$. Therefore, $\alpha_d = 0$. Now, consider the column player's payoff at column f. The only 2 non-zero entries of the payoff matrix are at rows f and d. Hence the column player can attain a payoff of 0 at this column. This is a contradiction to Claim 1.

Now, because the row player's payoff at every row $f \in F$ is r.

$$\sum_{h \in H} \mathcal{F}_{f,h} \cdot \beta_h + \beta_f = r.$$

Summing up these equations for all $f \in F$, we obtain $\sum_{h \in H} \beta_h + \sum_{f \in F} \beta_f = m \cdot r$. Because of the condition (12) for the column player, $r = 1$. □

Finally, we have the following claim.

Claim 3. *$\vec{\beta}_H$ is a dominating solution of (5-7).*

Proof. First, according to Claim 2, $r = 1$. Thus, for every row, the payoff of the row player is at most 1. Hence, $\vec{\beta}$ satisfies (8-11). This implies that $\vec{\beta}_H$ satisfies (5-7).

Let H^*, F^* be the set of columns at which the column player assigns positive β value. The payoff of the column player at these columns is c, and for any column $h \in H$, the payoff is at most c.

Consider a strategy, $\vec{\gamma}$, of the column player, by assigning equal weight to all columns in $H^* \cup F^*$. This means the value of each of these coordinates is $\frac{m}{|H^*|+|F^*|}$. Assuming the row player does not change $\vec{\alpha}$, the payoff of the column player remain c at $\vec{\gamma}$.

Consider another strategy, $\vec{\delta^h}$, where the column player puts all of the weight to column h. The column player's payoff, in this case, is at most c because of the payoff property by (14). This implies that $\vec{\alpha}^T \cdot \mathcal{C} \cdot \vec{\gamma} \geq \vec{\alpha}^T \cdot \mathcal{C} \cdot \vec{\delta^h}$. Therefore,

$$\vec{\alpha}^T \cdot (\mathcal{C} \cdot \vec{\gamma} - \mathcal{C} \cdot \vec{\delta^h}) \geq 0.$$

Together with $\vec{\alpha} \geq \vec{0}$, and $\sum_i \alpha_i = 1$, this implies that there must exist a row at which the value of the corresponding coordinate of $\vec{\alpha}$ is strictly greater than 0 and of $(\mathcal{C} \cdot \vec{\gamma} - \mathcal{C} \cdot \vec{\delta^h})$ is at least 0.

A positive coordinate of $\vec{\alpha}$ implies that the row player's payoff at that row is $r = 1$. This means that the corresponding constraint in the system (8-11) binds when $\vec{x} = \vec{\beta}_H$, $\vec{y} = \vec{\beta}_F$. We know that the constraint (11) cannot bind, therefore, this row will be in $F \cup G$. Assume this row is $f \in F$. The argument is similar for a row $g \in G$.

First, we will show that $\beta_f = 0$.

This will implies that the constraint at row f of the original inequality system (5-7) binds when $\vec{x} = \vec{\beta}_H$. That is, $\sum_{h \in H} \mathcal{F}_{f,h} \cdot \beta_h = 1$.

If $\beta_f > 0$, then $f \in F^*$ and the value of $\mathcal{C} \cdot \vec{\gamma}$ at row f is

$$\mathcal{C}_f \cdot \vec{\gamma} = \sum_{i \in H^* \cup F^*} \frac{\mathcal{C}_{f,i} \cdot m}{|H^*| + |F^*|} \leq \frac{\mathcal{C}_{f,f} \cdot m}{|H^*| + |F^*|} = -\frac{N^{\ell+1} \cdot m}{|H^*| + |F^*|} < -N^{\ell} \cdot m. \tag{15}$$

The last inequality holds because $|H^*| + |F^*| \leq \ell + 1 < N$.

On the other hand, the value of $C \cdot \vec{\delta^h}$ at row f is

$$\mathcal{C}_f \cdot \vec{\delta^h} = \mathcal{C}_{f,h} \cdot m \geq -N^{\ell} \cdot m \tag{16}$$

The last inequality holds because $\mathcal{C}_{f,h}$ is smallest when h is ranked the last in $\ll_f$.

However, (15) and (16) imply that $(\mathcal{C}_f \cdot \vec{\gamma} - \mathcal{C}_f \cdot \vec{\delta^h}) < 0$, a contradiction.

Now, we will show that for all $h \neq h' \in H^* \cap H_f, h \ll_f h'$.

Assume on the contrary that there exists $h' \in H^* \cap H_f$ and $h' \ll_f h$. Let k' and k be the rank of h' and h in $\ll_f$, respectively. We have $k' \geq k + 1$. We have

$$\mathcal{C}_f \cdot \vec{\gamma} = \sum_{i \in H^*} \mathcal{C}_{f,i} \frac{m}{|H^*| + |F^*|} \leq \mathcal{C}_{f,h'} \cdot \frac{m}{|H^*| + |F^*|} = -N^{k'} \frac{m}{|H^*| + |F^*|} < -N^{k'-1} \cdot m. \tag{17}$$

On the other hand, the value of $\mathcal{C} \cdot \vec{\delta^h}$ at row f is

$$\mathcal{C}_f \cdot \vec{\delta^h} = \mathcal{C}_{f,h} \cdot m = -N^{k} \cdot m \tag{18}$$

However, (17) and (18) together with $k' \geq k+1$, imply that $(\mathcal{C}_f \cdot \vec{\gamma} - \mathcal{C}_f \cdot \vec{\delta^h}) < 0$, a contradiction.

□

By showing $\vec{\beta}_H$ is a dominating solution of (5-7), we have shown the existence of a dominating solution. Notice that our proof is constructive. The computation of a dominating solution, however, depends on the computation of a two-person game. □

3.2 Rounding and Bounding Violation

We start with a definition of sparsity.

Definition 3. *A 0-1 matrix $\mathcal{M}$ is Δ-sparse, if every column of $\mathcal{M}$ contains at most Δ non-zero entries.*

We will use the following Theorem, which is based on iterative rounding. (See Nguyen et al. (2016) for proof.)

Theorem 2. *Given a Δ-sparse matrix $\mathcal{M}$ and matrix $\mathcal{F}$ satisfying (4). Let $\vec{x} \geq \vec{\mathbf{0}}$ be a fractional solution to $\{\mathcal{F}\vec{x} = \vec{\mathbf{1}}; \mathcal{M}\vec{x} \leq \vec{C}\}$, then there exists an integral $\vec{z} \geq 0$, such that $\mathcal{F}\vec{z} = \vec{\mathbf{1}}$, and*

$$\mathcal{M}\vec{z} \leq \lceil \vec{C} \rceil + (\Delta - 1) \cdot \vec{\mathbf{1}}.$$

Furthermore, if $x_i = 0$, then $z_i = 0$, and the algorithm to construct $\vec{z}$ from $\vec{x}$ is polynomial time in the size of $\mathcal{M}$ and $\mathcal{F}$.

In our application, we will start from a fractional allocation $x^*_{f,S}$ and round it to an integral one. The matrix $\mathcal{F}$ correspond to the constraint matrix of the families. $\mathcal{M}$ is created as follows.

Each row of the matrix $\mathcal{M}$ corresponds to a constraint for the total number of families of a given size. For example, if the maximum family size is 5, the number of rows of $\mathcal{M}$ will be $5n$, where recall that n is the number of games. The following constraint is created for each game g and a family size b.

$$\sum_{\substack{f,S:\ S \in G_f \\ g \in S,\ b_f = b}} x_{f,S} \leq \lceil \sum_{\substack{f,S:\ S \in G_f \\ g \in S,\ b_f = b}} x^*_{f,S} \rceil$$

Each columns of the $\mathcal{M}$ correspond to a variable $x_{f,S}$. Thus, if the largest bundle size is Δ, then $\mathcal{M}$ is Δ-sparse. Applying Theorem 2, we can obtain an integral solution from any fractional one, such that we add at most $\Delta - 1$ families of each size to each game.

In the sport-venue seating context, for example, the largest bundle is 6. Assume the fractional solution is to assign 999.5 families of size 1 and 1001.5 families of size 5 to a game. By putting constraints on each type of family, the integral solution will allocate to at most $\lceil 999.5 \rceil + 5 = 1005$ families of size 1 and at most $\lceil 100.5 \rceil + 5 = 1006$ families of size 5.

4 Competitive Stable Equilibrium

We introduce our main solution concept in this section. As discussed in the introduction, this concept is a natural combination of stability and competitive equilibrium with endowed income.

4.1 Competitive Stable Equilibrium

We recall that family $f \in F$ has a strict preference $\succ_f$ over bundles $S \in G_f$. For the games, we consider that the priority rule of each $g \in G$ depends only on the families. Each game g has a weak preference $\succeq_g$ over the families. Each family has a budget of d_f dollars.

In an equilibrium, each game is associated with a price p_g. Let $\vec{x} = \{x_{f,S}\}_{f \in F, S \in G_f}$ where each $x_{f,S} \in \{0,1\}$ denotes the vector of the bundle assignment for all families and $\vec{p}$ be the price vector of all the games. Let F_g denote the set of the families assigned to g according to $\vec{x}$.

Given $\vec{x}$ and $\vec{p}$, we define the *charge* c_g^f of family f for game g as follows.

(C1) f can get game g for free if g is not fully allocated, i.e. $c_g^f = 0$ if (3) does not bind.

(C2) f can get game g for free if $f \succ_g f'$ for some $f' \in F_g$, i.e. $c_g^f = 0$ if f is strictly more preferred by g to some families f' assigned to game g.

(C3) f cannot afford g if g is fully allocated, i.e. equality holds in (3), and $f' \succ_g f$ for all $f' \in F_g$, i.e. f is strictly less preferred by g compared to all families f' assigned to game g in $\vec{x}$. In this case $c_g^f = \infty$.

(C4) Otherwise, f needs to pay p_g dollars for game g, i.e. $c_g^f = p_g$. This case occurs when g is fully allocated in $\vec{x}$ and g does not have preference between f and the least preferred families that are assigned to g in $\vec{x}$.

A family can *afford* a bundle S if the total charge of family f for all games $g \in S$ is at most the family budget, that is, $\sum_{g \in S} c_g^f \leq d_f$. Given price $\vec{p}$, the *demanding bundle* S for family f is the bundle that is the most preferred in $\succ_f$ and affordable for f.

Now we are ready to define the *competitive stable equilibrium* (CSE).

Definition 4. *A* competitive stable equilibrium *(CSE) is a pair of price vector $\vec{p} \in \mathbb{R}^n_{\geq 0}$ and allocation vector $\vec{x}$ such that:*

1. *Feasibility: each agent consumes at most 1 bundle, i.e,* $\sum_{S \in G_f} x_{f,S} \leq 1$*; and for each* $f \in F$ *and* $\Omega(\vec{z}^g) \leq C_g$ *for each* $g \in G$*. Recall that* $x_{f,S} \in \{0, 1\}$ *and* z_b^g *is defined base on* (1)*.*

2. *Walras's law: if game* g *is not fully allocated, then* $p_g = 0$*.*

3. *Individual Rationality: each family gets its demanding bundle* S *according to* $\vec{x}$*, i.e.* $\sum_{g \in S} c_g^f \leq d_f$ *and for any* S' *such that* $\sum_{g \in S'} p_g \leq d_f$*,* $S \succ_f S'$*.*

To see why CSE is a useful solution concept, we analyze its stable and fair properties.

Definition 5. *Given a feasible assignment* $\vec{x}$ *of families to bundles of games, a family-bundle pair* (f, S) strongly blocks $\vec{x}$ *if*

- $x_{f,S} = 0$ *and* S *is preferred by* f *to the bundle assigned to* f*.*

- *for all* $g \in S$ *either the resource constraint at* g *does not bind, or* f *belongs to higher priority tier with another family currently assigned to* g*.*

Definition 6. *Given a feasible allocation* $\vec{x}$*, we say* f' envies f *if*

- f' *prefers* f*'s assigned bundle,* S_f*, to his own bundle, and*

- *for every* $g \in S_f$*, either* $f' \succeq_g f$*, or the resource constraint at* g *slacks.*

The allocation $\vec{x}$ *is* envy-free *if there no family that envies another.*

The following result comes directly from Definition 5.

Proposition 1. *Let* $(\vec{p}, \vec{x})$ *be a CSE, then* $\vec{x}$ *does not have any strong blocking pairs. Moreover, if agents are endowed with an equal budget, then* $\vec{x}$ *is envy-free.*

However, CSE does not always exist. The nonexistence of CSE holds even in a setting without priority, bundle preferences, and general resource constraints. A well-known example contains two agents, a diamond and a rock. Agents prefer the diamond to the rock and are given an equal budget of \$1. Perturbing budget by ε, however, will restore the existence of competitive equilibrium in this case. This motivates the following definition of ε-CSE, an approximate version of CSE.

Definition 7 (ε-CSE). *Given an instance* $(F, G, \{\succ_f\}_{f \in F}, \{\succeq_g, \Omega_g, C_g\}_{g \in G}, \vec{d})$*, a* $(\vec{p}, \vec{x})$ *is an* ε*-CSE if there exists* $\vec{d'} \geq \vec{0}$ *such that* $||\vec{d} - \vec{d'}||_\infty \leq \varepsilon$ *and* $(\vec{p}, \vec{x})$ *is a CSE with respect to* $\vec{d'}$*.*

Following Budish (2011), we will show that ε-CSE has the following approximate fairness property.

DEFINITION 8. *Given a feasible allocation $\vec{x}$, f'* strongly envies *f if for all $g \in S_f$, where S_f denotes the bundle assigned to f:*

- *f' prefers $S_f \setminus \{g\}$ to his own bundle, and*
- *either $f' \succeq_g f$, or the resource constraint at g slacks.*

THEOREM 3. *Let $(\vec{p}, \vec{x})$ be an ε-CSE, then $\vec{x}$ does not have any strong blocking pairs. Moreover, if agents are endowed with an equal budget of \$1, then for $\varepsilon \leq \frac{1}{2n}$, there will be no family that strongly envies another.*

Proof. If there is a strongly blocking pair (f, S), then f can obtain S with charge 0. This is a contradiction since f prefers S to his currently assigned bundle.

Assume f' strongly envies f, then for all $g \in S_f$, f' prefers $S_f \setminus \{g\}$ to his current bundle. But because f consumes his favourite bundle subject to his budget, the charge of bundle $S_f \setminus \{g\}$ is more than f''s budget. Because f' is weakly preferred at all games in S_f with positive charge, the charge to f' is not more than the charge to f at these games. Hence,

$$d_{f'} < \sum_{g' \in S_f \setminus \{g\}} c^f_{g'}.$$

Adding up these inequalities for all $g \in S_f$, we obtain

$$|S_f| d_{f'} \leq (|S_f| - 1) \sum_{g' \in S_f} c^f_{g'} = (|S_f| - 1) d_f.$$

Because $|S_f| \leq n$ and $d_f \leq 1 + \varepsilon \leq 1 + \frac{1}{2n}$, we have $d_{f'} \leq \frac{n-1}{n} d_f \leq \frac{n-1}{n}(1 + \frac{1}{2n}) < 1 - \frac{1}{2n}$. A contradiction to $\varepsilon < \frac{1}{2n}$ closeness of budgets. □

ε-CSE unifies existing methods in both one-sided and two-sided markets. For example, as special cases of Theorem 3, if the priority is strict, then an ε-CSE corresponds to a stable assignment. On the other hand, if all the families belong to the same priority tier and agents are endowed with an equal budget, our mechanism directly computes a competitive equilibrium with approximately equal budgets. In this case, our notion of strong envy is the same as the envy-free except for one item as defined in Budish (2011).

4.2 Our Mechanism

Suppose we have the set of families with their preferences and budget, the set of games with their preferences, capacities, and capacity functions, and an input parameter $\varepsilon > 0$. Formally, the input is a tuple $(F, G, \{\succeq_f\}_{f\in F}, \{\succeq_g, \Omega_g, C_g\}_{g\in G}, \vec{d}, \varepsilon)$. Our goal is to find an ε-CSE given this input. We propose the following mechanism, which returns an ε-CSE for *any* $\varepsilon > 0$.

1. Given input $(F, G, \{\succeq_f\}_{f\in F}, \{\succeq_g, \Omega_g, C_g\}_{g\in G}, \vec{d}, \varepsilon)$, we construct a general Scarf instance by the procedure `gen_Scarf`.

2. By Theorem 1, there exists a fractional dominating Scarf solution $\vec{x^*}$.

3. Round the solution via the rounding algorithm in Theorem 2, which shows that we can obtain an integral solution by adding at most $\Delta - 1$ families of each size to each game, where Δ is the size of the largest game bundle. Denote this integral solution by $\vec{\bar{x}}$.

4. Construct an ε-CSE $(\vec{p}, \vec{x})$ with the perturbed budget vector $\vec{d'}$ from $\vec{\bar{x}}$.

The Procedure `gen_Scarf` Given input $(F, G, \{\succeq_f\}_{f\in F}, \{\succeq_g, \Omega_g, C_g\}_{g\in G}, \vec{d}, \varepsilon)$, this procedure constructs the general Scarf matrix in Theorem 1 as follows.

Each family $f \in F$ is associated with a row of $\mathcal{F}$ and each game $g \in G$ is associated with a row of $\mathcal{G}$. Let $K := \max_{f\in F}\{|G_f|\}$ be the maximum number of bundles that a family is interested in. Let $\varepsilon' := \varepsilon/(2Kn)$. For family f, we set a different *perturbed budget* for each bundle $S \in G_f$. For the k-th most preferred bundle S, we set the perturbed budget of f to be $d_f(S) = d_f + 2(k-1)n\varepsilon'$, i.e. the better the bundle f gets, the lower the perturbed budget $d_f(S)$, and the budget differs by at least $2n\varepsilon'$ for different bundles.

Columns $h \in H$ are labelled $(f, S, \vec{q})$ for each $f \in F$, $S \in G_f$, and $\vec{q} \in \{0, \varepsilon', 2\varepsilon', ...\}^n$ such that $\sum_{g\in S} q_g \le d_f(S)$. In this case, we say that the price $\vec{q}$ is *feasible* for f and S[5].

In the $\mathcal{F}$ matrix, the row f' and column $(f, S, \vec{q})$ entry is one if and only if $f' = f$. In the $\mathcal{G}$ matrix, the row g and column $(f, S, \vec{q})$ entry is one if and only if $g \in S$ and $\vec{q}$ is feasible. We recall

[5] We note that is different from the charge of f, we do not have the assignment vector yet.

that H_f denotes the set of columns whose entry at row f is non-zero. The property (4) is satisfied because $\{H_f \mid f \in F\}$ forms a partition by this construction.

The preference ordering of each row $\ll_f$ and $\ll_g$ over the columns $(f, S, \vec{q})$ that is in the support of the row is defined as follows.

1. For a family row f: $\ll_f$ is based on the preference of family f over bundles first, then the increasing order of the bundle price, that is, $\sum_{g\in S} q_g$ when S is fixed. When bundle prices $\sum_{g\in S} q_g$ are the same, the tie is broken in a way consistent with a strict ordering.

2. For a game row g: $\ll_g$ is based on the preference of game g over family tiers first, then the decreasing order of the price q_g. That is, if $f =_g f'$, then g prefers the column whose price vector contains the higher price for game g. When the prices for game g are the same, and g does not have a preference between the two families, the tie is broken according to a fixed order (say lexicographical) over the families in the same tier. If both columns have the same family f and the same price q_g, then the game preference follows the preference of family f.

Let the capacity constraint be $\Phi_g = \Omega_g / C_g$. Now we have the general Scarf instance, i.e. the preferences $\ll_f$ and $\ll_g$ and the constraints (5-7).

Rounding the Scarf Solution Given a Scarf instance constructed by `gen_Scarf`, by Theorem 1, there exists a fractional dominating Scarf solution $\vec{x^*}$ of this instance. Given the solution $\vec{x^*}$ of Scarf, let $\vec{\bar{x}}$ be the rounded solution by Theorem 2. We round the solution $\vec{x^*}$ by the algorithm in Theorem 2 and the construction of matrix $\mathcal{M}$ in Section 3.2. Let $\vec{\bar{x}}$ be the integral solution after rounding and $\vec{\bar{z}}^g$ is obtained by (1). Then the perturbed capacity $C'_g = \Omega_g(\vec{\bar{z}}^g)$. As discussed in Section 3.2, the violation is measured by how many extra families of each size can be added to each game. Theorem 2 shows that it is at most $\Delta - 1$, where Δ is the size of the largest bundle.

Next, we show that the integral solution is a dominating solution after the rounding procedure with respect to the new capacity C'_g. This is because the rounding procedure ensures that the support of the rounded solution $\vec{\bar{x}}$ is contained within the support of the fractional solution $\vec{x^*}$, and if a constraint is binding under $\vec{x^*}$ it is still binding under $\vec{\bar{x}}$ with respect to the new capacity. We obtain the following lemma.

LEMMA 1. *The rounded solution $\vec{\bar{x}}$ is a dominating solution with respect to the new capacity C'.*

Proof. Without loss of generality, assume that all the family constraints bind; otherwise, it can be achieved by adding a dummy bundle that is less preferred than any other bundle. By Theorem 2, the integral dominating solution $\vec{\bar{x}}$ satisfies that $\mathcal{F}\vec{\bar{x}} = \vec{\mathbf{1}}$, and $\mathcal{M}\vec{\bar{x}} \leq \vec{C} + (\Delta - 1) \cdot \vec{\mathbf{1}}$. Furthermore, the support of $\vec{\bar{x}}$ is contained within the support of $\vec{x^*}$ since if $x_i^* = 0$ then $\bar{x}_i = 0$. Because $\mathcal{F}\vec{\bar{x}} = \vec{\mathbf{1}}$, all the family constraints bind. By definition of C'_g, all the game constraints also bind, that is, $\mathcal{G}\vec{\bar{x}} = \vec{C'}$. Therefore, $\vec{\bar{x}}$ is a dominating solution with respect to the perturbed capacity $\vec{C'}$. $\square$

From the Integral Scarf Solution to ε-CSE Suppose after rounding, we have an integral dominating solution $\vec{\bar{x}}$ for the general Scarf instance generated by `gen_Scarf` on input $(F, G, \{\succeq_f\}_{f\in F}, \{\succeq_g, \Omega_g, C'_g\}_{g\in G}, \vec{d}, \varepsilon)$. Let $x_{f,S} = \sum_{\vec{q}} \bar{x}_{f,S,\vec{q}}$ and $\vec{x} = \{x_{f,S}\}_{f\in F, S\in G_f}$ be the assignment vector. Let $F_g := \{f' \mid x_{f',S} = 1 \text{ and } g \in S\}$ be the set of families that are assigned to game g according to $\vec{x}$ and $\underline{F}_g := \{f' \mid f' \in F_g \text{ such that } \forall \tilde{f} \in F_g, \tilde{f} \succeq_g f'\}$ be the set of *lowest tier* families in F_g. We also obtain the corresponding price and perturbed budget for each $\bar{x}_{f,S,\vec{q}} = 1$. Let $p_g := \min\{q_g \mid \bar{x}_{f,S,\vec{q}} = 1 \text{ with } q_g \geq 0, g \in S \text{ and } f \in \underline{F}_g\}$ be the price for game g, and $\vec{p} = \{p_g\}_{g\in G}$ be the price vector. Since $\vec{\bar{x}}$ is integral, there is only one $\bar{x}_{f,S,\vec{q}} = 1$ for each f and we let the corresponding $d_f(S)$ be the perturbed budget of f. We show that $(\vec{p}, \vec{x})$ is an ε-CSE for the instance $(F, G, \{\succeq_f\}_{f\in F}, \{\succeq_g, \Omega_g, C'_g\}_{g\in G}, \vec{d})$.

Theorem 4. *$(\vec{p}, \vec{x})$ is an ε-CSE for the instance $(F, G, \{\succeq_f\}_{f\in F}, \{\succeq_g, \Omega_g, C'_g\}_{g\in G}, \vec{d})$.*

Proof. We recall that the definition of CSE consists of the feasibility requirements, Walras's law, and the property that each family gets its demanding bundle. Clearly, the feasibility requirements are satisfied by the general Scarf constraint (5) and (6).

In the integral dominating solution, we recall that when family f gets the k-th most preferred bundle S, then the perturbed budget of f is $d_f(S) = d_f + 2(k-1)n\varepsilon'$.

We start with showing Walras's law by the following lemma. The proof is provided in Appendix A.1.

Lemma 2. *If $p_g > 0$, then $\Omega_g(\vec{z}^g) = C'_g$ where z_b^g is defined in (1) based on $\vec{x}$.*

The remaining is to show that each family f gets its demanding bundle under the perturbed budget $d_f(S)$ where $x_{f,S} = 1$. We provide the proof in Appendix A.2.

Lemma 3. *If $x_{f,S} = 1$, then f prefers S to any bundle S' with charge $\sum_{g\in S'} c_g^f \leq d_f(S)$.*

We have shown that $(\vec{p}, \vec{x})$, obtained from the general Scarf solution $\vec{\bar{x}}$, satisfies all the three properties of CSE with the perturbed budgets. □

5 Reassign seats to sport venue

In this section, we apply our mechanism for reassigning seats in a sports venue. A sports venue is not restricted to American College Football, and it can also be Soccer, Basketball, Volleyball. College Football, NCAA sports, NFL, MLB, and other popular sports tickets were already sold out. With social distancing, only a fraction of people who already paid for the games can attend the games. For example, in a recent football game where Indianapolis Colts played against Green Bay Packers, there were 12,000 spectators where the Lucas Oil field's capacity is 70,000. This gives us a filling ratio of 17.14% because of social distancing. The Super Bowl 2021, that took place in the Raymond James Stadium in Tampa, Florida had around 25,000 people in the 65,000 capacity stadium yielding a 38% filling rate.

To illustrate role of choosing budgets and to keep it simple, we will consider a knapsack constraint for each game. Denote λ_f the number of seats needed for family f, the resource constraint is

$$\sum_f \lambda_f x_f \leq C.$$

Here C is the stadium's capacity and λ_f depends on the size of f. We will elaborate on these parameters in Section 5.3.

As discussed in the introduction, there is an inherent trade-off between efficiency and fairness because of the social distancing constraints. If priority is given to large-sized families, more people can be seated but leading to a high degree of unfairness. Choosing a "good" priority rule will be the key to balance this trade-off. For this purpose, we will assume all families belong to the priority tier and will compare the different implementations of prioritizing them. We consider two main methods: one is based on the mechanism we developed above, and the other is based on a draft mechanism (Budish and Cantillon, 2012).

5.1 Competitive Equilibrium Approach

The first mechanism we consider is directly based on our CSE method developed above. The remaining decision is to choose a budget for each family. Choosing a budget is a way to prioritizing families. Compared with the Draft mechanism described below, the advantage here is that prioritizing with a budget can be adjusted flexibly.

We consider two natural ways of budgeting.[6] First, because family f bought b_f tickets, we can assign a budget b_f to f. Second, because the social distancing constraint puts different coefficients for different family sizes, their consumption rates are different. Each person in a family f consumes $\frac{\lambda_f}{b_f}$ seats. When taking this into account, we will adjust the budget of the family f to be

$$d_f := b_f \cdot \frac{b_f}{\lambda_f} = \frac{(b_f)^2}{\lambda_f}.$$

5.2 Draft Mechanism Approach

A draft mechanism is a version of serial dictatorship, in which agents, in turn, select their best object in each round. To improve fairness, the order that agents are selected each round is the previous round's reverse order. We will use this idea to implement our mechanism.

Single Game

We start with the allocation of a single game and present two greedy-based algorithms as follows.

`Draft 1`: Allocate families to the games greedily based on a randomized order. (The randomized order can be thought of as a priority order based on club points or a random tie-breaking rule.)

`Draft 2`: Allocate families to the games greedily based on the family size first, then by the randomized order.

Remark. It is not hard to see that for a single game, `Draft 2` allocates more people than `Draft 1` does. Next, we consider the implementation of multiple games.

[6] We believe, however, by experimenting different ways of budgeting, one might find a better balance between efficiency and fairness, depending on the empirical setup. This direction is promising for many practical allocation problems but is out of the scope of this paper.

Multiple games

Given the preference over multiple games, the goal is to give as many people their better choice as possible under social distancing constraints.

The algorithm runs in multiple rounds. In each round, each game can have some capacity left. Each agent will have a part of their preference with some of their top choices cut off.

Step 1 For each game, consider the families who rank it first. If it is feasible to assign all of them, then assign them to the game. If not, then apply the algorithm for the single-game described above to select the set of families.

Next, families who got rejected by the first choice propose the second choice. At this point, some families that rank it first have been assigned to the game. If there is some available capacity, apply the single-game algorithm to select the families' set. If some families still get rejected, they apply to the third, the fourth choice, and so on until they get accepted, or there is no more game to propose to.

Step 2 After round 1, the only families that are not assigned to any games are the ones that are not interested in going to games that still have available seats. Among the families that are assigned, it is possible that not all families got their first choice. Let r_f be the rank of the game that family f is assigned to at this step.

Modify the families' ranking so that all the families assigned to the low ranked games are now on top.

Cut off the first r_f choices from the family's ranking of f so that the first choice is now $r_f + 1$.

Return to Step 1 with the new ranking of the families and the game with remaining capacities. Iterate this until there is no capacity left for each game or families have proposed to all games.

Remark. Because agents select games one by one, the mechanism cannot capture complementarities in preferences. The competitive equilibrium, on the other hand, takes into account this feature. Thus, theoretically, one can come up with an example where the draft mechanism performs infinitely worse than the competitive equilibrium approach. To obtain a fair comparison of different methods, the preferences we consider correspond to a linear scoring rule and avoid this issue. This is described in the next section.

5.3 Empirical Results

In this section, we compare CSE algorithms' performance with two different ways of budgeting described in the previous section with the two draft algorithms. Recall that the `CSE 1` algorithm sets the budget of family f to b_f and the `CSE 2` algorithm sets the budget of family f to b_f^2/λ_f. In our setting, `CSE 2` prioritizes families of bigger size more than `CSE 1` does since the ratio between the budget of bigger family size and the budget of smaller family size is larger in `CSE 2`.

Simulation Setup

In this setting, there are six games and 1000 families. Games do not have any priorities over families. We consider a specific setup of reassigning seats in a small stadium with 2650 seats. Here one possible configuration to maintain social distancing is to leave two empty seats and an empty row in between families. The table below provides data on the probability of family size and the number of seats needed. The average family size is 2.65. Therefore, the total number of seats can fit everyone without social distancing.

family size	seats needed λ	probability
1	6	15%
2	8	35%
3	10	30%
4	12	10%
5	14	10%

We consider the following way of generating family preferences over games and bundles.

1. Each family uniformly assigns a score between 0.2 and 1 at random to each game.

2. The preference of a family over games is based on the decreasing order of the scores.

3. The preference of a family over bundles is based on the decreasing order of bundle scores, where the bundle score is equal to the sum of the scores of all the games in the bundle.

We note that each ratio between the number of seats needed and the family size lies between 0.13 and 0.36. The estimated occupancy seating rate is about 0.3, so we consider bundles up to size 3.

Results

The following tables summarize the efficiency and fairness of all four algorithms. Table 2 shows that both CSE algorithms assign people to more games than the two draft algorithms do. The number of people assigned to one game is roughly the same for all four algorithms. Although fewer people are assigned to two games in CSE algorithms, many people in CSE algorithms get three games, and no one gets three games in the draft algorithms. In all algorithms, no person gets more than three games.

	`Draft 1`	`Draft 2`	`CSE 1`	`CSE 2`
1 game	760	706	774	700
2 games	1929	2037	1631	1176
3 games	0	0	245	624
4 games	0	0	0	0
5 games	0	0	0	0
6 games	0	0	0	0

Table 2: Distributions of people assigned to games

Table 3 illustrates that CSE algorithms outperform the two draft algorithms with respect to the number of people assigned to each game. Recall that the capacity of each game is 2650. On average over the six games, `Draft 1` occupies 28.6% of the capacity, `Draft 2` occupies 28.9% of the capacity, `CSE 1` occupies 30.0% of the capacity, and `CSE 2` occupies 31.1% of the capacity.

	`Draft 1`	`Draft 2`	`CSE 1`	`CSE 2`
Game 1	753	771	814	789
Game 2	755	757	811	792
Game 3	753	777	812	801
Game 4	755	738	748	817
Game 5	753	735	756	832
Game 6	773	816	830	893
Average	757	766	795	823

Table 3: Number of people assigned to each game

Tables 4 and 5 summarize the average bundle rank weighted by family size and the average bundle rank of each family size, respectively. Note that people in the same family have the same preference over bundles. The smaller the average bundle rank, the better the assignment. The average bundle ranks of CSE algorithms are slightly smaller than those of the draft algorithms. It means that, on average, people get better bundles in CSE algorithms. It is noteworthy that `Draft 1` randomly assigns families to games without priority while `Draft 2` prioritizes larger size families, so the average rank of each family size in `Draft 1` is roughly the same, and the large size families in `Draft 2` are generally assigned better bundles. For CSE algorithms, large size families are prioritized even more than `Draft 2`.

	`Draft 1`	`Draft 2`	`CSE 1`	`CSE 2`
Average bundle rank	8.53	8.16	8.00	7.00

Table 4: Average bundle rank (weighted by family size)

	`Draft 1`	`Draft 2`	`CSE 1`	`CSE 2`
Size 1	8.24	11.04	21.84	27.00
Size 2	9.19	12.07	19.33	20.26
Size 3	8.85	7.89	4.11	3.36
Size 4	8.32	5.25	2.13	1.38
Size 5	7.27	4.65	1.51	1.00

Table 5: Average bundle rank for each family size

Table 6 presents the average max envy and the average weighted envy.

	`Draft 1`	`Draft 2`	`CSE 1`	`CSE 2`
Average max envy	7.67	8.09	5.44	1.89
Average weighted envy	4.21	3.95	0.99	0.54

Table 6: Average envy

Tables 7 and 8 show the the average max envy and the average weighted envy for each family size, respectively. A family f envies another family f' if they have the same size and the rank of the bundle assigned to f is smaller than the rank of the bundle assigned to f'. It means that family f does not envy family f' if they do not have the same size (because their budgets are different) or f is assigned to a better bundle. The degree of envy of a family f with respect to another family f' is defined as the difference between the two ranks if f envies f', and 0 otherwise. The maximum envy of a family is equal to the maximum degree of envy over all other families of the same size. The average max envy and the average weighted envy are the averages of maximum envy over families and the average of maximum envy weighted by family size, respectively. Table 6, 7, and 8 illustrate that families in the two CSE mechanisms are less envious than families in the two draft mechanisms.

	Draft 1	Draft 2	CSE 1	CSE 2
Size 1	7.24	9.04	4.84	0.00
Size 2	8.19	11.07	10.33	3.26
Size 3	7.85	6.89	3.11	2.36
Size 4	7.32	4.25	1.13	0.38
Size 5	6.27	3.65	0.51	0.00

Table 7: Average max envy for each family size

	Draft 1	Draft 2	CSE 1	CSE 2
Size 1	4.09	3.85	1.16	0.00
Size 2	4.36	4.08	1.14	0.64
Size 3	4.20	2.64	1.04	0.96
Size 4	4.27	2.81	0.61	0.31
Size 5	3.80	2.56	0.39	0.00

Table 8: Average weighted envy for each family size

In all the simulations, the violation of capacity constraints is always less than 0.5 percent. It means that each game's capacity is adjusted by at most 14 seats, which is the number of seats required for a single-family of size 5.

6 Conclusion

Resource allocation with multi-unit demand, complex constraints, and weak preferences have a wide range of applications. The outcome of methods based on deferred acceptance algorithms depends on a specific tie-breaking rule and is not stable in the presence of combinatorial preferences. On the other hand, approaches based on competitive equilibrium do not accommodate strict priority and usually allow for lotteries that does not guarantee any measure of ex-post envy. We provide a new class of mechanisms that unifies the concepts of stability and approximate competitive equilibrium with indivisible goods. Our solution inherits the efficiency and fairness properties of both existing methods and extends them to a richer environment. Our mechanism is based on a nontrivial extension of Scarf's lemma to nonlinear constraints, which might be of independent interest. We apply our method to the problem of reassigning seats in sports events with social distancing. Our numerical simulations show that our approach outperforms standard mechanisms in both efficiency and fairness measures.

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A Missing proofs in Section 4

A.1 Proof of Lemma 2

LEMMA 2. *If $p_g > 0$, then $\Omega_g(\vec{z}^g) = C'_g$ where z^g_b is defined in* (1) *based on $\vec{x}$.*

Proof. For the sake of contradiction, suppose $\Omega_g(\vec{z}^g) < C'_g$. Suppose $p_g > 0$ and let f be a family with $\bar{x}_{f,S,\vec{q}} = 1$ where $q_g = p_g$. We focus on a feasible price vector $\vec{\tilde{q}}$ for family f, where $\tilde{q}_g = 0$ and $\tilde{q}_{g'} = q_{g'}$ for $g' \in S \setminus \{g\}$. Then the column $(f, S, \vec{\tilde{q}})$ is not dominated by $\vec{\bar{x}}$ because (1) f prefers $(f, S, \vec{\tilde{q}})$ to $(f, S, \vec{q})$ since $\sum_{g\in S} \tilde{q}_g < \sum_{g\in S} q_g$, (2) for each game $g' \in S \setminus \{g\}$, g' prefers $(f, S, \vec{\tilde{q}})$ to $(f, S, \vec{q})$ since when the two columns have the same family f and the same price for game g', the preference of game g' follows f, where f prefers $(f, S, \vec{\tilde{q}})$ to $(f, S, \vec{q})$, and (3) row g cannot dominate because it is not binding. This is a contradiction because $\vec{\bar{x}}$ is a dominating solution. □

A.2 Proof of Lemma 3

LEMMA 3. *If $x_{f,S} = 1$, then f prefers S to any bundle S' with charge $\sum_{g\in S'} c^f_g \leq d_f(S)$.*

Proof. We consider the relation between the charge for each family and the price. We recall the difference between affordability and feasibility. A bundle S is affordable for family f if the total charge does not exceed the perturbed budget, i.e. $\sum_{g \in S} c_g^f \leq d_f(S)$. The price $\vec{q}$ is feasible for f and S if the total price does not exceed the perturbed budget, i.e. $\sum_{g \in S} q_g \leq d_f(S)$. The high level strategy is to consider all the possible cases of the charging rule (C1)-(C4) and bound the difference between the charge c_g^f and the price q_g.

For case (C1), by Lemma 2, we have that each family f assigned to game g pays $c_g^f = q_g = 0$ for game g when g is not fully allocated.

Case (C2) happens when $f \in F_g \setminus \underline{F}_g$, i.e. f is not the least preferred family assigned to game g according to $\succeq_g$. We show that each non-lowest tier family assigned to g does not pay for game g.

LEMMA 4. *Given $\bar{x}_{f,S,\vec{q}} = 1$ and $g \in S$ such that $f \in F_g \setminus \underline{F}_g$, $q_g = 0$.*

Proof. For the sake of contradiction, suppose there exists $\vec{q}$ with $q_g > 0$ such that $\bar{x}_{f,S,\vec{q}} = 1$. We focus on a feasible price vector $\vec{\tilde{q}}$ for family f where $\tilde{q}_g = 0$ and $\tilde{q}_{g'} = q_{g'}$ for $g' \in S \setminus \{g\}$. Then the column $(f, S, \vec{\tilde{q}})$ is not dominated by $\vec{\bar{x}}$ because (1) f prefers $(f, S, \vec{\tilde{q}})$ to $(f, S, \vec{q})$ since $\sum_{g \in S} \tilde{q}_g < \sum_{g \in S} q_g$, (2) for each game $g' \in S \setminus \{g\}$, g' prefers $(f, S, \vec{\tilde{q}})$ to $(f, S, \vec{q})$ since when the two columns have the same family f and the same price for game g', the preference of game g' follows f, where f prefers $(f, S, \vec{\tilde{q}})$ to $(f, S, \vec{q})$, and (3) row g prefers $(f, S, \vec{\tilde{q}})$ to some $(f', S', \vec{q'})$ with $x_{f',S',\vec{q'}} = 1$ and $f' \in \underline{F}_g$ because $f \in F_g \setminus \underline{F}_g$ and $f \succ_g f'$. This is a contradiction because $\vec{\bar{x}}$ is a dominating solution. □

For case (C3), when g is fully allocated and $f' \succ_g f$ for any $f' \in F_g$, the charge $c_g^f = \infty$ so f cannot afford game f. This also implies that $x_{f,S} = 0$ because $f \notin F_g$.

For case (C4), when g has no preference between f and the lowest tier families assigned to g, we show that the difference between the charge and the price for f is small.

LEMMA 5. *Suppose $f =_g f'$ for any $f' \in \underline{F}_g$ and $\bar{x}_{f,S,\vec{q}} = 1$ where $g \in S$, then $q_g \leq p_g + \varepsilon'$.*

Proof. For the sake of contradiction, suppose $f =_g f'$ for any $f' \in \underline{F}_g$, $\bar{x}_{f,S,\vec{q}} = 1$ where $g \in S$, and $q_g > p_g + \varepsilon' = c_g^f + \varepsilon'$. We focus on the feasible price vector $\vec{\tilde{q}}$ for family f where $\tilde{q}_g = p_g + \varepsilon'$ and $\tilde{q}_{g'} = q_{g'}$ for $g' \in S \setminus \{g\}$. Then the column $(f, S, \vec{\tilde{q}})$ is not dominated by $\vec{x}$ because (1) f prefers $(f, S, \vec{\tilde{q}})$ to $(f, S, \vec{q})$ since $\sum_{g \in S} \tilde{q}_g < \sum_{g \in S} q_g$, (2) for each game $g' \in S \setminus \{g\}$, g' prefers $(f, S, \vec{\tilde{q}})$ to $(f, S, \vec{q})$ since when the two columns have the same family f and the same price for game g', the preference of game g' follows f, where f prefers $(f, S, \vec{\tilde{q}})$ to $(f, S, \vec{q})$, and (3) row g prefers $(f, S, \vec{\tilde{q}})$

to some $(f', S', \vec{q'})$ where $\bar{x}_{f',S',\vec{q'}} = 1$, $g \in S'$, $f' =_g f$, and $q'_g = p_g$ because $\tilde{q}_g > p_g = q'_g$. This is a contradiction because $\vec{\bar{x}}$ is a dominating solution. □

Given the relation between the feasible price and the charge, we are ready to show that each family f gets its demanding bundle under the perturbed budget $d_f(S)$ where $x_{f,S} = 1$.

Clearly, f can afford S with perturbed budget $d_f(S)$ by the Scarf construction. For the sake of contradiction, suppose there exists an affordable bundle $S' \in G_f$ for f such that $S' \succ_f S$ and $\sum_{g \in S'} c_g^f \leq d_f(S)$. We focus on the feasible price vector $\vec{\tilde{q}}$ for family f, where $\tilde{q}_g = p_g + \varepsilon'$ for $g \in S' \cap S$ and $\tilde{q}_{g'} = p_{g'} + 2\varepsilon'$ for $g' \in S' \setminus S$. We note that S' under price $\vec{\tilde{q}}$ is feasible for f because $d_f(S') - d_f(S) \geq 2n\varepsilon'$. We show that $(f, S', \vec{\tilde{q}})$ is not dominated by $\vec{\bar{x}}$. Clearly, the column $(f, S', \vec{\tilde{q}})$ is not dominated at row f because f prefers S' to S.

In order to find a row that dominates $(f, S', \vec{\tilde{q}})$, we only consider fully allocated games, where its constraint (3) binds. Let $\vec{q}$ be such that $\bar{x}_{f,S,\vec{q}} = 1$. We consider the following cases:

1. If $g \in S' \cap S$, then the column $(f, S', \vec{\tilde{q}})$ is not dominated at row g. By Lemma 5, $\tilde{q}_g = p_g + \varepsilon' \geq q_g$. Either one of the following holds (1) $\tilde{q}_g > q_g$ or (2) when the two columns have the same family f and the same price for game g, the preference of game g follows f, where f prefers $(f, S', \vec{\tilde{q}})$ to $(f, S, \vec{q})$. In either case, row g prefers column $(f, S', \vec{\tilde{q}})$ to column $(f, S, \vec{q})$.

2. For every $g \in S' \setminus S$, we consider the following three cases:

   (a) Suppose g strictly prefers all families in F_g to f. In this case, f cannot afford g because $c_g^f = \infty$ which contradicts to our assumption that f can afford g. This implies that $(f, S', \vec{\tilde{q}})$ is not dominated at row g.

   (b) If $g \in F_g \setminus \underline{F}_g$, then g prefers column $(f, S', \vec{\tilde{q}})$ to some other column $(f', S'', \vec{q'})$ where $f' \in \underline{F}_g$ and $g \in S''$, i.e. there is a lower tier family f' assigned to g, so $(f, S', \vec{\tilde{q}})$ is not dominated at row g.

   (c) The remaining case is $f =_g f'$ for any $f' \in \underline{F}_g$. In this case row g prefers column $(f, S', \vec{\tilde{q}})$ to some other column $(f', S'', \vec{q'})$ with $q'_g \leq p_g + \varepsilon'$ by Lemma 5. This is because $\tilde{q}_g = p_g + 2\varepsilon' > q'_g$. $(f, S', \vec{\tilde{q}})$ is not dominated at row g.

None of the rows can dominate column $(f, S', \vec{\tilde{q}})$. This contradicts to the assumption that $\vec{\bar{x}}$ is a dominating solution. □